Banach-Hilbert Spaces, Vector Measures and Group Representations

Tsoy-Wo Ma

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BANACH–HILBERT SPACES, VECTOR MEASURES AND GROUP REPRESENTATIONS
Enlarged Edition of Classical Analysis on Normed Spaces

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Preface

This book provides an elementary introduction to classical analysis on normed spaces with special attention to nonlinear topics such as fixed points, calculus and ordinary differential equations. In this second edition, a new approach to vector measures on δ-rings based on dominated convergence is introduced. Matrix-representations of groups are also included because mean-values of almost periodic functions behave similar to translation-invariant integrals and are available in infinite dimensional Banach spaces. This book is for beginners who want to get through the basic material as soon as possible and then do their own research immediately. It assumes only general knowledge in finite dimensional linear algebra, simple calculus, elementary complex analysis and in the last part also elementary group theory. The treatment is essentially self-contained except chapter 27 which may be skipped without discontinuity. With sufficient details, even an undergraduate with mathematical maturity should have no trouble to work through it alone. Various chapters can be integrated into parts of a Master Degree Program by course work organized by any regional university. Restricted to $\mathbb{R}^n$ rather than normed spaces, selected chapters can be used for a course in advanced calculus. We also hope that Engineers and Physicists would find this book to be a handy reference in classical analysis especially our approach to vector measures. High school teachers may be interested to enrich their programs by including the generalization of triangles and tetrahedra as treated in our chapter 4. Some special features are highlighted below.

Banach-Hilbert Spaces

• Sequences can be interpreted as samples taken per unit time. It seems to be more intuitive to use them as description of topological properties.
• Simplicial Complexes are treated in details for potential school projects if restricted to $\mathbb{R}^2$, $\mathbb{R}^3$.
• Transition from $\mathbb{R}^n_1$ to finite dimensional spaces is analytic, §5-1.9,12.
• Explicit formula for no retraction is given in §5-2.4.
• Infinite dimensional topological results are developed without homology.
• Higher derivatives in addition to first derivatives are represented by matrices.
• Higher Chain-Products Formulas §§10-6.2,3 are expressed more naturally in polynomials rather than in symmetric multilinear maps.
• Local solution interval to initial valued problem is independent of initial data.
• Dependence on initial conditions are in global setting. As an informal illustration, suppose a commodity can last one year in a laboratory. Local theory

says that its mass production should work for at least a few seconds but our global theory ensures a period of at least 300 days.

• Tensor products of vectors and linear maps are defined separately but we prove that they are consistent in §15-5.11.

• To the best of our knowledge, tensor products of operators on Hilbert spaces §§15-7.9 to 16 are our contribution.

Vector Measures

• Existing instructors do not have to learn new trick in order to demonstrate their leadership in helping a new generation of scientists to equip with better tools in *vector* measures-integrals.

• Our proposed approach can replace most existing courses in scalar measure theory because the treatment is self-contained without assuming Egorov's Theorem or semivariations from scalar-theory.

• Complex vector lattices are used as framework for measures and means.

• Breakable vector lattices ensure that order bounded linear forms are linear combinations of positive linear forms. This unifies several proofs in §§17-3.4, 24-6.9, 27-3.2, 31-3.2. Some results of real vector lattices are extended to complex breakable vector lattices.

• Semirings are the starting points of *all* our measures. Finite variation is characterized in terms of order §17-3.6 and absolute convergence, §17-4.5.

• Measures are defined on δ-rings so that they need not be bounded. Sets in δ-rings are called *decent sets*. They correspond to bounded Borel sets in $\mathbb{R}^n$.

• Measures are of finite variation in order to use breakable vector lattices.

• Functions of finite variation are related to Stieltjes measures.

• Simple proof of certain complex charge to be of finite variation is given in §18-1.4.

• Restriction to finite-valued outer measures allows the approximation of extension to decent sets by values on sets in semirings, §18-3.3,4,6. As a result, integrals of decent functions are defined.

• A set with a δ-ring is called a δ-space. Measurable sets are defined by localization and are independent of all measures.

• Measurable functions in general are finite-valued and are defined everywhere but μ-measurable functions depend heavily on a particular measure μ, §22-3.4.

• Approximation by simple functions has the additional property of increasing modulus, §19-4.4.

• Explanation §19-5.1,3 why measurable (vector) maps should not be defined trivially as sequential limits of simple maps as in most literatures so that continuous maps are measurable.

• Extension of sign-function to vector maps, §19-6.2.
• Weak and weak-star measurability of maps are unified in §19-6.7.
• Inner regularity defines the integrals of simple functions, §20-1.3.
• Positive measures are defined for all measurable sets, §20-2.2, while vector measures are defined only on integrable sets, §21-1.6.
• Integrals of vector maps are defined by Dominated Convergence, §21-2.8.
• L_p-spaces are in the context of vector maps and vector measures, §21-4.2,4,6.
• L_∞ is defined without measure, §21-5.1.
• Various modes of convergence are defined for vector maps.
• Algebra of measures are developed in §§20-5, 21-8.2.
• Products of vector measures do not require σ-finiteness, §22-2.2
• Product spaces are in the context of vector measures, §22-3.9,19.
• Reduction of elementary operations in linear algebra into two cases, §23-5.3.
• Absolute continuity is characterized at the level of semirings §§24-2.7,9.
• Polar form §24-4.4 reduces complex measures to positive measures.
• The condition of σ-finiteness for Radon-Nikodym theorem together with the vector version of concentration of continuous linear forms §24-6.6 removes the σ-finiteness from the duality of L_p-spaces for $1 < p < \infty$, §24-6.7.
• Cantor set and function are developed on the familiar decimal system, §25-4.
• Spectral measures are treated as a continuation of functional calculus.
• Simple classical technique is employed for monotone convergence, §26-1.7. Our approach is intuitive as shown by the proof of spectral theorem, §26-6.4.
• Spectral measures are extended from semirings to σ-algebras, §26-2.
• Regularity of measures on locally compact spaces are defined in terms of variation to accommodate vector measures.

Group Representations

• Our notation of mean-values strongly indicates the resemblance to integrals but monotone convergence theorem fails for mean-values, §§31-1.3,4. We wish to draw the attention of the community that something similar to translation-invariant integrals has been available on infinite dimensional locally convex spaces although we restrict ourselves to Banach spaces in this book.
• Further development on top of von Neumann's almost periodic functions, abbreviated as ap-functions, has close relation to group representations.
• Restriction to *matrix* representations of groups avoids unnecessary formality.
• Matrix-valued maps are used whenever possible.
• On $\mathbb{R}$, we may be interested only in continuous objects because continuous characters are of the exponential form $e^{i\theta x}$ but our groups have no topology. We get by with comfortable almost periodic functions, abbreviated

as cap-functions. We propose the study of saturated closed invariant ideals, §§30-3.2,3,5,8,9,20.

• Develop product groups based on representations, §31-1.6.

• Means on groups are defined as continuous linear forms on cap-functions. Rich properties §§31-2.4,5,6 deserve better attention from the community.

• As the dual spaces, monotone convergence theorem holds, §§31-3-10, 11.

• $\ell_1(G)$ may be the infinite dimensional counter part of $L_1(\mathbb{R}^n)$ but the establishment of $\ell_p(G)$ versus $L_p(\mathbb{R}^n)$ is an open challenge to readers.

Web-page of this book has been in service since 1997:

http://www.maths.uwa.edu.au/~twma/free/norm/norm.htm

or http://maths.uwa.edu.au/~twma/free/norm/norm.htm

Every paragraph is prefixed with a unique identification. For example, §§1-2.3, 4.5,6 means chapter 1, section 2 paragraph 3 and section 4 paragraphs 5,6. Notations are introduced or recalled at the early stage of a section and then will be used throughout the section unless further specification is mentioned. Theorems without proofs and exercises have the same meaning to us. Exercises are normally illustrative but not tricky because we believe that your time may be used more profitably to do your own research.

References are given at the end of every chapter. Each reference is identified by the family name of the first author only and, if necessary, also by the year of publication. If necessary again, the first letter of one or several words from the title will be included. This method is purely for convenience because it is easy to understand and is independent of the enumeration of references in different books and papers. In order to arouse the curiosity of the beginners, the selection of references is based on informative titles rather than the significance in the history of development. To enrich their cultural background, readers are advised to look up the titles even they may not have the time or facility to study the literatures. Our references offer starting points for further study or to do research in various areas. For example, polymeasures, multimeasures and special functions on groups are not covered in this book but limited references are included. We also include a few references on unbounded measures because our measures need not be bounded. Each reference is normally mentioned only once in the book although it may be related to several chapters. In order to cut down the number of references, if a series of papers on related topics is published by the same person, only the most recent one available in our record is normally quoted. It should give enough information if readers want to trace their earlier work.

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Introduction

1. This book essentially consists of four parts:

Chapters 1 to 12 in Normed spaces;
Chapters 13 to 15 in Hilbert spaces and tensor products;
Chapters 16 to 27 in vector measures on δ-rings and
Chapters 28 to 31 in group representations.

Within the first part, nonlinear analysis by topological method is covered in Chapters 4,5 and nonlinear analysis involving differentiation in Chapters 8 to 11. Readers with suitable background should start with any of Chapters 4, 8, 13, 16 or 28. They can also work on these chapters concurrently. *Informal* introduction is given here while the chapters tend to be concise and precise. Maps in this book generally need *not* be scalar-valued but functions normally are scalar-valued. Complex functions mean that they need not be real. Characteristic functions are denoted by ρ instead of χ so as to avoid possible confusion with x, X in hand-writing.

Banach Spaces

2. The first three chapters of this book provide the necessary background for any course in analysis. Sequences are used consistently to characterize continuity, closures, completeness, precompactness and compactness. The fact that closed bounded sets are compact is proved by upper and lower limits rather than bisecting infinite sets. One dimensional intermediate value theorem, fixed point theorem and structure of open sets §2-10.9 follow as a result of connectedness. Partition of unity is developed at the end of first chapter and will be used twice in §§5-3.3, 12-4.5. After finite dimensional normed spaces are characterized, standard criteria of compactness in infinite dimensional spaces (Ascoli's theorem) and also approximation of continuous functions (Stone-Weierstrass theorem) are given by the end of chapter 3. This would quickly cover the topological background required by advanced calculus and complex variable at the undergraduate level. It would be interesting to see the connection of functional analysis and neural network, e.g. [Cotter]. For general history of functional analysis, consult [Dunford], [Dieudonne-81] and [Musielak].

3. Affine approximations on simplicial complexes have been playing an active role in computation with computers as indicated for example by [Kearfott],

[Talman], [Todd], [Eaves], [Mara]. Because surfaces can be approximated by gluing triangles together and solids by tetrahedra, perhaps Mechanical Engineers and also experts in physical chemistry, e.g. [Bytheway] may be interested to know more about its theory. Restricting to one, two and three dimensional spaces, e.g. [Steenrod], [Shashkin], chapter 4 also offers an opportunity for high-school projects. Convex sets §4-2 will be required in the subsequent context. However if you are not interested in topological method, skip to chapter 6. Alternatively, you read only the statement of §4-10.8, and then skip to chapter 5. We treat chapter 4 thoroughly. Proofs on geometrical independence in §4-1 are rarely found in any existing textbooks. In §4-4.11, vertices are proved to be extreme points. The rest of the chapter is devoted to the construction of simplicial approximations. Good background has been laid for you to continue the study in simplicial homology theory which is beyond the scope of this book. Consult [James], [Fan-90] for history.

4. Instead of waiving our hands demanding acceptance with faith of topological invariance, Borsuk-Ulam theorem is transported from $\mathbb{R}_1^n$ in §5-1.4 to finite dimensional normed space §5-1.12 with scaling homeomorphism. Brouwer's fixed point theorem is derived from Borsuk-Ulam theorem with explicit formula in §5-2.4. Retraction theorem is proved by elementary technique which is within the capacity of undergraduates. A general fixed point theorem on convex sets is given in §5-3.5. The treatment of compact fields is simplified from [Granas-62] and only Tietze's extension theorem is used to develop homotopy extension theorem. By the end of chapter 4, we have covered most of the traditional applications of classical algebraic topology but in a more general context of infinite dimensional spaces as in [Granas-62]. The whole chapter is within the reach of undergraduates without asking them to take anything for granted. Consult for example, [Steinlein], [Fan-99] and [Jaworowski] for further information about Borsuk-Ulam theorem.

5. Standard material of linear functional analysis is developed in chapters 6 and 7. Hahn-Banach extension theorems guarantee that there are sufficient amount of continuous linear forms to separate convex sets. This is applied to derive Krein-Milman theorem. It also allows us to reduce certain cases from infinite to one dimensional spaces, e.g. §§8-3.3, 4.6, etc. Uniform boundedness theorem ensures that weakly bounded sets are norm-bounded. One of its applications is given in §8-4.5. References to show that open map and closed graph theorems cannot be generalized to bilinear maps are included. Chapter 7 covers topics such as embedding into bidual spaces, duality of quotient spaces

and subspaces, direct sums, transposes, reflexive spaces and weak convergence.

6. Vector-valued maps of a scalar variable are introduced in chapter 8. We support the simple and elegant way to develop integration theory starting with step maps. As soon as the fundamental theorem of calculus is proved, we can evaluate integrals as antiderivatives and this is what we use most of the time. At this point, we expect the readers to have basic knowledge of complex analysis including criteria of holomorphic functions and Cauchy's integral formula. These results are treated in the context of Banach spaces. Laurent series expansion including Taylor series as special case, is done from scratch. Liouville's theorem follows from the characterization of polynomials from entire maps. Resolvent map is an important example of holomorphic vector maps. It is used to show that the spectrum of an operator on complex Banach space is non-empty. Chapter 8 finishes with holomorphic maps of an operator defined by Cauchy's integral formula together with a practical formula for functions of square matrices without Jordan forms. See e.g. [Taylor-71] for history and [Fan-96], [Sharma] for recent development.

7. Chapters 9, 10 are devoted to advanced calculus. Most undergraduate textbooks restrict themselves to scalar-valued functions of several variables. Based on our numerical examples §§9-3.8, 5.6, 7, 10-5.6, 15-4.13, you may be interested to standardize the notations of higher derivatives of maps and also polynomials from $\mathbb{R}^n$ to $\mathbb{R}^m$ in terms of matrices. Transition from scalar to vector variable for differential theory is given in §9-1.2. Integral and uniform mean-value theorems of §§9-2.7,8 later become Taylor's formula and its corollary §§10-5.2,3. Inverse and implicit mapping theorems are proved by contraction which is also an important tool in numerical analysis due to its ability to estimate the error. Local properties of differentiable maps §9-6 are restricted to finite dimensional maps because of the simplicity of using determinants although they have been extended to infinite dimensional spaces. The theorem on Lagrange multiplier is modified from [Sagan] without the assumption on the rank of certain matrix. It is well-known that a special case of §10-5.4 is an important tool in differentiable manifolds, e.g. to prove the Morse's lemma. The higher chain formula §10-6.2 is expressed in the natural setting of polynomials rather than multilinear maps in unnecessary generality. Our proof involves only simple combinatorial method. Consult e.g. [Ma-01] for inverse mapping theorem on locally convex spaces.

8. Chapter 11 deals with initial value problem $x' = f(t, x)$. Existence of common solution interval for all initial conditions near a given point is proved

in §11-1.6 which is used to derive ***global*** continuity of initial condition in §11-5.4 where the *size of solution interval* is guaranteed by an interval (α, β). Its importance is illustrated by an example §11-1.10. As a result of continuity, the domain Ω_f in §11-5.5 of the flow associated with the vector field f is open and the flow φ is smooth §11-5.10. Since we do not carry the topic any further, we do not introduce the concept of flow in the context. Our boundary theorem §11-2.6 holds in infinite dimensional spaces. Linear differential equations are studied in §11-3 and §11-4. Two estimates of the solution to a linear equation are given in §11-3.4. Fulmer's method of finding the exponential function of a matrix is by method of differential equations. Finally, topological method is used to generalize Peano's theorem to infinite dimensional spaces. Consult e.g. [Lobanov] for locally convex spaces. Just like compact fields, we prefer to suitable rather than maximum generality in an undergraduate course. We believe that at this point, students are well equipped to study differential topology which is beyond the scope of this book. Most of this chapter work for complex Banach spaces even we restrict ourselves to the real case.

9. Chapter 12 deals with compact linear operators. Fredholm alternative §12-2.9 is stated in a form identical to a characterization of non-singular matrices in linear algebra. Readers should look up some references in order to have a feeling of the importance of compact linear operators because it is probably the simplest tractable infinite dimensional linear operators. For example, their spectra are null sequences, §12-3.4. The proof of a lemma for the existence of hyperinvariant subspace is slightly shortened by the order of the operators K, T in §12-4.5. For an extension of Lomonosov's techniques to non-compact operators, see [Simonivc].

Hilbert Spaces

10. Systematic exposition of Hilbert spaces is given in two subsequent chapters. They should be read concurrently with the corresponding topics in Banach spaces. The link among operators, sesquilinear forms and quadratic forms is established first. The second half-chapter studies various types of operators based on the star operation. Geometric properties of subspaces are characterized in terms of algebraic equations of projectors. It is a pity that my teaching duty was governed by official course outlines otherwise I would start with C^*-algebras.

11. The spectrum of an operator on Hilbert space is studied in chapter 14. For a self-adjoint operator, the connection between its quadratic form, its spectrum

and its norm is given by §14-2.12. Diagonal operators generalize diagonal matrices of which the properties are completely determined by the diagonal coefficients. It turns out that every compact normal operator is diagonable. As an important consequence §14-5.14, every compact operator on a Hilbert space can be approximated by finite dimensional operators in norm. This is not true in Banach spaces, e.g. [Enflo-73], [Szankowski]. The chapter ends with a functional calculus of self-adjoint operators and polar decomposition. Consult [Gohberg-00] for traces and determinants on Hilbert spaces and Konig-75 on Banach spaces.

12. Our tensor products of vector spaces are constructed within the framework of product and dual spaces without the naive concept of formal sums. The notation $f \otimes g$ is justified in §15-4.1. In contrast to the current system, we adopt the reverse lexical order in §15-4.3 because of the matrix representations of multilinear maps, e.g. §15-4.9, 12. Tensor products of linear maps are defined in a way completely different from tensor product of vectors. We prove that they are consistent under the natural injection in §15-5.11. It is obvious that tensor products of normal operators are normal. Motivated by finite dimensional spaces, the converse is given in §15-7 for several classes of normal operators which to the best of our knowledge were our contribution.

Vector Measures on δ-Rings

13. My interest in measure theory was inspired by [Apostol, pp207-212], [Zaanen-59, p42], [Kolmogorov], [Loomis, 12C] in early sixties and my teaching assignment from mid-seventies until 1986. After a long gap, I began to work in vector measures started with [Dinculeanu-67] and [Diestel-77] during 2000 and 2001. In addition to our own new results, most steps in the following treatment are modifications with *improvement* of known facts but our final overall version seems to be unique in this area. We introduce breakable vector lattices to unify proofs, inner regularity to define measures from decent sets. Then we extend to integrable sets and finally define vector integrals by dominated convergence property. We do not need Egorov's Theorem, semivariations, Vitali-Hans-Saks Theorem, Nikodym's Theorem in order to develop the vector version of measures. We also explain why measurable vector maps should not simply be defined as sequential limits of simple maps if we want continuous maps on locally compact spaces to be measurable. Through our innovative approach at no extra cost, existing staffs can easily help a new generation of scientists to have better tools of integration used practically everyday in our lives. If

measure theory on σ-rings is an abstract generalization of Lebesgue measure then measures on δ-rings are the counter parts of Stieltjes measures. Finite-valued signed measures on σ-rings must be bounded but measures on δ-rings need not be bounded. See [Hawkins], [Chae] for history of early development. All our measures are of finite variation. The following is a more detailed introduction.

14. Conjugation rather than the popular complexification is introduced in Chapter 16. Our definition of complex vector lattices is easier to verify and sufficient to provide service to this book although most of the concrete examples also satisfy the popular axioms. We also introduce breakable vector lattices as generalization of *real* vector lattices, §16-3.12. Its existence is justified by the order duals §16-5.5 and the unification of proofs in §§17-3.4, 24-6.9, 27-3.2, 31-3.2.

15. Semi-intervals and semi-rectangles should be used as guiding examples of semirings in Chapter 17. Every finite family of semi-rectangles can be decomposed into disjoint unions of semi-rectangles. This is formalized analytically into Semiring Formula, §17-1.9 and developed into Step Mapping Theorem, §17-2.3. Vector charges and vector integrals are introduced in §17-2. Algebraic method §17-2.10,11 identifying charges and linear forms simplifies the proof of geometrical results, §17-2.12. We characterize finite variation in terms of order-boundedness §17-3.6 and also absolute convergence, §17-4.5 by modifying [Munster]. Admissible bilinear map ensures the continuity connecting vector measures and integrands. Countable additivity of charges is related to monotone convergence in set-level §17-5.7 and at function-level, §17-5.10. All measures in this book are of finite variation. Their variations are defined in §17-3.4 and proved to be measure in §17-6.5. Being of finite variation of maps on $\mathbb{R}$ is defined in terms of charges so that we can apply measure theory to get the old results later. For an approach without the abstract theory of this chapter, see [Apostol; Thm 8-14, also Ch. 9]. For application to stochastic processes, see for example [Dinculeanu-00s].

16. A δ-space is a set with a δ-ring. Sets in the δ-ring are called decent sets for convenience. In Chapter 18, we give a simple proof that countably additive scalar charges are of finite variation. Uniqueness of extension of vector measures from semirings to generating δ-rings is proved in §18-1.11. The actual extension of positive measures is by standard method of outer measures. In order to have rich algebraic operations among complex measures, it is necessary

to start off §18-2.3 with a finite-valued function on $\mathbb{S}$. Classical results from [Kolmogorov] are put into abstract setting.

17. Motivated by earlier Chinese edition of [Xia], measurable sets are defined by localization independent of measures. We try to avoid μ^*-measurable sets which are obtained from an outer measure μ^*. It follows with standard properties of measurable functions. Approximation of measurable functions by simple functions with increasing modulus prepares the ground to define integrals by dominated convergence. Measurable complex functions and vector maps take values in $\mathbb{R}$, $\mathbb{C}$ or Banach spaces but measurable real functions are allowed to take $\pm\infty$ as value in order to define integrals. Example §19-5.3 explains why we have to define measurable maps §19-5.5 by localization in order to include continuous maps on locally compact spaces, §27-1.4. The characterization §19-5.4 is obtained by modification of [Kuttler, 23.1]. Uniform approximation of measurable maps by sequences of simple maps is given in §19-6.4,5. Our lemma §19-6.7 takes care of both weak and weak-star measurability at the same time. Finally we prove that sequantial limits of measurable maps are measurable.

18. Upper functions are measurable extended-valued positive functions. The integrals of upper functions with respect to positive measures are developed in Chapter 20. Measurable functions are approximated by simple functions but our measures are defined only on decent sets. The crucial bridging step is by inner regularity, §20-1.3. Positive measures are defined on measurable sets §20-2.2 but vector measures on integrable sets only, §21-1.6. Equality almost everywhere and σ-finiteness allow us remove infinity from values of integrable functions so that their sums are defined. The important features of integration theory takes its most primitive form in §20-3. Characterization of σ-finite sets §20-4.9 in terms of decent sets will be used repeatedly. Working with two positive measures is motivated by [Taylor-65].

19. Our unique approach to vector integration with respect to a vector measure μ is carried out in Chapter 21. Since the variation $|\mu|$ is a positive measure, μ-measurable sets are defined and can be approximated by decent sets. This is the basis of extending a vector measure μ to all integrable sets. For a measurable map f, its variation $|f|$ is an upper function and the integral $\int |f| \; d|\mu|$ defines the scope of integrable maps of which their integrals $\int f d\mu$ are defined by dominated convergence §21-2.8. Important results of vector L_p-spaces for $1 \leq p < \infty$ required by classical harmonic analysis is given

in §21-4. They include Dominated Convergence Theorem, Monotone Convergence Theorem, Integration Term by Term and differentiation under integral sign. Density theorem §21-4.6 incorporates increasing modulus. Vector L_∞-spaces are introduced without any measure because of the need in spectral measures in §26-4. All modes of convergence are for vector maps. This Chapter concludes with integration on subspaces and comparison with improper Riemann integrals.

20. Because our machinery begins with semirings, they are applicable to the product spaces directly in Chapter 22. Product measures does not require σ-finiteness, §22-2.2. To have nice formula $|\mu \otimes \nu| = |\mu| \otimes |\nu|$, we have to assume one of them to be scalar, §22-2.4. To proof the Fubini's Theorem, most text book takes the advantage of $0\infty = 0$ to shorten the proofs. For vector measures, we have to trace back bolts and nuts of our machine in §22-3.2. Exercise §22-3.7 is a modified version of [Bartle]. We may identify the L_p-space on the product space $X \times Y$ as the tensor product of L_p-spaces on X, Y respectively by §22-3.19.

21. Outer measures are recalled in Chapter 23 because it is handy to check if a set is null. The reduction of three elementary operations in linear algebra into two §23-5.3 can shorten many proofs including our §23-5.5. Because our measurable sets are independent of measures, it requires a little work to show that smooth images of measures sets are measurable §23-5.7. Change-variable of multiple integral is influenced by [Cohn, pp170-175].

22. Chapter 24 is devoted to Radon-Nikodym derivatives and duality of L_p-spaces. The first section on the relationship between $hd\mu$ and μ is an improvement of [Garnir-72, pp142-157]. Absolute continuity of measures is brought to the level of semirings §24-2.7,8 so that it can link to absolutely continuous maps, §24-2.9. Convergence in L_p is characterized in terms of convergence in measures and equicontinuity at the empty set. Positive and negative sets are initiated as tools but Hahn Decomposition Theorem is derived after Radon-Nikodym Theorem for simplicity. Polar form §24-4.4 reduces complex measures to positive measures. Tight restriction to σ-finite sets for Radon-Nikodym Theorem will be used later. The isometry of L_p with L'_q, §24-6.2,12 are modification of [Dinculeanu-67, pp229,234]. Continuous linear forms on vector L_p-spaces for $1 < p < \infty$ concentrates on σ-finite sets, §24-6.6. This result is a vector version obtained from [Bartle]. The duality of L_p-spaces for $1 < p < \infty$, §24-6.7 does not require σ-finiteness

any more. Sharper results for scalar measures make use of our breakable vector lattices, §24-6.10. Sufficient condition for Banach spaces to have Radon-Nikodym property is reorganized from [Diestel-77, ch 3].

23. In Chapter 25, cubes rather than general measurable sets as in [Rudin-74], are used for the geometrical expression of Radon-Nikodym derivatives because they are simple, intuitive and probably general enough for applications in physics. We prefer to Vitali cover instead of Raising Sun Lemma because Vitali Covering Theorem is consistent with our approach via semi-intervals. Lemma §25-1.10 is modified from [Cohn, p181]. The general theory is applied to the specific situation of the real line §25-2 and we start with factorizing pulse functions. Our Cantor set and function are developed on the familiar decimal system, §25-4.

24. The root of spectral measure theory on Hilbert space H in Chapter 26 is the product formula §26-2.2 based on §13-9.8 that if the sum of projectors is a projector then the products of any two summands are zero. In §26-2.5,6 we explain why spectral measures cannot be derived as a special case of vector measures. Spectral measures on semirings are extended in §26-2 to decent sets by finite variation and finally to measurable sets by inner regularity. Spectral integrals are defined by dominated convergence,§26-3.2. Commutativity is transferred from measures on decent sets to spectral integrals, §26-2.12,3.10,11. Null sets are defined in terms of μ_{xy} for $x, y \in H$ rather than one single measures but the ground has been prepared in §21-5. Properties unique to spectral measures are developed in §26-4. Product spectral measures prepare the amalgamation of spectral measures of self-adjoint operators to normal operators. For a specific operator A, $f(A)$ is defined in §14-6 for every continuous function f. In the first section of this chapter, we extend to semi-continuous functions to obtain a spectral measure on semi-intervals leading to spectral representation of A. Finally, isolated eigenvalues provide an interpretation of spectral measures in terms of diagonable matrices.

25. For integration on locally compact spaces in Chapter 27, we define regularity in terms of valuations of vector measures. Positive linear forms on the spaces of continuous functions with compact support are identified with regular positive measures and through breakable vector lattices, order-bounded linear forms on with regular measures. Finally the duals of continuous functions on compact spaces are identified with regular measures.

Almost Periodic Functions and Group Representations

26. Almost periodic functions appear in older books such as [Loomis] and [Yosida] but practically vanish in most of recent texts. We want to promote them because mean-values behave like translation-invariant integrals while Haar integrals demand local compactness which is not available in infinite dimensional groups or spaces. Probably special functions or harmonic analysis [Gong] could be developed in infinite dimensional cases.

27. We motivate the readers with a simple example that the sum of two periodic functions need not be periodic and almost periodic functions, abbreviated as ap-functions, have closed relation to group representations. We follow [vonNeumann] and [Maak] closely to introduce mean-values, convolutions and eigen expansion in terms of projectors. We restrict ourselves to matrix representations because of their simplicity and richness. Matrix notation is used whenever possible. Chapters 28,29 should cover the contents of most undergraduate courses in this area. Last two chapters are what we have done on top of [vonNeumann]. For a history of ap-functions, see [Levitan].

28. In real life, we are interested only in continuous unitary representations such as $e^{i\theta x}$ in one dimensional case. We have to define the scope of ap-functions that we work comfortably. Motivated by the duality of compact groups, we introduce saturated closed invariant ideals of comfortable almost periodic functions on groups G, abbreviated as cap-functions. Chapter 30 deals with the duality between cap-functions and representations. Finally, we point out the special cases of additive groups of normed spaces and compact groups.

29. Although mean-values behave like integrals but Monotone Convergence Theorem fails §31-1.3,4. The final Chapter starts with representations of product groups. The mean space $M(G)$ is defined as the dual of $C_\infty(G)$ of cap-functions. It turns out that $M(G)$ has rich structures including convolution, variation and mean-values. With dual order, Monotone Convergence Theorem and Fatou's Lemma hold for means. Parallel to integration theory, we embed cap-functions into $M(G)$ and its closure $\ell_1(G)$ acts like the counter part of L_1-spaces. Due to the restriction of §30-3.9, we have are unable to develop something on $\ell_p(G)$. Look up our web-page for recent development.

30. We hope that the uniqueness of this book could fill in a gap among the current literatures.

twma-2001.

Chapter 1

Metric Spaces

1-1 Standard Finite Dimensional Vector Spaces

1-1.1. Most treatments of functional analysis are applicable to both real and complex cases. In order to unify our notation, let $\mathbb{K}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Write $i^2 = -1$. The conjugate of a complex number z will be denoted by z^-, the real part by $\text{Re}(z)$ and the imaginary part by $\text{Im}(z)$ respectively. Scalar-valued maps are normally called *functions* in this book.

1-1.2. Let E be a vector space. A function $x \to \|x\|$ from E into $\mathbb{R}$ is called a *norm* on E if for all $x, y \in E$, we have

(a) $\|x\| \geq 0$, positive ;

(b) $\|x\| = 0$ iff $x = 0$, non-degenerate ;

(c) $\|x + y\| \leq \|x\| + \|y\|$, triangular inequality ;

(d) $\|\lambda x\| = |\lambda| \; \|x\|$, for every $\lambda \in \mathbb{K}$, scalar multiplication.

A vector space together with a given norm is called a *normed space*. Norms generalize the concept of absolute values of numbers.

1-1.3. **Example** The vector space $\mathbb{K}^n$ consists of columns of n numbers in $\mathbb{K}$ but for convenience we shall frequently write them as rows: $x = (x_1, x_2, \cdots, x_n)$ where x_j is the j-th coordinate of x. The same notation will be applied to other letters without further specification. For each $x \in \mathbb{K}^n$, let

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}.$$

It can be easily verified that they are norms on $\mathbb{K}^n$. We shall write $\mathbb{K}^n_1$ and $\mathbb{K}^n_\infty$ to indicate the normed spaces with specific norms in use. Note that we implicitly assume $n \geq 1$.

1-1.4. **Schwartz's Inequality** For all $x, y \in \mathbb{K}^n$, let $< x, y > = \sum_{j=1}^n x_j y_j^-$. Then we have $| < x, y > | \leq \|x\| \; \|y\|$. The number $< x, y >$ is called the *inner product* of x, y. Chapter 13 on Hilbert spaces should be read concurrently.

Proof. Consider the special case when all coordinates x_j, y_j are positive. Since

$$\|x\|^2 + 2t < x,y > +t^2\|y\|^2 = \sum_{j=1}^n (x_j + ty_j)^2 \geq 0$$

for all real number t, the discriminant of the above positive definite quadratic form in a real variable t must be negative, i.e. $< x,y >^2 - \|x\|^2\|y\|^2 \leq 0$ which gives the result. The general case is obtained from the following simple calculation: $| < x,y > | \leq \sum_{j=1}^n |x_j|\ |y_j| \leq \|x\|\ \|y\|$. □

1-1.5. **Example** The expression

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} = \sqrt{< x,x >}, \quad \forall\, x \in \mathbb{K}^n$$

defines a norm on $\mathbb{K}^n$ which is called the *usual norm* or the Euclidean norm. According to coordinate geometry, $\|x\|$ is the distance from the origin to x. We shall write $\mathbb{K}_2^n$ to indicate this norm. Whenever no norm is mentioned explicitly, the usual norm is assumed.

Proof. We shall prove the triangular inequality only and leave the other verification as an exercise. Observe that

$$(\|x+y\|\ )^2 = \sum_{j=1}^n (x_j + y_j)(x_j + y_j)^- = \sum_{j=1}^n \{x_j x_j^- + x_j y_j^- + x_j^- y_j + y_j y_j^-\}$$
$$= \sum_{j=1}^n \{x_j x_j^- + 2\mathrm{Re}(x_j y_j^-) + y_j y_j^-\} \leq \sum_{j=1}^n \{|x_j|^2 + |y_j|^2 + 2|x_j|\ |y_j^-|\}$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\ \|y\| = (\|x\| + \|y\|)^2.$$

Taking square root gives $\|x+y\| \leq \|x\| + \|y\|$. □

1-1.6. **Exercise** Let E be a normed space. The set $\{x \in E : \|x\| = 1\}$ is called the *unit sphere* of E. Sketch the unit spheres of the normed spaces $\mathbb{R}_1^2$, $\mathbb{R}_2^2$ and $\mathbb{R}_\infty^2$.

1-1.7. **Exercise** Let E, F be normed spaces. For every vector (x,y) in the product vector space $E \times F$, let

$$\|(x,y)\|_1 = \|x\| + \|y\|,$$

$$\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$$

and $$\|(x,y)\|_2 = \sqrt{\|x\|^2 + \|y\|^2}.$$

Prove that these are all norms on $E \times F$. The product set $X \times Y$ together with one of the above norms is called a *product normed space*.

1-1.8. **Exercise** Prove the inequality : $\big|\ \|x\| - \|y\|\ \big| \leq \|x - y\|$ for all x, y in a normed space.

1-2 Convergent Sequences in Metric Spaces

1-2.1. In analysis, we are interested in the concept of approximation and convergence which will be described in term of distance between two points in spaces. A metric space does not require any algebraic structure. However apart from the discrete metric spaces, practically all metric spaces in this book can be regarded as subsets of normed spaces.

1-2.2. Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if for all $x, y, z \in X$, we have
(a) $d(x, y) \geq 0$, positive;
(b) $d(x, y) = 0$ iff $x = y$, non-degenerate;
(c) $d(x, y) = d(y, x)$, symmetric;
(d) $d(x, z) \leq d(x, y) + d(y, z)$, triangular inequality.
The ordered pair $X[d]$ is called a *metric space*. For simplicity we shall use the symbol d for all metrics whenever there is no ambiguity. Hence we write X instead of $X[d]$.

1-2.3. **Example** Let X be a non-empty set. For all $x, y \in X$, let $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ otherwise. Then d is obviously a metric on X called *discrete metric*. In this case, $X[d]$ is called a *discrete metric space*.

1-2.4. **Exercise** Let X be a subset of a normed space E. For all $x, y \in X$, let $d(x, y) = \|x - y\|$. Show that d is a metric on X. In particular, every normed space is a metric space.

1-2.5. Let X be a set. A map from the subset $\{1, 2, 3, \cdots\}$ of integers into X is called a *sequence* in X. Because we always emphasize on the image of the function we shall adopt the notation $\{x_n : n \geq 1\}$ or simply $\{x_n\}$. For definiteness and convenience, we always assume that the starting index is 1 unless it is specified otherwise.

1-2.6. Let X be a metric space. A sequence $\{x_n\}$ in X is said to *converge to a point* $b \in X$ if for every $\varepsilon > 0$, there is an integer p such that for every $n \geq p$, we have $d(x_n, b) \leq \varepsilon$. In this case, the point b is called the *limit* of $\{x_n\}$. We shall write $x_n \to b$ or $\lim x_n = b$ as $n \to \infty$. Clearly a sequence $\{x_n\}$ converges to $b \in X$ iff the sequence $\{d(x_n, b)\}$ of real numbers converges to $0 \in \mathbb{R}$.

1-2.7. **Theorem** Every convergent sequence $\{x_n\}$ has a unique limit.

Proof. Suppose to the contrary that $a \neq b$ are limits of a convergent sequence $\{x_n\}$. Then for $\varepsilon = \frac{1}{3}d(a, b) > 0$, there are integers p, q such that $d(x_n, a) \leq \varepsilon$

for all $n \geq p$ and $d(x_n, b) \leq \varepsilon$ for all $n \geq q$. Let $n = p + q$. Then the following contradiction establishes the proof:

$$3\varepsilon = d(a,b) \leq d(a,x_n) + d(x_n,b) = d(x_n,a) + d(x_n,b) \leq \varepsilon + \varepsilon = 2\varepsilon. \qquad \square$$

1-2.8. A sequence $\{y_n\}$ is called a ***subsequence*** of $\{x_n\}$ if there is a sequence of integers $n(1) < n(2) < n(3) < \cdots$ such that $y_j = x_{n(j)}$ for all j. Since all indices of sequences in this book start with 1, we have $n(j) \geq j$ for all j.

1-2.9. **Theorem** If $x_n \to b$ and if $\{y_n\}$ is a subsequence of $\{x_n\}$ then $y_n \to b$.

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to b$, there is an integer p such that for all $n \geq p$ we have $d(x_n, b) \leq \varepsilon$. With the same notation of last paragraph, for every $j \geq p$, we have $n(j) \geq j \geq p$ and hence $d(y_j, b) = d(x_{n(j)}, b) \leq \varepsilon$. This proves $y_n \to b$. $\square$

1-2.10. **Exercise** Various intervals denoted by circular and square brackets are described by the following examples : $[2,3) = \{x \in \mathbb{R} : 2 \leq x < 3\}$, $(3,4] = \{x \in \mathbb{R} : 3 < x \leq 4\}$, $[4,3] = \{x \in \mathbb{R} : 4 \leq x \leq 3\} = \emptyset$ and $(3,3) = \{x \in \mathbb{R} : 3 < x < 3\} = \emptyset$. Is the sequence $\{\frac{1}{n}\}$ convergent in the metric spaces $\mathbb{R}$, $[0,1]$ and $(0,1]$ respectively?

1-2.11. **Exercise** Let $x_n = (\frac{1}{n}, \sqrt{n+1} - \sqrt{n})$ and $y_n = (3, (-1)^n)$. Prove or disprove that they are convergent in the normed spaces $\mathbb{R}^2_1$, $\mathbb{R}^2_2$ and $\mathbb{R}^2_\infty$ respectively.

1-2.12. **Exercise** Prove that if $x_n = a \in X$ for all n, then $x_n \to a$.

1-2.13. **Exercise** Prove that in a discrete metric space X, if $x_n \to a$ then there is an integer p such that for all $n \geq p$, we have $x_n = a$.

1-3 Continuous Maps

1-3.1. Let X, Y be metric spaces and let $f : X \to Y$ be a given map. Then f is said to be ***continuous at a point*** $b \in X$ if for every sequence $x_n \to b$ in X, we have $f(x_n) \to f(b)$ in Y. Plotting the sequence $\{x_n\}$ and its image $\{f(x_n)\}$ should give an intuitive idea that when x is near b, $f(x)$ must be near $f(b)$. The map f is said to be ***continuous on*** X if it is continuous at every point of X. In calculus, typical examples of continuous functions include polynomials, exponential functions and trigonometric functions. At the end of this chapter, we shall prove that every metric space has plenty of continuous functions.

1-3.2. **Theorem** Let X, Y, Z be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are given maps.
(a) If f is continuous at $b \in X$ and if g is continuous at $f(b)$ then the composite map gf is continuous at $b \in X$.
(b) If f is continuous on X and if g is continuous on Y then the composite map gf is continuous on X.

Proof. Let $x_n \to b$ in X. Since f is continuous at b, we have $f(x_n) \to f(b)$. Since g is continuous at $f(b)$, we get $g[f(x_n)] \to g[f(b)]$, i.e. $(gf)(x_n) \to (gf)(b)$. This proves (a). Part (b) becomes an easy exercise. □

1-3.3. **Exercise** Show that every map from a discrete metric space into a metric space is continuous.

1-3.4. Let X, Y be metric spaces. For all (x, y) and (a, b) in the product set $X \times Y$, let

$$d_1[(x, y), (a, b)] = d(x, a) + d(y, b);$$

$$d_2[(x, y), (a, b)] = \sqrt{d(x, a)^2 + d(y, b)^2}\ ;$$

and
$$d_\infty[(x, y), (a, b)] = \max\{d(x, a), d(y, b)\}.$$

It is routine to verify that d_1, d_2, d_∞ are metrics on $X \times Y$. The product set $X \times Y$ together with one of the metrics d_1, d_2, d_∞ is called a *product metric space.*

1-3.5. **Theorem** Let (x_n, y_n) and (a, b) be points in the product metric space $X \times Y$. The sequence $\{(x_n, y_n)\}$ converges to (a, b) in $X \times Y$ iff $x_n \to a$ in X and $y_n \to b$ in Y.

Proof. We shall prove part the case d_1 but leave the cases d_2, d_∞ as exercises. Suppose $(x_n, y_n) \to (a, b)$ in $X \times Y$. Then we have

$$0 \leq d(x_n, a) \leq d_1[(x_n, y_n), (a, b)] \to 0$$

as $n \to \infty$. Hence $x_n \to a$ in X. Similarly $y_n \to b$ in Y. Conversely, suppose $x_n \to a$ in X and $y_n \to b$ in Y. Then

$$d_1[(x_n, y_n), (a, b)] = d(x_n, a) + d(y_n, b) \to 0$$

as $n \to \infty$. Therefore $(x_n, y_n) \to (a, b)$ in $X \times Y$. □

1-3.6. **Exercise** Let X, Y, Z be metric spaces. Show that the projection $\pi : X \times Y \to X$ given by $\pi(x, y) = x$ is continuous. Prove that a map $f : Z \to X \times Y$ is continuous iff both coordinate maps $\pi f : Z \to X$ and

$\varphi f : Z \to Y$ are continuous where $\varphi : X \times Y \to Y$ is the projection onto the second coordinate.

1-3.7. **Exercise** Prove that every convergent sequence $\{x_n\}$ in a normed space is bounded, i.e. there is $M > 0$ such that $\|x_n\| \leq M$ for all n. Note that bounded sets in metric spaces will be defined later in §2-2.1. Also see §2-2.9.

1-3.8. **Theorem** (a) The addition on a normed space E is a continuous map from the product space $E \times E$ into E.
(b) The scalar multiplication is a continuous map from the product space $\mathbb{K} \times E$ into E.

Proof. (a) Let $(x_n, y_n) \to (a, b)$ be a convergent sequence in $E \times E$. Then $x_n \to a$ and $y_n \to b$. Observe that

$$\|(x_n + y_n) - (a + b)\| = \|(x_n - a) + (y_n - b)\| \leq \|x_n - a\| + \|y_n - b\| \to 0,$$

as $n \to \infty$. Therefore the addition is continuous.
(b) Let $\lambda_n \to \alpha$ in $\mathbb{K}$ and $x_n \to a$ in E be convergent sequences. Then $\{\lambda_n\}$ is bounded in $\mathbb{K}$. There is $M > 0$ such that all $|\lambda_n| \leq M$. Now observe that

$$\begin{aligned}&\|\lambda_n x_n - \alpha a\| = \|\lambda_n(x_n - a) + (\lambda_n - \alpha)a\| \\ &\leq |\lambda_n|\ \|x_n - a\| + |\lambda_n - \alpha|\ \|a\| \leq M\|x_n - a\| + |\lambda_n - \alpha|\ \|a\| \to 0\end{aligned}$$

as $\to \infty$. Therefore the scalar multiplication is continuous. □

1-3.9. **Exercise** Prove that the function $x \to \|x\|$ is continuous on E.

1-3.10. **Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Then f is continuous at a point $b \in X$ iff for every $\varepsilon > 0$, there is $\delta > 0$ such that whenever $x \in X$ satisfies $d(x, b) \leq \delta$, we have $d(f(x), f(b)) \leq \varepsilon$.

Proof. ($\Rightarrow$) Let f be continuous at $b \in X$. Suppose to the contrary that $\exists\ \varepsilon > 0, \forall\ \delta > 0, \exists\ x \in X,\ d(x, b) \leq \delta$, and $d(f(x), f(b)) > \varepsilon$. Taking $\delta = 1/n$, there is $x_n \in X$ such that $d(x_n, b) \leq 1/n$ and $d(f(x_n), f(b)) > \varepsilon$. Therefore $x_n \to b$ but $f(x_n) \not\to f(b)$. Consequently, f cannot be continuous at b.
($\Leftarrow$) Assume $x_n \to b$ in X. Let $\varepsilon > 0$ be given. Find $\delta > 0$ such that $d(x, b) \leq \delta$ implies $d(f(x), f(b)) \leq \varepsilon$. Choose p so that $\forall\ n \geq p, d(x_n, b) \leq \delta$, that is $d(f(x_n), f(b)) \leq \varepsilon$. Therefore $f(x_n) \to f(b)$. Consequently f is continuous at b.

1-3.11. **Exercise** Let X, Y, Z be metric spaces and let (a, b) be a point in the product space $X \times Y$. Prove that a map $f : X \times Y \to Z$ is continuous at (a, b) iff for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $d(x, a) \leq \delta$ in X and $d(y, b) \leq \delta$ in Y we have $d[f(x, y), f(a, b)] \leq \varepsilon$.

1-3.12. **Exercise** Let E be a normed space with more than one point and let d be the metric associated with the norm. Prove that d cannot be the discrete metric.

1-4 Open Sets

1-4.1. An alternative way to describe the concept of nearness without norms or distances in more general context is to use open sets. Full development along this line is called *general topology* which is beyond our scope. Only essential properties of open sets will be introduced.

1-4.2. Let X be a metric space. Suppose $a \in X$ and $r > 0$. Then the set $\mathbb{B}(a, r) = \{x \in X : d(x, a) < r\}$ is called the *open ball* with center a and radius r. Similarly, the closed ball is defined as the set $\overline{\mathbb{B}}(a, r) = \{x \in X : d(x, a) \le r\}$. By a ball, we always mean an open ball. We may drop r such as $\mathbb{B}(a)$ if the radius is not critical in the context.

1-4.3. **Lemma** If $a \in \mathbb{B}(x, \alpha) \cap \mathbb{B}(y, \beta)$, then there is $\delta > 0$ such that

$$\mathbb{B}(a, \delta) \subset \mathbb{B}(x, \alpha) \cap \mathbb{B}(y, \beta).$$

Proof. Let $\delta = \min\{\alpha - d(x, a), \beta - d(y, a)\}$. Take any $z \in \mathbb{B}(a, \delta)$. Then we have $d(z, a) < \delta \le \alpha - d(x, a)$. Hence $d(z, x) \le d(z, a) + d(a, x) < \alpha$, i.e. $z \in \mathbb{B}(x, \alpha)$. Similarly, $z \in \mathbb{B}(y, \beta)$. This completes the proof. Beginners should sketch the picture of balls with their radii. □

1-4.4. A subset M of X is said to be *open* if for every $x \in M$, there is some ball $\mathbb{B}(x)$ contained in M. As a result of last theorem when $x = y$ and $\alpha = \beta$, every open ball is open.

1-4.5. **Theorem** (a) Both $\emptyset$ and X are open.
(b) If M, N are open then $M \cap N$ is open.
(c) If $\{M_i : i \in I\}$ is a family of open sets then the union $\bigcup_{i \in I} M_i$ is open.

Proof. Take any $x \in M \cap N$. There are balls A, B with the same center x such that $A \subset M$ and $B \subset N$. There is another ball C with center x such that $C \subset A \cap B$. Hence $C \subset M \cap N$. Since $x \in M \cap N$ is arbitrary, $M \cap N$ is open. This proves (b). The rest is left as an exercise. □

1-4.6. **Exercise** Prove that a sequence $\{x_n\}$ in X converges to $b \in X$ iff for every open set V containing b, there is an integer p such that for all $n \ge p$, we have $x_n \in V$.

1-4.7. Let M be a subset of a metric space X. Then a point $x \in X$ is called an *interior point* of M if there is a ball $\mathbb{B}(x)$ contained in M. The set of all interior points of M is called the *interior* of M. It is denoted by M^o.

1-4.8. **Theorem** (a) If A is an open subset of M, then we have $A \subset M^o$.
(b) M^o is the largest open subset of M.
(c) M is open iff $M = M^o$.
(d) $M^{oo} = M^o$.

Proof. (a) Suppose A is an open subset of M. Take any $x \in A$. Since A is open, there is a ball $\mathbb{B}(x) \subset A$. By $A \subset M$, we have $\mathbb{B}(x) \subset M$. Therefore, x is an interior point of M, i.e. $x \in M^o$. This proves $A \subset M^o$.
(b) Clearly M^o is a subset of M by definition. Take any $x \in M^o$. There is a ball $\mathbb{B}(x) \subset M$. Since $\mathbb{B}(x)$ is an open subset of M, it follows from (a) that $\mathbb{B}(x) \subset M^o$. Because $x \in M^o$ is arbitrary, the set M^o is open. It follows from (a) that M^o is the largest one.
(c,d) These are left as exercises. □

1-4.9. **Exercise** Describe the interiors of the sets $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ and $\{(x, y) \in \mathbb{R}^2 : y \geq x^2\}$ respectively.

1-4.10. **Exercise** Show that every subset of a discrete metric space is open.

1-4.11. **Exercise** Prove that a finite product of open sets is open.

1-4.12. **Exercise** Prove that the only non-empty open vector subspace of a normed space E is E itself.

1-5 Closures of Sets

1-5.1. Let M be a subset of a metric space X. Then a point y is called a *closure point* of M if there is a sequence $\{x_n\}$ in M which converges to y. The set of all closure points of M is called the *closure* of M. It is denoted by $\overline{M}$.

1-5.2. **Theorem** A point $y \in X$ is a closure point of M iff every ball $\mathbb{B}(y)$ contains a point of M.

Proof. Assume y is a closure point of M. Then there is sequence $\{x_n\}$ in M which converges to y. Let $\mathbb{B}(y, r)$ be a given ball with center y. There is an integer p such that for all $n \geq p$, we have $d(x_n, y) \leq r/2$. In this case, the ball $\mathbb{B}(y, r)$ contain the point x_p of M. Conversely, suppose every ball $\mathbb{B}(y, \frac{1}{n})$

contains a point of M, say x_n. Then $\{x_n\}$ is a sequence in M which converges to y. □

1-5.3. **Exercise** Prove that a point $y \in X$ is a closure point of M iff every open set containing y also contains a point of M.

1-5.4. A subset M of a metric space X is said to be *closed* if it contains all its closure points, i.e. $\overline{M} \subset M$.

1-5.5. **Theorem** (a) $\overline{M}$ is a closed set containing M.
(b) If H is any closed set containing M then $\overline{M} \subset H$. Therefore $\overline{M}$ is the smallest closed set containing M.
(c) M is closed iff $M = \overline{M}$.
(d) $\overline{\overline{M}} = \overline{M}$.

Proof. (a) Let x be a closure point of $\overline{M}$. Consider any ball $\mathbb{B}(x)$. There is $y \in \mathbb{B}(x) \cap \overline{M}$. Since the ball $\mathbb{B}(x)$ is open, there is another ball $\mathbb{B}(y) \subset \mathbb{B}(x)$. Since y is a closure point of M, there is $z \in \mathbb{B}(y) \cap M$. Thus, $z \in \mathbb{B}(x) \cap M$. Therefore that x is a closure point of M, i.e. $x \in \overline{M}$. This proves that $\overline{M}$ is a closed set. By considering the constant sequences, clearly we get $M \subset \overline{M}$.
(b) Take any $y \in \overline{M}$. There is sequence $\{x_n\}$ in M convergent to y. From $M \subset H$, $\{x_n\}$ is also a sequence in H convergent to y, i.e. $y \in \overline{H}$. Since H is closed, we have $y \in H$. This proves (b).
Parts (c,d) are left as exercises. □

1-5.6. **Theorem** A subset M of a metric space X is closed iff $X \setminus M$ is open.

Proof. Assume that M is closed. Suppose to the contrary $X \setminus M$ is not open. There is $x \in X \setminus M$ such that for every $r > 0$, $\mathbb{B}(x, r) \not\subset X \setminus M$, i.e. $\mathbb{B}(x, r) \cap M \neq \emptyset$. Thus $x \in \overline{M}$. Since M is closed, we have $x \in M$ which contradicts the choice of x. Therefore $X \setminus M$ is open. Conversely, assume that $X \setminus M$ is open but M is not closed. Then there is $x \in \overline{M} \setminus M$. Hence X belongs to the open set $X \setminus M$. There is a ball $\mathbb{B}(x) \subset X \setminus M$, i.e. $\mathbb{B}(x) \cap M = \emptyset$. On the other hand, since x is a closure point of M, we have $\mathbb{B}(x) \cap M \neq \emptyset$. This contradiction establishes the proof. □

1-5.7. **Corollary** Both the empty set and the whole space are closed sets. Finite unions of closed sets are closed. Arbitrary intersections of closed sets are closed.

1-5.8. **Exercise** Prove $\overline{A \times B} = \overline{A} \times \overline{B}$ for all subsets A, B of metric spaces X, Y respectively. Prove that finite products of closed sets are closed.

1-5.9. **Exercise** Find the closure of an open ball $\mathbb{B}(x, 1)$ in a discrete metric space. What is the closed ball $\overline{\mathbb{B}}(x, 1)$?

1-5.10. **Exercise** Prove that every finite subset of a metric space is closed.

1-5.11. **Exercise** Prove that the closure of a vector subspace of a normed space is a vector subspace.

1-5.12. **Exercise** Describe the closures of the sets $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ and $\{(x, y) \in \mathbb{R}^2 : y > x^2\}$ respectively.

1-5.13. **Exercise** Show that the set $\{(\cos t, \sin t, t) : t \in \mathbb{R}\}$ is closed in $\mathbb{R}^3$.

1-5.14. **Exercise** Find the closure of the set $\{(\cos \frac{1}{t}, \sin \frac{1}{t}, t) : t > 0\}$ in $\mathbb{R}^3$.

1-5.15. **Example** For every subset A of a metric space X, we have

$$\overline{A} = \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \mathbb{B}(a, 1/n) = \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \overline{\mathbb{B}}(a, 1/n).$$

Proof. Let $x \in \overline{A}$. Choose $a_j \in A$ with $a_j \to x$ as $j \to \infty$. For every n, there is j such that $d(x, a_j) < 1/n$, i.e. $x \in \mathbb{B}(a_j, 1/n)$. We have proved that $\overline{A} \subset \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \mathbb{B}(a, 1/n)$. Next, let $x \in \bigcap_{n=1}^{\infty} \bigcup_{a \in A} \overline{\mathbb{B}}(a, 1/n)$. For every n, there is $a_n \in A$ such that $x \in \overline{\mathbb{B}}(a_n, 1/n)$, i.e. $d(a_n, x) \leq 1/n$. Hence $a_n \to x$ with $a_n \in A$, i.e. $x \in \overline{A}$. ☐

1-6 Characterization of Continuity

1-6.1. An important way to prove a set to be open or closed is by inverse images of continuous maps. A natural question at this stage is whether every metric space has a continuous function. The answer will be provided by the distance function. Glue Theorem will offer a nice way to piece continuous maps together.

1-6.2. **Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Then the following statements are equivalent.
(a) f is continuous on X.
(b) The inverse image of every closed set in Y is closed in X.
(c) The inverse image of every open set in Y is open in X.
(d) For every subset A of X, we have $f(\overline{A}) \subset \overline{f(A)}$.

Proof. ($a \Rightarrow b$) Let M be a closed set in Y and let $a \in X$ be a closure point of $f^{-1}(M)$. Then there are $x_n \in f^{-1}(M)$ satisfying $x_n \to a$. Since f is continuous at a, we have $f(x_n) \to f(a)$ in Y. Now $f(x_n) \in M$ implies that

$f(a)$ is a closure point of the closed set M. Hence $f(a) \in M$, i.e. $a \in f^{-1}(M)$. Therefore $f^{-1}(M)$ is closed.

($b \Rightarrow c$) It follows immediately by taking complements.

($c \Rightarrow a$) Let $a \in X$ and $\varepsilon > 0$ be given. Since the inverse image of the open set $\mathbb{B}(f(a), \varepsilon)$ is an open set containing the point a, there is a ball $\mathbb{B}(a, 2\delta) \subset f^{-1}\mathbb{B}(f(a), \varepsilon)$. Now suppose $d(x, a) \leq \delta$. Then $x \in \mathbb{B}(a, 2\delta)$. Hence $x \in f^{-1}[\mathbb{B}(f(a), \varepsilon)]$, i.e. $f(x) \in \mathbb{B}(f(a), \varepsilon)$, or, $d(f(x), f(a)) \leq \varepsilon$. Therefore f is continuous at every point of X, i.e. continuous on X.

($a \Rightarrow d$) Let $a \in \overline{A}$. There are $x_n \in A$ convergent to a. Since f is continuous, we have $f(x_n) \to f(a)$. Hence $f(a) \in \overline{f(A)}$. Therefore $f(\overline{A}) \subset \overline{f(A)}$.

($d \Rightarrow b$) Let M be a closed set in Y. Define $A = f^{-1}(M)$. Then we have

$$f(\overline{A}) \subset \overline{f(A)} \subset \overline{ff^{-1}(M)} \subset \overline{M} \subset M$$

i.e. $\overline{f^{-1}(M)} \subset \overline{A} \subset f^{-1}(M)$. Therefore $f^{-1}(M)$ is closed. □

1-6.3. **Exercise** Show that the set $\{(x, y) \in \mathbb{R}^2 : ye^{-x} \sin(x + y) > x \cos xy\}$ is open in $\mathbb{R}^2$ and the set $\{(x, y) \in \mathbb{R}^2 : ye^{-x} \sin(x + y) \geq x \cos xy\}$ is closed in $\mathbb{R}^2$.

1-6.4. **Theorem** Let $X[d]$ be a metric space. Then the distance function $d : X \times X \to \mathbb{R}$ is continuous on the product space. In particular, $d(a, x)$ is a continuous function in x.

Proof. It follows immediately from $|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b)$. □

1-6.5. **Exercise** Show that the sphere $\{x \in X : d(a, x) = r\}$, the closed ball $\overline{\mathbb{B}}(a, r)$, and the set $\{x \in X : d(a, x) \geq r\}$ are closed. Along the same line, prove that the open ball $\mathbb{B}(a, r)$ and the set $\{x \in X : d(a, x) > r\}$ are open.

1-6.6. Let $X[d]$ be a metric space and H a subset of X. Then the restriction $d|_H$ of the metric d onto H is a metric on H. It is called the *relative metric.* The metric space $H[d|_H]$ is called a *subspace* of X. For simplicity, we shall write d instead of $d|_H$.

1-6.7. **Exercise** Let $X = \{(x, y) \in \mathbb{R}^2 : |x| \leq 2, |y| \leq 2\}$ be equipped with the relative metric from $\mathbb{R}^2$. Sketch the open ball in X with center $(1, 1)$ and radius 2.

1-6.8. Let X, Y be metric spaces and $f : X \to Y$ be a given map. Suppose H is a subset of X. Then f is said to be *continuous on* H if the restriction $f|_H$

is continuous on the metric subspace H. Clearly, if f is continuous on X then f is continuous on H. For simplicity, we write f instead of $f|_H$.

1-6.9. **Exercise** Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ for $x \leq 1$ and $f(x) = 1$ for $x > 1$ is discontinuous on $\mathbb{R}$ but continuous on the subset $(0,1) \cup (1,2)$.

1-6.10. Let Y be a subspace of a metric space X. Write $x_n \to a$ in Y if all x_n, a are in Y and $d(x_n, a) \to 0$. A subset V of Y is said to be *closed* (respectively open) *in* Y if V is closed (respectively open) in the metric subspace Y. Note that Y is open in itself but need not be open in X.

1-6.11. **Lemma** Let $A \subset B$ be two subsets of a metric space X. If A is closed in B and if B is closed in X, then A is closed in X.

Proof. Let $x_n \in A$ and $y \in X$. Suppose $x_n \to y$ in X. Since $A \subset B$, we have $x_n \in B$. Since B is closed in X, we have $y \in B$. Now y is a closure point of A on the subspace B. Because A is closed in B, y belongs to A. Therefore A is closed in X.

1-6.12. **Exercise** State and prove a result for open sets similar to the last lemma.

1-6.13. **Glue Theorem** Let X, Y be metric spaces and $f : X \to Y$ be a given map. Suppose $X = M \cup N$ is the union of two closed subsets M, N. If f is continuous on both M, N separately, then f is continuous on X.

Proof. Let V be a closed subset of Y. Since f is continuous on M, the set $(f|_M)^{-1}(V) = M \cap f^{-1}(V)$ is closed in M. Since M is closed in X, $M \cap f^{-1}(V)$ is closed in X. Similarly, $N \cap f^{-1}(V)$ is closed in X. Therefore

$$f^{-1}(V) = [M \cap f^{-1}(V)] \cup [N \cap f^{-1}(V)]$$

is closed in X. Consequently, f is continuous on X.

1-6.14. **Exercise** Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ for $x \leq 0$ and $f(x) = x^2$ for $x > 0$ is continuous on $\mathbb{R}$.

1-7 Duality of Closure-Interior Operators

1-7.1. Open and closed sets are complement to each other as in §1-5.6. The following theorem extends this duality to operators.

1-7.2. **Theorem** For every subset M of a metric space X, we have $M'^{-\prime} = M^o$ and $M'^{o\prime} = M^-$ where $M' = X \setminus M$ denotes the complement of M.

Proof. Observe that $x \in M'^{-\prime}$ iff $x \in M'^-$ is false. The negation of $\forall\, r > 0, \mathbb{B}(x,r) \cap M' \neq \emptyset$ is the statement: $\exists\, r > 0, \mathbb{B}(x,r) \cap M' = \emptyset$, that is $\mathbb{B}(x,r) \subset M$. This is equivalent to $x \in M^o$. Therefore $M'^{-\prime} = M^o$. Replacing M by its complement M', we obtain the second identity. □

1-7.3. **Exercise** Let $M = (1,2] \cup \{\frac{1}{n} : n > 1\}$ be the union of a semi-interval and a sequence in the real line. How many new sets can you obtain by constructing interior, closure and complement repeatedly?

1-7.4. Let M be a subset of a metric space X. A point $x \in X$ is called a *boundary point* of M if every ball $\mathbb{B}(x)$ contains a point in M and also a point not in M. The set of all boundary points of M is called the *boundary* of M and is denoted by ∂M. A point $x \in X$ is called an *exterior point* of M if there is a ball disjoint from M. The set of all exterior points of M is called the *exterior* of M and is denoted by $ext(M)$.

1-7.5. **Exercise** Prove that $\partial M = M^- \setminus M^o = M^- \cap M'^-$. Hence show that the boundary of a set is closed. Also prove that M^o and ∂M form a partition of M^-.

1-7.6. **Exercise** Prove that $ext(M) = M^{-\prime} = M'^o$. Hence deduce that $ext(M)$ is open. Show that $ext(M)$ and M^- form a partition of the whole space X.

1-7.7. **Example** Every non-empty open interval contains a rational number and an irrational number.

Proof. Let $a < b$ be the endpoints of the given interval. Choose any integer $n > \frac{1}{b-a}$ and mark the points $0, \pm\frac{1}{n}, \pm\frac{2}{n}, \pm\frac{3}{n}, \cdots$ on the real line $\mathbb{R}$. It is obvious that the interval (a,b) has to contain a rational number of the form $\frac{m}{n}$. Repeating the same process with $0, \pm\frac{1}{n\sqrt{2}}, \pm\frac{2}{n\sqrt{2}}, \pm\frac{3}{n\sqrt{2}}, \cdots$, we prove the case for irrational. □

1-7.8. **Exercise** Find the closure, interior, boundary and exterior of the set of rational numbers in the interval $(0,1)$.

1-7.9. **Exercise** Find the closure, interior, boundary and exterior of the set of points $(\frac{1}{m}, \frac{1}{n}) \in \mathbb{R}^2$ where m, n run over all non-zero integers.

1-7.10. **Exercise** Find the closure, interior, boundary and exterior of the closed ball of $\mathbb{R}_2^2$ with center at the origin and radius $\frac{1}{2}$. Repeat the same problem when $\mathbb{R}_d^2$ is given the discrete metric.

1-8 Partition of Unity

1-8.1. Intuitively, through a partition of unity, a point x in an abstract metric space is described by a vector $(\alpha_1(x), \alpha_2(x), \cdots, \alpha_n(x))$ in $\mathbb{R}^n$ where α_j are continuous functions of x. Compactness in Chapter 2 will allow us to reduce an arbitrary open cover to a finite cover so that partition of unity can be applied. For example, see §§5-3.3, 12-4.5. We start off with the distance function.

1-8.2. Let A be a non-empty subset of a metric space X. The *distance from a point* $x \in X$ to A is defined by $d(x, A) = \inf_{a \in A} d(x, a)$.

1-8.3. **Lemma** For all $x, y \in X$, we have $|d(x, A) - d(y, A)| \leq d(x, y)$. As a result, $d(x, A)$ is a continuous function of $x \in X$. Consequently, we have sufficient amount of continuous functions on every metric space.

Proof. For each $a \in A$, we have $d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$. Taking infimum over $a \in A$, we obtain $d(x, A) \leq d(x, y) + d(y, A)$, that is, $d(x, A) - d(y, A) \leq d(x, y)$. Interchanging x, y, we obtain the required inequality. □

1-8.4. **Lemma** $d(x, A) = 0$ iff x is a closure point of A.

Proof. Suppose $d(x, A) = 0$. For every $n \geq 1$, there is $a_n \in A$ such that $d(x, a_n) \leq \frac{1}{n}$. Hence $a_n \to x$. Therefore x is a closure point of A. Conversely, let x be a closure point of A. There is a sequence $\{a_n\}$ in A convergent to X. Hence we have $0 \leq d(x, A) \leq d(x, a_n) \to 0$ as $n \to \infty$. Therefore $d(x, A) = 0$. □

1-8.5. **Theorem** Let A, B be disjoint closed subsets of a metric space X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof. Note that if one of A, B is empty, then a constant function would do the job. So, assume both A, B are non-empty. Firstly, we claim $d(x, A) + d(x, B) > 0, \forall\, x \in X$. In fact, suppose to the contrary that for some $x \in X, d(x, A) + d(x, B) = 0$. Then $d(x, A) = d(x, B) = 0$. Hence x is a closure point for both A, B. Since A, B are closed, x belongs to both A, B. This contradicts the fact that A, B are disjoint. Therefore the following function is well-defined:

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \forall\, x \in X.$$

Clearly it is a required continuous function on X. □

1-8.6. **Exercise** Prove that the function $g(x) = [1 - f(x)]a + f(x)b$ where $a, b \in \mathbb{R}$ is continuous on X and satisfies $g(A) = a$, $g(B) = b$.

1-8.7. **Corollary** Let A be a closed set and V an open set in X. If $A \subset V$ then there is an open set W satisfying $A \subset W \subset \overline{W} \subset V$.

Proof. Since A and $X \backslash V$ are disjoint closed sets, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(X \setminus V) = 1$. Then $W = f^{-1}(-\infty, \frac{1}{2})$ is an open set containing A and $\overline{W} \subset f^{-1}(-\infty, \frac{1}{2}] \subset V$. □

1-8.8. **Lemma** Let A be a closed subset of a metric space X and let $\{V_j : 1 \leq j \leq n\}$ be an open cover of A, i.e. $A \subset \bigcup_{j=1}^{n} V_j$ and all V_j are open. Then there are closed subsets B_j of A such that $A = \bigcup_{k=1}^{n} B_k$ and $B_j \subset V_j$ for each $1 \leq j \leq n$.

Proof. It suffices to prove the case for $n = 2$. Let U, V be open sets such that $A \subset U \cap V$. Let $M = A \backslash V$ and $N = A \backslash U$. Then both M, N are closed sets and they are disjoint. There is a continuous function f on X such that $f(M) = 0$ and $f(N) = 1$. Then both $P = f^{-1}(-\infty, \frac{1}{2})$ and $Q = f^{-1}(\frac{1}{2}, \infty)$ are open sets containing M, N respectively. Therefore $E = A \setminus Q$ and $F = A \setminus P$ are closed subsets of A. Clearly we have $E \cup F = A \setminus (P \cap Q) = A$. Furthermore, observe that $E = A \setminus Q \subset A \setminus N \subset U$. Similarly we obtain $F \subset V$. This completes the proof. It would be helpful if you sketch a picture to go along with the above constructions. □

1-8.9. Let X be a metric space and $f : X \to \mathbb{K}$ a given function. Then the *support* of f is defined to be the closure of the set $\{x \in X : f(x) \neq 0\}$. It is denoted by supp($f$). Let A be a closed subset of X and $\{V_j : 1 \leq j \leq n\}$ an open cover of A. A sequence of continuous functions $\alpha_j : X \to [0, 1]$ is called a *partition of unity* on A subordinated to $\{V_j\}$ if the following conditions hold:

(a) $\sum_{j=1}^{n} \alpha_j(a) = 1$ for all $a \in A$.

(b) $\sum_{j=1}^{n} \alpha_j(x) \leq 1$ for all $x \in X$.

(c) for each j, the support of α_j is contained in V_j.

1-8.10. **Example** Let $A = (0, 4]$, $X = (0, \infty)$, $U = (0, 3)$ and $V = (1, 6)$. Clearly

$\{U, V\}$ is an open cover of the closed subset A of X. Let

$$\alpha(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 2 - x, & \text{if } 1 < x \leq 2, \\ 0, & \text{if } 2 \leq x, \end{cases}$$

and

$$\beta(x) = \begin{cases} 1 - \alpha(x), & \text{if } 0 < x \leq 4, \\ 5 - x, & \text{if } 4 \leq x \leq 5, \\ 0, & \text{if } 5 \leq x. \end{cases}$$

Find the supports of α, β. Show that $\{\alpha, \beta\}$ is a partition of unity on A subordinated to $\{U, V\}$.

1-8.11. **Theorem** Let A be a closed subset of a metric space X. Then for every open cover $\{V_j : 1 \leq j \leq n\}$ of A, there is a partition of unity on A subordinated to $\{V_n\}$.

Proof. Let $V_0 = X \setminus A$. Then $\{V_j : 0 \leq j \leq n\}$ is an open cover of the whole space X. Let B_j be closed sets in X such that $X = \bigcup_{k=0}^{n} B_k$ and for all j, $B_j \subset V_j$. Now for each j, there is an open set W_j such that $B_j \subset W_j \subset \overline{W_j} \subset V_j$ and also there is a continuous function $f_j : X \to [0, 1]$ such that $f_j(B_j) = 1$ and $f_j(X \setminus W_j) = 0$. Take any $x \in X$. Then $x \in B_j$ for some $0 \leq j \leq n$, i.e. $f_j(x) = 1$. Hence $\sum_{k=0}^{n} f_k(x) \geq 1$. Therefore the functions $\alpha_j : X \to [0, 1]$ given by $\alpha_j(x) = \dfrac{f_j(x)}{\sum_{k=0}^{n} f_k(x)}$ are well-defined and continuous on X. Observe that if $\alpha_j(x) \neq 0$, then $x \notin W_j$, i.e. $x \in W_j$. Hence $\text{supp}(\alpha_j) \subset \overline{W_j} \subset V_j$ for each $j = 0, 1, 2, \cdots, n$. In particular, if $\alpha_0(x) \neq 0$, we have $x \in V_0$, i.e. $x \notin A$. Thus $\alpha_0(A) = 0$. Now it is obvious to verify all other conditions for $\{\alpha_j : 1 \leq j \leq n\}$ to be a partition of unity on A subordinated to $\{V_j : 1 \leq j \leq n\}$. □

1-99. **References and Further Readings** : Dunford, Taylor-58, Kreyszig, Yosida and Meise.

Chapter 2

Complete, Compact and Connected Sets

2-1 Cauchy Sequences

2-1.1. The tail of a convergent sequence is eventually near its limit and hence becomes small as we throw away sufficiently many initial terms. The concept of sequences with small tails will be formalized as Cauchy sequences. It turns out that convergence is equivalent to being Cauchy and possessing a convergent subsequence or a cluster point.

2-1.2. Let X be a metric space. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for every $\varepsilon > 0$, there is an integer p such that for all $m, n \geq p$, we have $d(x_m, x_n) \leq \varepsilon$.

2-1.3. **Theorem** Every convergent sequence is Cauchy.

Proof. Let $x_n \to b$ in a metric space X. Then for every $\varepsilon > 0$, there is an integer p such that for all $n \geq p$, we have $d(x_n, b) \leq \frac{1}{2}\varepsilon$. Now take any $m, n \geq p$. Observe that

$$d(x_m, x_n) \leq d(x_m, b) + d(b, x_n) = d(x_m, b) + d(x_n, b) \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Therefore the given sequence is Cauchy. □

2-1.4. **Theorem** Every subsequence of a Cauchy sequence is Cauchy.

Proof. Let $\{y_j\}$ be a subsequence of a Cauchy sequence $\{x_n : n \geq 1\}$ in a metric space X. Then for every $\varepsilon > 0$, there is an integer p such that for all $m, n \geq p$, we have $d(x_m, x_n) \leq \frac{1}{2}\varepsilon$. There is a sequence of integers $n(1) < n(2) < n(2) < \cdots$ such that $y_j = x_{n(j)}$ for all j. Now for all $j, k \geq p$, we have $n(j) \geq p$ and $n(k) \geq p$. Hence we have $d(y_j, y_k) = d(x_{n(j)}, x_{n(k)}) \leq \varepsilon$. Therefore $\{y_j\}$ is Cauchy. □

2-1.5. **Theorem** Let $\{x_n\}$ be a Cauchy sequence in a metric space X. If it has a subsequence $\{y_j\}$ convergent to some $b \in X$, then the original sequence $\{x_n\}$ also converges to b.

Proof. Let $\varepsilon > 0$ be given. Since $\{x_n : n \geq 1\}$ is Cauchy, there is an integer $p \geq 1$ such that for all $m, n \geq p$, we have $d(x_m, x_n) \leq \frac{1}{2}\varepsilon$. There is a sequence of integers $n(1) < n(2) < n(2) < \cdots$ such that $y_j = x_{n(j)}$ for all j. Since $y_j \to b$, there is an integer $q \geq 1$ such that for all $j \geq q$ we have $d(y_j, b) \leq \frac{1}{2}\varepsilon$. Let $k = p + q$. Take any $m \geq n(k)$. Then both $n(k), m \geq p$. Hence, $d(x_m, x_{n(k)}) \leq \frac{1}{2}\varepsilon$. Since $k \geq q$, we have $d(x_{n(k)}, b) \leq \frac{1}{2}\varepsilon$. Consequently, $d(x_m, b) \leq \varepsilon$ for all $m \geq n(k)$. Therefore $x_n \to b$ as $n \to \infty$. □

2-1.6. **Exercise** Let $x_n = 1/n$ for all $n \geq 1$. Prove that it is convergent on the real line $\mathbb{R}$ and hence it is a Cauchy sequence. Prove that it is not convergent on its subspace $(0, 1)$. Hence a Cauchy sequence in certain metric space need not be convergent.

2-1.7. Let X be a metric space and $\{x_n\}$ a sequence in X. Then a point $b \in X$ is called a *cluster point* of $\{x_n\}$ if for every $\varepsilon > 0$, for every integer p, there is $n \geq p$ such that $d(x_n, b) \leq \varepsilon$. Clearly the limit of convergent sequence is a cluster point.

2-1.8. **Theorem** A point $b \in X$ is a cluster point of a sequence $\{x_n\}$ iff there is a subsequence convergent to b.

Proof. ($\Rightarrow$) Let b be a cluster point of $\{x_n : n \geq 1\}$. Then there is $n(1) > 1$ such that $d(x_{n(1)}, b) \leq 1/1$. Similarly, there is $n(2) > n(1)$ such that $d(x_{n(2)}, b) \leq 1/2$. By induction, there is $n(j) > n(j-1)$ such that $d(x_{n(j)}, b) \leq 1/j$. Now $\{x_{n(j)}\}$ is a subsequence convergent to b.

($\Leftarrow$) Let $\{x_{n(j)}\}$ be a subsequence of $\{x_n\}$ such that $x_{n(j)} \to b$ as $j \to \infty$. Let $\varepsilon > 0$ and $p \geq 1$ be given. Since the subsequence converges to b, there is j_0 such that for every $j \geq j_0$ we have $d(x_{n(j)}, b) \leq \varepsilon$. Let $k = j_0 + p$. Then $n(k) \geq k \geq j_0$ and hence $d(x_{n(k)}, b) \leq \varepsilon$. Also $n(k) \geq k \geq p$. Therefore b is a cluster point of $\{x_n\}$.

2-1.9. **Exercise** In $\mathbb{R}^2$, let $x_n = (\frac{1+n}{n} \sin \frac{1}{4}n\pi, \cos \frac{1}{2}n\pi)$. Show that $\{x_n\}$ is not a Cauchy sequence. Find all cluster points and for each cluster point b, construct a subsequence convergent to b.

2-2 Bounded Sets

2-2.1. Let M be a non-empty subset of a metric space X. Then its *diameter* is defined by $diam(M) = \sup\{d(x, y) : x, y \in M\}$. The set M is said to be *bounded* if its diameter is finite, i.e. $diam(M) < \infty$. A sequence is said to be *bounded* if its range is bounded.

2-2.2. **Theorem** $diam(\overline{M}) = diam(M)$.

Proof. It is easy to prove that $diam(M) \leq diam(\overline{M})$. If M is unbounded, then both sides are ∞. Without loss of generality, assume $diam(M) < \infty$. Take any $x, y \in \overline{M}$. For every $\varepsilon > 0$, we select $a \in \mathbb{B}(x, \varepsilon) \cap M$ and $b \in \mathbb{B}(y, \varepsilon) \cap M$. Hence, $d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq diam(M) + 2\varepsilon$. Taking supremum over all $x, y \in \overline{M}$, we have $diam(\overline{M}) \leq diam(M) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $diam(\overline{M}) \leq diam(M)$. □

2-2.3. **Theorem** Every Cauchy sequence $\{x_n\}$ is bounded.

Proof. For $\varepsilon = 1$ there exists an integer p such that for all $m, n \geq p$ we have $d(x_m, x_n) \leq \varepsilon$. Let $r = \max\{d(x_j, x_k) : 1 \leq i, k \leq p\}$. Take any $m, n \geq 1$. If both $m, n \geq p$, then $d(x_m, x_n) \leq 1 \leq 1 + r$. If both $m, n \leq p$, then $d(x_m, x_n) \leq r \leq 1 + r$. If $m \geq p$ and $n \leq p$, then

$$d(x_m, x_n) \leq d(x_m, x_p) + d(x_p, x_n) \leq 1 + r.$$

Therefore the diameter of $\{x_n\}$ is bounded by $1 + r$. □

2-2.4. **Exercise** Prove that finite unions of bounded sets are bounded. Also prove that subsets of bounded sets are bounded.

2-2.5. **Exercise** Find the diameter of the open unit ball in $\mathbb{R}^2$ and also the diameter of the triangle with vertices $(0,0), (0,1), (1,0)$.

2-2.6. **Exercise** Prove that the diameter of a closed ball $\overline{\mathbb{B}}(a, r)$ is no more than $2r$. What is the diameter of an open unit ball in a discrete metric space?

2-2.7. **Exercise** Prove that a set is bounded iff it can be covered by a ball.

2-2.8. **Exercise** Let M be a subset of a product metric space $X \times Y$. Prove that M is bounded iff both projections of M into X, Y are bounded.

2-2.9. **Exercise** Prove that a subset X of a normed space E is bounded iff there is $M > 0$ such that $\|x\| \leq M$ for all $x \in X$.

2-3 Upper and Lower Limits

2-3.1. In this section, we shall make full usage of the order structure of the real line to establish the fundamental properties of bounded real sequences.

2-3.2. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Its *lower* and *upper limits* are defined by $\liminf_{n\to\infty} x_n = \sup_{k\geq 1} \inf_{n\geq k} x_n$ and $\limsup_{n\to\infty} x_n = \inf_{k\geq 1} \sup_{n\geq k} x_n$ respectively. Since every non-empty set of real numbers which is bounded above must has a supremum, both lower and upper limits of a bounded sequence in $\mathbb{R}$ always exist.

2-3.3. **Exercise** Prove the following statements.

(a) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

(b) $\liminf_{n\to\infty}(-x_n) = -\limsup_{n\to\infty} x_n$ and $\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$.

2-3.4. **Exercise** Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of real numbers. Prove the following.

(a) If $x_n \leq y_n$ for all n, then $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$ and $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$.

(b)
$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}(x_n + y_n) \leq \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\ &\leq \limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.\end{aligned}$$

Also prove similar results for product if all $x_n, y_n \geq 0$.

2-3.5. **Lemma** Let $\{x_n\}$ be bounded real sequence and let $a \in \mathbb{R}$ be given. Then $a = \limsup x_n$ iff the following conditions hold:

(a) For every $\varepsilon > 0$ there is an integer p such that for all $n \geq p$, we have $x_n \leq a + \varepsilon$.

(b) For every $\varepsilon, p > 0$, there is $n \geq p$ such that $a - \varepsilon \leq x_n$.

Proof. Since $a = \inf_{p\geq 1} \sup_{n\geq p} x_n$, there is p such that $\sup_{n\geq p} x_n \leq a+\varepsilon$ which is (a). Now for given $\varepsilon, p > 0$, we have $a - \varepsilon < \sup_{n\geq p} x_n$ and hence (b) follows from definition of supremum. The converse is left as an exercise. □

2-3.6. Similar statement for lower limits also hold. The following results follow immediately from last lemma.

2-3.7. **Theorem** The upper and lower limits are cluster points.

2-3.8. **Theorem** A bounded sequence $\{x_n\}$ converges to some $a \in \mathbb{R}$ iff

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a.$$

2-3.9. **Corollary** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

2-3.10. **Corollary** Bounded monotonic sequences in $\mathbb{R}$ are convergent.

Proof. Suppose $x_n \le x_{n+1} \le a$ for all n. It follows from monotonicity that $\sup_n x_n = \sup_n \inf_{m \ge n} x_m = \liminf x_n$. On the other hand, we also have $\liminf x_n \le \limsup x_n \le \sup_n x_n$. Therefore they are all equal and hence $\lim x_n = \sup_n x_n$. Similar proof works for decreasing sequences. □

2-3.11. **Exercise** Prove that the upper limit of a bounded real sequence is the largest cluster point.

2-3.12. **Exercise** On the real line, let $x_n = [(-1)^n + \frac{1}{n}] \sin \frac{1}{4} n\pi$. Find all cluster points, upper and lower limits of the sequence $\{x_n\}$. For each cluster point b, construct a subsequence convergent to b.

2-4 Complete Sets

2-4.1. In order that a beautiful theory can be developed, it is necessary to require that there is sufficient amount of convergent sequences. Complete sets formalize this requirement.

2-4.2. Let A be a subset of a metric space X. Then A is said to be ***complete*** if every Cauchy sequence in A converges to a limit which *belongs to* A.

2-4.3. **Theorem** If A is a complete subset of X then it is closed in X.

Proof. Let y be a closure point of A. There is a sequence $\{x_n\}$ in A convergent to y. This sequence is Cauchy in X but also Cauchy in A. Since A is complete, it converges to some point $z \in A$. Since $\{x_n\}$ converges to both y, z in X, we have $y = z \in A$. Therefore A is closed. □

2-4.4. **Theorem** If A is a closed subset of a complete metric space X, then A is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in A. Then it is also a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a limit $b \in X$. Since all $x_n \in A$, b is a closure point of A. Because A is closed, we have $b \in A$. This completes the proof. □

2-4.5. **Theorem** Let X, Y be metric spaces and $\{(x_n, y_n)\}$ be a sequence in the product space $X \times Y$. Then $\{(x_n, y_n)\}$ is Cauchy iff both $\{x_n\}$ and $\{y_n\}$ are Cauchy.

Proof. Regardless which metric is used in the product space, we always have $d(x, a) \le d[(x, y), (a, b)] \le d(x, a) + d(y, b)$. The result follows immediately from definitions. □

2-4.6. **Theorem** If X, Y are complete metric spaces, then so is the product space $X \times Y$.

Proof. Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$. Then both $\{x_n\}$ and $\{y_n\}$ are Cauchy and hence converge to some $a \in X$ and $b \in Y$ respectively. Therefore $(x_n, y_n) \to (a, b)$ in $X \times Y$. Consequently, $X \times Y$ is also a complete metric space. □

2-4.7. **Theorem** Every $\mathbb{K}^n$ is complete.

Proof. Every Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ is bounded and hence it has a convergence subsequence. Therefore the original sequence $\{x_n\}$ is convergent. Consequently $\mathbb{R}$ is complete. As a direct result of last theorem, all $\mathbb{R}_1^n$, $\mathbb{R}_2^n$ and $\mathbb{R}_\infty^n$ are complete. Since $\mathbb{C} = \mathbb{R}_2^2$ is complete, all $\mathbb{C}_1^n$, $\mathbb{C}_2^n$ and $\mathbb{C}_\infty^n$ are also complete. □

2-4.8. **Exercise** Is the set of all integers a complete subset of the real line? Is the set Q of all rational numbers complete in $\mathbb{R}$?

2-4.9. **Exercise** Let $X = \{(x, y) \in \mathbb{R}^2 : |x| < 2\}$. Is it a complete metric space? Is the ball $\mathbb{B} = \{(x, y) \in X : x^2 + y^2 \leq 1\}$ a complete subset of X?

2-4.10. **Lemma** Let $\{x_n\}$ be a sequence in X and let $T_n = \{x_k : k \geq n\}$. Then a point $y \in X$ is a cluster point of $\{x_n\}$ iff $y \in \bigcap_{n=1}^{\infty} \overline{T}_n$. Intuitively, T_n are the tails of the given sequence.

Proof. $(\Rightarrow)$ Assume $x \in \bigcap_{n=1}^{\infty} \overline{T}_n$. Let $\varepsilon > 0$ and $p \geq 1$ be given. Since $y \in \overline{T}_p$, the ball $\mathbb{B}(y, \varepsilon)$ contains a point of T_p, i.e. there is $n \geq p$ such that $d(x_n, y) < \varepsilon$. Therefore y is a cluster point of $\{x_n\}$.
$(\Leftarrow)$ Let y be a cluster of $\{x_n\}$. Fix any n. Consider any open ball $\mathbb{B}(y, \varepsilon)$. There is $m \geq n$ such that $d(x_m, y) \leq \frac{1}{2}\varepsilon$, i.e. $x_m \in \mathbb{B}(y, \varepsilon) \bigcap T_n$. Since $\varepsilon > 0$ is arbitrary, $y \in \overline{T}_n$. Since n is free, we have $y \in \bigcap_{n=1}^{\infty} \overline{T}_n$. □

2-4.11. **Theorem** Let X be a metric space. Then the following statements are equivalent.
(a) X is complete.
(b) Let $A_{n+1} \subset A_n$ be a decreasing sequence of non-empty closed sets. If their diameters tend to zero then $\bigcap_{n=1}^{\infty} A_n$ contains exactly one point. This is called the *nested property* which will be used in §6-7.3.
(c) A sequence $\{x_n\}$ in X satisfying $d(x_{n+1}, x_n) \leq 1/2^n$ for all n, is convergent.

Proof. $(a \Rightarrow b)$ For each n, take any $x_n \in A_n$. Since $diam(A_n) \to 0$, for every $\varepsilon > 0$ there is $p \geq 1$ such that for every $n \geq p$, we have $diam(A_n) \leq \varepsilon$.

Choose any $m, n \geq p$. Since $\{A_n\}$ is decreasing, we have $x_m \in A_m \subset A_p$ and $x_n \in A_n \subset A_p$. Hence, $d(x_m, x_n) \leq diam(A_p) \leq \varepsilon$. Therefore $\{x_n\}$ is Cauchy in the complete space X. It converges to some limit $a \in X$. For each fixed k, $\{x_{n+k} : n \geq 1\}$ is a sequence in the closed set A_k and it converges to a. Hence $a \in A_k$. Since k is arbitrary, we have $a \in \bigcap_{k=1}^{\infty} A_k$. Finally, suppose $a, b \in \bigcap_{k=1}^{\infty} A_k$. Then for each n, we have $a, b \in A_n$. Therefore we obtain $0 \leq d(a, b) \leq diam(A_n) \to 0$. Consequently $d(a, b) = 0$, i.e. $a = b$.

($b \Rightarrow c$) Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \leq 1/2^n$ for all n. Define $T_n = \{x_k : k \geq n\}$ and $A_n = \overline{T}_n$. Observe that

$$\begin{aligned} d(x_n, x_{n+i}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+i-1}, x_{n+i}) \\ &\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots \leq \frac{1}{2^{n-1}}; \end{aligned}$$

and
$$d(x_{n+i}, x_{n+j}) \leq d(x_n, x_{n+i}) + d(x_n, x_{n+j}) \leq \frac{1}{2^{n-2}}.$$

Since i, j are arbitrary, we have $diam(A_n) = diam(T_n) \leq 1/2^{n-2} \to 0$. By (b), there is $a \in \bigcap_{n=1}^{\infty} A_n$. Hence a is a cluster point of $\{x_n\}$. Therefore $\{x_n\}$ has a convergent subsequence. Since $\{x_n\}$ is Cauchy, it is convergent.

($c \Rightarrow a$) Let $\{x_n\}$ be a Cauchy sequence in X. Then for every $\varepsilon > 0$, there is an integer p such that for all $m, n \geq p$, we have $d(x_m, x_n) \leq \varepsilon$. In particular, for $\varepsilon = 1/2$, there is $n(1)$ such that for all $m, n \geq n(1)$, we have $d(x_m, x_n) \leq 1/2$. Inductively, for $\varepsilon = 1/2^j$, there is $n(j) > n(j-1)$ such that for all $m, n \geq n(j)$, we have $d(x_m, x_n) \leq 1/2^j$. Then $\{x_{n(j)}\}$ is a subsequence of $\{x_n\}$. Furthermore, we have $d(x_{n(j+1)}, x_{n(j)}) \leq 1/2^j$. By (c), the subsequence $\{x_{n(j)}\}$ converges. Therefore $\{x_n\}$ is convergent. Consequently, X is complete.

2-4.12. **Exercise** Find a decreasing sequence $A_{n+1} \subset A_n$ of non-empty closed subsets of a complete metric space X such that $\bigcap_{n=1}^{\infty} A_n$ is empty.

2-4.13. **Exercise** Find a decreasing sequence $A_{n+1} \subset A_n$ of non-empty closed subsets of a metric space X such that the intersection $\bigcap_{n=1}^{\infty} A_n$ is empty and $\lim_{n \to \infty} diam\ A_n = 0$.

2-4.14. **Exercise** Prove that a decreasing sequence $A_{m+1} \subset A_m$ of closed balls in $\mathbb{K}^n$ has non-empty intersection.

2-5 Precompact Sets

2-5.1. Let X be a metric space. A subset M is said to be ***precompact*** or totally bounded if for every $\varepsilon > 0$, there is a finite subset J of M such that

$M \subset \bigcup_{a \in J} \mathbb{B}(a, \varepsilon)$. Intuitively, precompact sets can be approximated by finite sets.

2-5.2. **Theorem** Closures of precompact sets are precompact.

Proof. Let M be a precompact subset of X. For every $\varepsilon > 0$, there is a finite subset J of M such $M \subset \bigcup_{j \in J} \mathbb{B}(j, \varepsilon)$. Let x be any closure point of M. There is $a \in M$ such that $d(x, a) \leq \varepsilon$. There is $j \in J$ such that $d(a, j) < \varepsilon$. Hence $d(x, j) < 2\varepsilon$. Therefore $\overline{M} \subset \bigcup_{j \in J} \mathbb{B}(j, 2\varepsilon)$. Consequently, $\overline{M}$ is precompact.□

2-5.3. **Theorem** Every Cauchy sequence $\{x_n\}$ in X is precompact.

Proof. Let $\varepsilon > 0$ be given. There is $p \geq 1$ such that for all $m, n \geq p$, we have $d(x_m, x_n) \leq \frac{1}{2}\varepsilon$. Then $\{x_n\}$ is covered by the open balls $\{\mathbb{B}(x_i, \varepsilon) : 1 \leq i \leq p\}$. Therefore the range of the Cauchy sequence is precompact. □

2-5.4. **Diagonal Process** Let X be a set. For each integer $p \geq 1$, let $\{x_n^p : n \geq 1\}$ be a sequence in X. Suppose each $\{x_n^{p+1} : n \geq 1\}$ is a subsequence of $\{x_n^p : n \geq 1\}$. Then the diagonal $\{x_n^n : n \geq p\}$ is a subsequence of $\{x_n^p : n \geq 1\}$ for each p. If we list each sequence $\{x_n^p : n \geq 1\}$ horizontally, the interpretation of diagonal becomes obvious.

2-5.5. **Lemma** Let $A, B_1, B_2, \cdots, B_k$ be subsets of a given set X. Suppose $A \subset \bigcup_{i=1}^k B_i$. If $\{x_n\}$ is a sequence in A, then there is a subsequence $\{y_n\}$ and an index i such that $y_n \in B_i, \forall n$.

Proof. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $E_i = \{n \in \mathbb{N} : x_n \in B_i\}$. Since $A \subset \bigcup_{i=1}^k B_i$, we have $\mathbb{N} \subset \bigcup_{i=1}^k E_i$. Since $\mathbb{N}$ is infinite, some E_i is also infinite. There is a sequence $n(1) < n(2) < \cdots$ in E_i. Consequence $\{x_{n(j)}\}$ is a subsequence of $\{x_n\}$ and also $x_{n(j)} \in B_i$ for all j. □

2-5.6. **Theorem** Let A be a subset of a metric space X. Then A is precompact iff every sequence in A has a Cauchy subsequence.

Proof. Let A be a precompact set. Assume $\{x_n\}$ is a given sequence in A. Then there is a finite subset J_1 of A such that $A \subset \bigcup_{x \in J_1} \mathbb{B}(x, 1)$. There exist $a_1 \in J_1$ and a subsequence $\{x_n^1\}$ of $\{x_n\}$ such that all $x_n^1 \in \mathbb{B}(a_1, 1)$. Suppose a subsequence $\{x_n^{p-1} : n \geq 1\}$ of $\{x_n\}$ has been defined by induction. There is a finite subset J_p of A such that $A \subset \bigcup_{x \in J_p} \mathbb{B}(x, 1/p)$. There are $a_p \in J_p$ and a subsequence $\{x_n^p : n \geq 1\}$ of $\{x_n^{p-1} : n \geq 1\}$ such that all $x_n^p \in \mathbb{B}(a_p, 1/p)$. Following the diagonal process, define $y_n = x_n^n$. Then $\{y_n : n \geq p\}$ is a subsequence of $\{x_n^p : n \geq 1\}$. Consequently, $\{y_n : n \geq 1\}$ is a subsequence of the original sequence $\{x_n\}$. Let $\varepsilon > 0$ be given. Choose $p > \frac{2}{\varepsilon}$. Since all $x_n^p \in \mathbb{B}(a_p, \frac{1}{p})$, we have $d(x_m^p, x_n^p) \le d(x_m^p, a_p) + d(a_p, x_n^p) < \frac{2}{p}$. Since $\{x_n^n : n \ge p\}$ is a subsequence of $\{x_n^p : n \ge 1\}$, we get $d(y_m, y_n) < 2/p = \varepsilon$, $\forall m, n \ge p$. Therefore $\{y_n\}$ is a Cauchy subsequence of $\{x_n\}$. Conversely, assume that every sequence in A has a Cauchy subsequence. Suppose to the contrary that A is not precompact. In particular, A is non-empty, say $x_1 \in A$. There is $\varepsilon > 0$ such that for all finite subset j of A, we have $A \not\subset \bigcup_{a \in J} \mathbb{B}(a, \varepsilon)$. Since $A \not\subset \mathbb{B}(x_1, \varepsilon)$, there is $x_2 \in A$ but $d(x_1, x_2) \ge \varepsilon$. Inductively, since $A \not\subset \bigcup_{j=1}^{n-1} \mathbb{B}(x_j, \varepsilon)$, there is $x_n \in A$ but $d(x_n, x_j) \ge \varepsilon, \forall 1 \le j \le n-1$. Then $\{x_n\}$ is a sequence in A. Consequently, it has a Cauchy subsequence $\{y_n\}$. There is $k \ge 1$ such that for all $i, j \ge k$ we have $d(y_i, y_j) \le \frac{\varepsilon}{2}$. There is a sequence of integers $n(1) < n(2) < \cdots$ such that $y_j = x_{n(j)}$ for all j. Now $\varepsilon < d(x_{n(k)}, x_{n(k+1)}) = d(y_k, y_{k+1}) \le \frac{1}{2}\varepsilon$ is a contradiction that completes the proof. □

2-5.7. **Theorem** Products of precompact sets are precompact.

Proof. Let A, B be precompact sets of metric spaces X, Y respectively. Take any sequence $\{(x_n, y_n)\}$ in $A \times B$. Then $\{x_n\}$ and $\{y_n\}$ are sequences in the precompact sets A, B respectively. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy subsequences of $\{x_n\}$ and $\{y_n\}$ respectively. Then $\{(a_n, b_n)\}$ is a Cauchy subsequence of $\{(x_n, y_n)\}$. Therefore $A \times B$ is precompact. □

2-5.8. **Exercise** Prove that every precompact set is bounded. Prove that every subset of a precompact set is precompact.

2-5.9. **Corollary** Every bounded subset of $\mathbb{K}^n$ is precompact.

Proof. Let B be a bounded subset of $\mathbb{R}$. Then every sequence in B has a convergent subsequence which is also Cauchy. Hence every bounded subset of $\mathbb{R}$ is precompact. Now let M be a bounded subset of $\mathbb{R}^n$. Then the projection M_j of M to its j-th coordinate is bounded and hence precompact. As a subset of the precompact set $\Pi_{j=1}^n M_j$, M is precompact in $\mathbb{R}^n$. Next, since $\mathbb{C} = \mathbb{R}^2$, bounded sets are identical with precompact sets. Repeating the same argument for $\mathbb{R}^n$, we obtain the result for $\mathbb{C}^n$. □

2-5.10. **Exercise** Show that the real line equipped with the discrete metric is bounded but not precompact.

2-5.11. **Exercise** Prove that the projection of a precompact set in a product metric space $X \times Y$ to X is precompact.

2-5.12. **Exercise** Is the set of all integers precompact? Is the set of all rationals in $(0, 1)$ precompact?

2-6 Compactness

2-6.1. In a precompact set A, every sequence has a Cauchy subsequence which need not converge in A. It is natural to introduce the concept of compactness to safeguard the existence of limits. It turns out that compactness plays a vital role in almost every aspect of analysis especially in infinite dimensional spaces, e.g. Chapter 5 below. Open covers provide an alternative way to utilize the compactness and also allow us to extend the concept to general topological spaces. To reduce an arbitrary open cover to countable subcover, we need the concept of separable sets.

2-6.2. Let A, B be subsets of a metric space X. Then B is said to be *dense in* A if $A \subset \overline{B}$. The set A is said to be *separable* if A has a countable dense subset.

2-6.3. **Theorem** Continuous images of separable sets are separable.

Proof. Let X, Y be metric spaces. Let $f : X \to Y$ be a continuous map and A a separable subset of X. There is a countable dense subset B of A. Now $f(B)$ is a countable subset of $f(A)$. Since $f(A) \subset f(\overline{B}) \subset \overline{f(B)}$, the set $f(B)$ is countable dense in $f(A)$. Therefore $f(A)$ is separable. □

2-6.4. **Exercise** Prove that finite products, countable unions and subsets of separable sets are separable.

2-6.5. **Exercise** Prove that $\mathbb{K}^n$ is separable.

2-6.6. **Theorem** Every precompact set A in a metric space X is separable.

Proof. For each integer $n \geq 1$, there is a finite subset J_n of A such that $A \subset \bigcup_{x \in J_n} \mathbb{B}(x, \frac{1}{n})$. Let $B = \bigcup_{n=1}^{\infty} J_n$. Then B is a countable subset of A. To show $A \subset \overline{B}$, take any $y \in A$. For each n, there is $x_n \in J_n$ such that $y \in \mathbb{B}(x_n, \frac{1}{n})$. Then $\{x_n\}$ is a sequence in B. Since $d(x_n, y) \leq \frac{1}{n}$, we have $x_n \to y$. Therefore $y \in \overline{B}$. Hence B is a countable dense subset of A. Therefore A is separable. □

2-6.7. Let A be a subset of a metric space X. Then A is said to be *compact* if every sequence in A has a subsequence convergent to some limit which is also in A. Let $\{V_i : i \in I\}$ be a cover of A. Then $\{V_i : i \in I\}$ is called an *open* cover of A if every V_i is open. The cover $\{V_i : i \in I\}$ is said to be *finite* (respectively *countable*) if the index set I is finite (respectively countable). A subfamily of $\{V_i : i \in I\}$ which remains to be a cover of A is called a *subcover*.

2-6.8. **Theorem** Let A be a subset of a metric space X. If A is separable, then every open cover $\{V_i : i \in I\}$ of A has a countable subcover.

Proof. Let K be a countable dense subset of A. Take ant $x \in A$. Since $\{V_i : i \in I\}$ is a cover of A, some index $i(x) \in I$ satisfies $x \in V_{i(x)}$. Because $V_{i(x)}$ is open, there is a *rational* number $r(x) > 0$ such that $\mathbb{B}[x, r(x)] \subset V_{i(x)}$. Since $A \subset \overline{K}$, x is a closure point of K. There is some point $b(x) \in K \cap \mathbb{B}[b(x), \frac{1}{2}r(x)]$, that is $x \in \mathbb{B}[b(x), \frac{1}{2}r(x)]$. Let $\mathcal{B} = \{\mathbb{B}[b(x), \frac{1}{2}r(x)] : x \in A\}$. Since both K and the set of all rationals are countable, so is $\mathcal{B}$. Let $B_1, B_2, \cdots$ be an enumeration of $\mathcal{B}$. For each $n = 1, 2, \cdots$, there is $x_n \in A$ such that $B_n = \mathbb{B}[b(x), \frac{1}{2}r(x)]$. We claim that $\{V_{i(x_n)} : n \geq 1\}$ is a cover of A. In fact, take any $y \in A$. There is n such that $y \in B_n$. Therefore $y \in \mathbb{B}[b(x_n), \frac{1}{2}r(x_n)]$, that is $d[y, b(x_n)] < \frac{1}{2}r(x_n)$. Since $b(x_n) \in \mathbb{B}[x_n, \frac{1}{2}r(x_n)]$, we have $d[x_n, b(x_n)] < \frac{1}{2}r(x_n)$. As a result, $d(y, x_n) < r(x_n)$, i.e. $y \in \mathbb{B}[x_n, r(x_n)] \subset V_{i(x_n)}$. Consequently, $\{V_{i(x_n)} : n \geq 1\}$ is a countable open subcover. □

2-6.9. **Theorem** Let A be a subset of a metric space X. Then the following statements are equivalent.

(a) A is compact.

(b) A is complete and precompact.

(c) Every open cover has a finite subcover.

(d) Every countable open cover has a finite subcover.

Proof. $(a \Rightarrow b)$ Let $\{x_n\}$ be a Cauchy sequence in A. Since A is compact, there is a subsequence $\{y_n\}$ convergent to some limit $b \in A$. Hence the original sequence $\{x_n\}$ is also convergent to b. Therefore A is complete. To show that A is precompact, let $\{x_n\}$ be any sequence in A. Since A is compact, there exists a convergent subsequence $\{y_n\}$. Since $\{y_n\}$ is also Cauchy, A is precompact.

$(b \Rightarrow c)$ Since A is precompact, A is separable. Every open cover $\mathcal{C}$ of A has a countable open subcover $\{V_j : j \geq 1\}$. Suppose to the contrary that $A \not\subset \bigcup_{j=1}^{n} V_j$ for every n. There is $x_n \in A$ but $x_n \notin \bigcup_{j=1}^{n} V_j$. Since A is precompact, $\{x_n\}$ has a Cauchy subsequence. Since A is complete, the subsequence converges to some $b \in A$. So, there is a sequence of integers $n(1) < n(2) < n(2) < \cdots$ such that $x_{n(j)} \to b$. Since $\{V_n\}$ is a cover of A, we have $b \in V_k$, for some k. Because V_k is open, there is $\mathbb{B}(b, \varepsilon) \subset V_k$. By convergence of subsequence, there is p such that for all $j \geq p$, we have $d(x_{n(j)}, b) \leq \frac{1}{2}\varepsilon$, or $x_{n(j)} \in \mathbb{B}(b, \varepsilon) \subset V_k$. Take any $j \geq p + k$. Since $j \geq p$, we get $x_{n(j)} \in V_k$. It follows from the choice of $\{x_n\}$, we obtain $x_{n(j)} \notin \bigcup_{i=1}^{n(j)} V_i$. Now $n(j) \geq j \geq k$ gives $x_{n(j)} \notin V_k$. This contradiction shows that some n satisfies $A \subset \bigcup_{i=1}^{n} V_i$. Therefore the given cover $\mathcal{C}$ has a finite subcover.

$(c \Rightarrow d)$ It is obvious.

$(d \Rightarrow a)$ Let $\{x_n\}$ be a sequence in A. Suppose to the contrary that every subsequence of $\{x_n\}$ cannot converge to any point in A. Hence, no point of A can be a cluster point of $\{x_n\}$, that is, $A \cap \bigcap_{n=1}^{\infty} \overline{K}_n = \emptyset$ where $K_n = \{x_j : j \geq n\}$. Now each $V_n = X \setminus \overline{K}_n$ is open in X and $\{V_n : n \geq 1\}$ forms a countable open cover of A. Hence $A \subset \bigcup_{n=1}^{p} V_n$ for some p. Since $K_{n+1} \subset K_n$, we have $V_n \subset V_{n+1}$ and so $A \subset V_p$. Now $x_p \in A$. Thus $x_p \in V_p$, or $x_p \notin \overline{K}_p$, and so $x_p \notin K_p$ which is a contradiction. Therefore the original sequence must have a subsequence convergent to some point which is in A. This completes the proof. □

2-6.10. **Corollary** Every compact set in a metric space is closed and bounded.

Proof. Complete sets are closed and precompact sets are bounded. □

2-6.11. **Exercise** A subset of $\mathbb{K}^n$ is compact iff it is closed and bounded.

2-6.12. **Exercise** Show that the real line equipped with the discrete metric is closed and bounded but not compact.

2-6.13. **Theorem** Every closed subset B of a compact set A is compact.

Proof. Let $\{x_n\}$ be a sequence in B. Then $\{x_n\}$ is also a sequence in the compact set A. So, $\{x_n\}$ has a subsequence $\{y_n\}$ convergent to some point $a \in A$. Since all $y_n \in B$, a is a closure point of B. Since B is closed in X, we have $a \in B$. Therefore B is compact. □

2-6.14. **Theorem** Products of compact sets are compact.

Proof. See the proof for precompact sets. □

2-6.15. **Exercise** Let $\{A_i : i \in I\}$ be a non-empty family of compact subsets of a metric space X. Prove that if for every finite subset J of I, the finite intersection $\bigcap_{j \in J} A_j$ is non-empty then the total intersection $\bigcap_{i \in I} A_i$ is also non-empty.

2-7 Continuous Maps on Compact Spaces

2-7.1. In elementary calculus, every continuous real function on a closed bounded interval has an absolute maximum which is used to prove the Rolle's Theorem and then to derive the mean-value theorem. This is just one of many standard applications of compactness in analysis.

2-7.2. **Theorem** Continuous images of compact sets are compact.

Proof. Let X, Y be metric spaces. Let $f : X \to Y$ be a continuous map and A a compact subset of X. Let $\{H_i : i \in I\}$ be an open cover of $f(A)$. Then $\{f^{-1}(H_i) : i \in I\}$ is an open cover of the compact set A. It has a finite subcover $\{f^{-1}(H_j) : j \in J\}$ of A. Hence, $\{H_j : j \in J\}$ is a finite subcover of $f(A)$. Therefore $f(A)$ is compact. □

2-7.3. **Theorem** Let f be a continuous real function on a metric space X. If X is compact and non-empty, then there are $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$. In other words, f attains its maximum and minimum values.

Proof. Since the continuous image $f(X)$ of the compact space X is a compact subset of $\mathbb{R}$, it is closed and bounded in $\mathbb{R}$. Let $\alpha = \inf f(X)$. For every $n \geq 1$, there is $x_n \in X$ such that $f(x_n) \leq \alpha + \frac{1}{n}$. Since X is compact, there is a subsequent $\{x_{n(j)}\}$ convergent to some $a \in X$. Letting $j \to \infty$ in the expression: $f[x_{n(j)}] \leq \alpha + \frac{1}{n(j)}$, we have $f(a) \leq \alpha \leq f(x)$ for all $x \in X$. Therefore f attains it minimum at a. Similarly, it also attains its maximum.□

2-7.4. Let X, Y be metric spaces. Then a bijection $f : X \to Y$ is called a *homeomorphism* if both f, f^{-1} are continuous.

2-7.5. **Corollary** Let $f : X \to Y$ be a continuous bijection. If X is compact, then f is a homeomorphism.

Proof. It suffices to show that $g = f^{-1}$ is continuous. Let A be a closed subset of X. Since X is compact, so is A. Now $f(A)$ is compact and hence closed. Therefore the inverse image $g^{-1}(A) = f(A)$ of a closed set under g is closed. Consequently, $f^{-1} = g$ is continuous. □

2-7.6. **Lemma** Let X, Y, Z be metric spaces and let $f : X \times Y \to Z$ be a continuous map. If Y is compact then for every $a \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $d(x, a) < \delta$ and $y \in Y$ we have $d[f(x, y), f(a, y)] < \varepsilon$. This handy lemma will be used later, e.g. §§8-3.2, 9-3.9, 11-5.8.

Proof. Take any $t \in Y$. Since f is continuous at $(a,t) \in X \times Y$, there is $\delta_t > 0$ such that for all $x \in \mathbb{B}(a, \delta_t)$ and $y \in \mathbb{B}(t, \delta_t)$ we have $d[f(x,y), f(a,t)] < \frac{1}{2}\varepsilon$. By compactness, there is a finite subset J of Y such that $Y \subset \bigcup_{t \in J} \mathbb{B}(t, \delta_t)$. Define $\delta = \min\{\delta_t : t \in J\}$. Now assume $d(x,a) < \delta$ and $y \in Y$. Select $t \in J$ such that $y \in \mathbb{B}(t, \delta_t)$. Then $d(x,a) < \delta_t$ and $d(y,t) < \delta_t$. The following inequality completes the proof:

$$d[f(x,y), f(a,y)] \leq d[f(x,y), f(a,t)] + d[f(a,t), f(a,y)] < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \qquad \square$$

2-7.7. **Exercise** Let X be the real line equipped with the discrete metric and $\mathbb{R}$ the real line with usual metric. Show that the identity map from X onto $\mathbb{R}$ is a continuous bijection which is not a homeomorphism.

2-7.8. **Exercise** A subset of a metric space is said to be *relatively compact* if its closure is compact. Prove that every relatively compact set is precompact. Prove that in a complete metric space, every precompact set is relatively compact.

2-7.9. **Exercise** Prove that a subset in a product space is relatively compact iff its projection onto factor spaces are relatively compact.

2-7.10. **Exercise** Prove that linear combinations of compact sets in normed spaces are compact. Prove that linear combinations of relatively compact sets are relatively compact.

2-7.11. **Exercise** Two sets of $\mathbb{R}^2$ are given by $A = \{(t, \sin\frac{1}{t}) : 0 < t \leq 1\}$ and $B = \{(0,t) : 0 \leq t \leq 1\}$. Determine whether A, B and $A \cup B$ are compact.

2-7.12. **Exercise** Prove that the projection of a product metric space $X \times Y$ onto the coordinate space X is an open map, i.e. the image of an open set is open. Show that it does not carry every closed set onto a closed set. Also find an example of a continuous map which carries every closed set onto a closed set but not every open set onto an open set.

2-7.13. Let A, B be non-empty subsets of a metric space X. Then the *distance* between A, B is defined by $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$. Clearly we have, $d(A,B) = d(B,A) = \inf\{d(x,B) : x \in A\}$.

2-7.14. **Theorem** Let A, B be non-empty subsets of a metric space X. If A is compact, then there is $a \in A$ such that $d(a,B) = d(A,B)$. Furthermore if A, B are disjoint closed sets, then $d(A,B) > 0$.

Proof. The function $d(x, B)$ is continuous in x on a compact set A. □

2-7.15. **Theorem** Let A, B be non-empty compact subsets of a metric space X. Then there are points $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.

Proof. Since both A, B are compact, the product set $A \times B$ is compact in the product space $X \times Y$. Since the continuous function $d : A \times B \to \mathbb{R}$ attains its minimum, there is some $(a, b) \in A \times B$ such that $d(a, b) \leq d(x, y)$ for all $(x, y) \in A \times B$. Therefore $d(a, b) = d(A, B)$. □

2-7.16. **Exercise** Let $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ and B the x-axis of $\mathbb{R}^2$. Find the distance $d(A, B)$.

2-8 Uniform Continuity

2-8.1. Uniform continuity is a global property because uniformly continuous map carries any pair of near points into near points. One of many standard applications is to show that continuous functions are integrable. Since every continuous linear map is uniformly continuous, we shall apply the unique extension theorem below to construct regulated integrals for vector-valued maps in Chapter 8.

2-8.2. Let X, Y be metric spaces and $f : X \to Y$ a given map. Then f is said to be ***uniformly continuous*** if for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$, $d(x, y) \leq \delta$ implies $d[f(x), f(y)] \leq \varepsilon$. Clearly, every uniformly continuous map is continuous. For a normed space, it is easy to prove that the function $x \to \|x\|$ is uniformly continuous. For a metric space, the distance function $d(x, A)$ from a point x to a non-empty set A is uniformly continuous in x.

2-8.3. **Exercise** Write explicitly the negation of uniform continuity. Prove that the function x^2 is continuous but not uniformly continuous on $\mathbb{R}$.

2-8.4. **Theorem** Let X, Y be metric spaces and $f : X \to Y$ a continuous map. If X is compact, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. For each $a \in X$, there is $\delta_a > 0$ such that $d(x, a) \leq \delta_a$ implies $d[f(x), f(a)] \leq \frac{1}{2}\varepsilon$. Now $\{\mathbb{B}(a, \frac{1}{2}\delta_a) : a \in X\}$ is an open cover of the compact space X. There is a finite subset J of X such that $X \subset \bigcup_{a \in J} \mathbb{B}(a, \frac{1}{2}\delta_a)$. Let $\delta = \frac{1}{2}\min\{\delta_a : a \in J\}$. Then $\delta > 0$. Suppose $x, y \in X$ satisfying $d(x, y) \leq \delta$. Then there is $a \in J$ such that $x \in \mathbb{B}(a, \frac{1}{2}\delta_a)$, i.e. $d(x,a) < \frac{1}{2}\delta_a$. Now, $d(y,a) \leq d(x,y)+d(x,a) \leq \delta + \frac{1}{2}\delta_a \leq \delta_a$. Hence we have $d[f(x), f(a)] \leq \frac{1}{2}\varepsilon$ and $d[f(y), f(a)] \leq \frac{1}{2}\varepsilon$. Thus, $d[f(x), f(y)] \leq \varepsilon$. Therefore f is uniformly continuous. □

2-8.5. **Lemma** Let X, Y be metric spaces. Let $f : X \to Y$ be a uniformly continuous map. Then:
(a) if $\{x_n\}$ is a Cauchy sequence in X, then its image $\{f(x_n)\}$ is Cauchy in Y.
(b) if E is a precompact subset of X, then $f(E)$ is precompact in Y.

Proof. The first follows immediately from definition and the second from the characterization of precompactness in terms of Cauchy sequences. □

2-8.6. **Lemma** Let f, g be continuous maps from a metric space X into a metric space Y. If $f = g$ on a dense subset M of X, then $f = g$ on X.

Proof. Take any $a \in X$. There is a sequence $x_n \in M$ such that $x_n \to a$. Since f, g are continuous, we have $f(a) = \lim f(x_n) = \lim g(x_n) = g(a)$. Therefore we have $f = g$ on X. □

2-8.7. A sequence $\{z_n\}$ is said to be *merged from* two given sequences $\{x_n\}$ and $\{y_n\}$ in X if $z_{2n-1} = x_n$ and $z_{2n} = y_n$ for all $n \geq 1$.

2-8.8. **Unique Extension Theorem** Let X, Y be metric spaces. Let f be a map defined on a subset M of X into Y. If f is uniformly continuous on M and if Y is complete, then there is a unique uniformly continuous extension of f over the closure of M.

Proof. Take any point $a \in \overline{M}$. There is a sequence $\{x_n\}$ in E convergent to a. Then $\{x_n\}$ is Cauchy. By uniform continuity, $\{f(x_n)\}$ is Cauchy in the complete metric space Y and hence it converges to some limit, denoted by $g(a)$. We claim that $g(a)$ is independent of the choice of the sequence $\{x_n\}$. Suppose $\{y_n\}$ is another sequence in M convergent to a. Let $\{z_n\}$ be the sequence merged from $\{x_n\}$ and $\{y_n\}$. Then $\{z_n\}$ converges to a and as above $\{f(z_n)\}$ converges in Y. Hence its subsequence $\{g(x_n)\}$ and $\{g(y_n)\}$ converge to the same limit. Therefore a map $g : \overline{M} \to Y$ is well-defined. If $a \in M$ then the constant sequence $\{a\}$ converges to a and hence $g(a) = f(a)$. Consequently g is an extension of f. To prove that g is uniformly continuous, let $\varepsilon > 0$ be given. There is $\delta > 0$ such that whenever $d(x,y) \leq \delta$ in M we have $d[f(x), f(y)] \leq \varepsilon/3$. Now take any $a, b \in \overline{M}$ satisfying $d(a,b) \leq \delta/3$. There are sequences $\{x_n\}$ and $\{y_n\}$ in M convergent to a and b respectively. There is an integer p such that for all $n \geq p$ we have $d(x_n, a) \leq \delta/3$, $d(y_n, b) \leq \delta/3$, $d[f(x_n), g(a)] \leq \varepsilon/3$ and

$d[f(y_n), g(b)] \leq \varepsilon/3$. Since $d(x_p, y_p) \leq d(x_p, a) + d(a, b) + d(b, y_p) \leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$ we have $d[f(x_p), f(y_p)] \leq \frac{\varepsilon}{3}$. Hence

$$d[g(a), g(b)] \leq d[g(a), f(x_p)] + d[f(x_p), f(y_p)] + d[f(y_p), g(b)] \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The uniqueness follows from last lemma. This completes the proof. □

2-8.9. **Exercise** Show that the continuous function $\frac{1}{x}$ on $(0, 1)$ has no continuous extension over $[0, 1]$.

2-8.10. **Exercise** Prove that the projections from a product metric space $X \times Y$ onto its factor spaces X, Y are uniformly continuous.

2-9 Connected Sets

2-9.1. Let X be a metric space. A pair (A, B) of non-empty subsets of X is called a *disconnection* if $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$. A subset M of X is said to be *disconnected* if there is a disconnection (A, B) such that $M = A \cup B$. A subset of X is said to be *connected* if it is not disconnected. A set H is said to be *contained in a disconnection* (A, B) if $H \subset A \cup B$.

2-9.2. **Theorem** A subset M of the real line $\mathbb{R}$ is connected iff M is an interval, i.e. $a, b \in M$ and $a \leq c \leq b$ implies $c \in M$.

Proof. ($\Rightarrow$) Suppose to the contrary that $a, b \in M$, $a \leq c \leq b$ and $c \notin M$. Then we get $a < c < b$. Let $A = M \cap (-\infty, c)$ and $B = M \cap (c, \infty)$. Since $c \notin M$, we have $M \subset (\infty, c) \cup (c, \infty)$, or

$$M = M \cap \{(-\infty, c) \cup (c, \infty)\} = \{M \cap (-\infty, c)\} \cup \{M \cap (c, \infty)\} = A \cup B.$$

Observe that $A \cap \overline{B} \subset (-\infty, c) \cap \overline{(c, \infty)} = (-\infty, c) \cap [c, \infty) = \emptyset$. Hence $A \cap \overline{B} = \emptyset$. Similarly, $B \cap \overline{A} = \emptyset$. Since both A, B are non-empty, M is disconnected.
($\Leftarrow$) Suppose M is disconnected. There are non-empty sets A, B such that $M = A \cup B, A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$. Let $a \in A$ and $b \in B$. Since $A \cap B = \emptyset$, we may assume $a < b$. Define $H = \{x \in A : x \leq b\}$ and $c = \sup H$. Then $a, b \in M$ and $a \leq c \leq b$. Since M is an interval, $c \in M$. There are two cases. Firstly, suppose $c \in A$. Since $A \cap \overline{B} = \emptyset$, c does not belong to the closed set $\overline{B}$. There is $\varepsilon > 0$ such that $\mathbb{B}(c, 2\varepsilon) \cap \overline{B} = \emptyset$. Hence $c + \varepsilon \notin B$ and $a \leq c + \varepsilon \leq b$. Since $c = \sup H$, $c+\varepsilon \notin A$. We proved that $a, b \in M, a \leq a+\varepsilon \leq b$ but $c+\varepsilon \notin M$. This establishes a contradiction. Secondly, suppose $c \in B$. Since $B \cap \overline{A} = \emptyset$, c does not belong to the closed set $\overline{A}$. There is $\varepsilon > 0$ such that $\mathbb{B}(c, 2\varepsilon) \cap \overline{A} = \emptyset$. Take any $x \in H$. We have $x \in A$ and $x \leq c$. Since $[c - \varepsilon, c] \cap A = \emptyset$, we obtain

$x \le c - \varepsilon$. Taking supremum over $x \in H$, we get $c \le c - \varepsilon$ which is another contradiction. Therefore M must be connected. □

2-9.3. **Theorem** Continuous images of connected sets are connected.

Proof. Let X, Y be metric spaces and $f : X \to Y$ a continuous map. Assume E is a connected subset of X. Suppose to the contrary that $f(E)$ is disconnected. There is a disconnection (P, Q) such that $f(E) = P \cup Q$. Let $A = E \cap f^{-1}(P)$ and $B = E \cap f^{-1}(Q)$. Then A, B are non-empty sets satisfying $E = A \cup B$. Suppose to the contrary that there is $x \in A \cap \overline{B}$. Then $f(x) \in P$ and $f(x) \in f(\overline{B}) \subset \overline{f(B)} \subset \overline{Q}$. Hence $f(x) \in P \cap \overline{Q}$ which is a contradiction. Therefore $A \cap \overline{B} = \emptyset$ and similarly $B \cap \overline{A} = \emptyset$. Consequently, E is disconnected. This completes the proof. □

2-9.4. **Intermediate Value Theorem** Let $a < b$ be in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) < 0$, then there is $x \in (a, b)$ such that $f(x) = 0$.

Proof. The interval $[a, b]$ is connected. The continuous image $f([a, b])$ is also connected and hence it is an interval. Since $f(a)f(b) < 0$, the number zero is between $f(a)$ and $f(b)$, i.e. $0 \in f([a, b])$, or for some $x \in [a, b]$, $f(x) = 0$. □

2-9.5. **Bisection Method** Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)f(b) < 0$. Define $a_1 = a$ and $b_1 = b$. Inductively, suppose $[a_{n-1}, b_{n-1}]$ is defined. Bisect $[a_{n-1}, b_{n-1}]$ into subintervals $[a_{n-1}, c]$ and $[c, b_{n-1}]$ where $c = \frac{1}{2}(a_{n-1} + b_{n-1})$ is the mid-point. If $f(c) = 0$, then we have a solution already. Suppose $f(c) \ne 0$. If $f(a_{n-1})f(c) < 0$ then define $a_n = a_{n-1}$ and $b_n = c$, else define $a_n = c$ and $b_n = b_{n-1}$. Therefore $f(a_n)f(b_n) < 0$. By intermediate value theorem, we know that there is a solution between a_n and b_n. Therefore the error of assuming $\frac{1}{2}(a_n + b_n)$ to be a solution is no more than $(b - a)/2^n$. Alternatively, you may cut an interval into more than two subintervals. This is a slow but very simple and reliable way to find approximate numerical solution of a nonlinear equation $f(x) = 0$.

2-9.6. **Fixed Point Theorem** Let $a < b$ be in $\mathbb{R}$. Every continuous function $f : [a, b] \to [a, b]$ has a fixed point.

Proof. Let $g(x) = x - f(x)$. If $g(a) = 0$ or $g(b) = 0$ then a or b is a fixed point of f. Assume that both $g(a), g(b)$ are non-zero. Since $f([a, b]) \subset [a, b]$, we have $g(a) < 0$ and $g(b) > 0$. Hence $g(x) = 0$ for some $x \in (a, b)$. Consequently $f(x) = x$. □

2-9.7. **Exercise** Let $y = f(x) : [0,1] \to [0,2]$ and $x = g(y) : [0,2] \to [0,1]$ be continuous functions. Prove that there is $(a,b) \in [0,1] \times [0,2]$ such that $b = f(a)$ and $a = g(b)$. Sketch the graphs of f, g on the rectangle $[0,1] \times [0,2]$ and interpret the location of (a,b).

2-9.8. **Lemma** If M is a connected set contained in a disconnection (P,Q), then either $M \subset P$ or $M \subset Q$.

Proof. Suppose to the contrary that $M \not\subset P$ and $M \not\subset Q$. Define $A = M \cap P$ and $B = M \cap Q$. If possible, assume $A = \emptyset$. Then

$$M = M \cap (P \cup Q) = (M \cap P) \cup (M \cap Q) = A \cup (M \cap Q) = M \cap Q,$$

i.e. $M \subset Q$ which is a contradiction. Hence $A \neq \emptyset$ and similarly $B \neq \emptyset$. Now observe $M = M \cap (P \cup Q) = (M \cap P) \cup (M \cap Q) = A \cup B$. Since $A \cap \overline{B} \subset P \cap \overline{Q} = \emptyset$, we have $A \cap \overline{B} = \emptyset$. Similarly, $B \cap \overline{A} = \emptyset$. Therefore M is disconnected. This contradiction shows either $M \subset P$ or $M \subset Q$. □

2-9.9. **Theorem** Let $\{M_i : i \in I\}$ be a family of connected sets in a metric space X. If the intersection $\bigcap_{i \in I} M_i$ is non-empty, then the union $\bigcup_{i \in I} M_i$ is also connected.

Proof. Let $M = \bigcup_{i \in I} M_i$. Suppose to the contrary that there is a disconnection (P,Q) satisfying $M = P \cup Q$. Take any $a \in \bigcap_{i \in I} M_i$. Then $a \in M$. Without loss of generality, let $a \in P$. For each $i \in I$, M_i is a connected subset of the disconnection (P,Q). So, we obtain either $M_i \subset P$ or $M_i \subset Q$. Now $a \in M_i \cap P$. Since P, Q are disjoint, we have $a \notin Q$. Hence $M_i \not\subset Q$. Therefore we have $M_i \subset P$ for all $i \in I$. Consequently, $M \subset P$. Thus $Q = Q \cap (P \cup Q) = Q \cap M \subset Q \cap P = \emptyset$ gives a contradiction. Therefore M is connected. □

2-9.10. **Exercise** Let X be a metric space. A continuous map φ from an interval into X is called a *curve* in X. If $\varphi : [a,b] \to X$ is a continuous map, then $f(a), f(b)$ are said to be *joined* by the curve φ. Show that a subset M of X is connected if any two points in M can be joined by a curve in M.

2-9.11. **Exercise** Prove that all ball in a normed space are connected. Prove that every normed space is connected.

2-9.12. **Exercise** Show that any two points of a connected open subset M of a normed space can be joined by a curve in M.

2-10 Components

2-10.1. Let X be a metric space. A subset C of X is called a *component* of X if it is a maximal connected subset of X. More precisely, C is a connected subset of X and if A is any connected set containing C, then $A = C$.

2-10.2. **Theorem** Let M, N be subsets of a metric space X such that $M \subset N \subset \overline{M}$. If M is connected, then so is N.

Proof. Suppose to the contrary that N is disconnected. There is a disconnection (P, Q) such that $N = P \cup Q$. Since M is a subset of the disconnection (P, Q), we have either $M \subset P$ or $M \subset Q$. If $M \subset P$, then

$$Q = Q \cap N \subset Q \cap \overline{M} \subset Q \cap \overline{P} = \emptyset,$$

i.e. $Q = \emptyset$ which is a contradiction. Similarly, $M \subset Q$ leads to another contradiction. Therefore N is connected. □

2-10.3. **Theorem** Every component is closed.

Proof. Let C be a component of a metric space X. Then $\overline{C}$ is a connected set containing C. Hence $C = \overline{C}$. Therefore C is closed. □

2-10.4. **Exercise** Let $X = (01) \cup (1, 2)$. Show that $(0, 1)$ is closed in X but not closed in $\mathbb{R}$. Find the components of X.

2-10.5. **Exercise** Find the components of the metric space Q of all rational numbers.

2-10.6. **Exercise** Find closure of the subset $M = \{(x, y) : y = \sin \frac{1}{x}, x > 0\}$ of $\mathbb{R}^2$. Show that the union X of M and the y-axis is connected. Is it possible to join the points $(0, 3)$ and $(\frac{1}{\pi}, 0)$ by a curve in X?

2-10.7. **Theorem** The family of all components of a metric space X forms a partition of X.

Proof. Let $a \in X$ be given. Let $\mathcal{E}$ be the family of all connected sets containing a. Then $\mathcal{E}$ is non-empty because the singleton $\{a\}$ is in $\mathcal{E}$. Let C be the union of all sets in $\mathcal{E}$. Then C is connected. Furthermore, if B is a connected set containing C, then B is a connected set containing a; hence $B \in \mathcal{E}$, and therefore $B \subset C$. Consequently, C is maximal connected set. As a result, every point of X is covered by some component. Next, let M, N be two components of X. Suppose that $M \cap N \neq \emptyset$. Then $M \cup N$ is a connected set containing the component M. By maximality, we have $M = M \cup N$. Similarly, $N = M \cup N$.

Hence, $M = N$. Therefore two components are either disjoint or identical. This completes the proof. □

2-10.8. **Lemma** Let $\mathbb{D}$ be a family of disjoint non-empty open sets in a metric space X. If X is separable then $\mathbb{D}$ is countable.

Proof. Let $\{a_n : n \geq 1\}$ be an enumeration of a countable dense set in X. Take any $A \in \mathbb{D}$. Given that A is non-empty, there is $x \in A$. Since A is open, there is a ball $\mathbb{B}(x) \subset A$. Being dense set, there is $a_n \in \mathbb{B}(x)$. Define $f(A) = n$. Note that the choice of n depends on A but it need not be unique. We have defined an integer function f on $\mathbb{D}$. If $f(A) = f(B) = n$, then $a_n \in A \cap B$ and hence $A = B$ because sets in $\mathbb{D}$ are disjoint. Therefore f is an injection of $\mathbb{D}$ into the set of integers. Consequently, $\mathbb{D}$ is countable. □

2-10.9. **Theorem** Every open subset M of $\mathbb{R}$ is a countable union of disjoint open intervals.

Proof. Let C be a component of M. Since M is open, for each $x \in M$, there is $\varepsilon > 0$ such that the ball $B = (x-\varepsilon, x+\varepsilon)$ is a subset of M. Since $x \in B \cap C$ and both B, C are connected, $B \cup C$ is a connected set containing the component C. Hence $C = B \cup C$, i.e. $B \subset C$. Therefore C is open in $\mathbb{R}$. Because C is connected, it is an open interval. Now the family $\mathbb{D}$ of all components of M consists of disjoint open intervals. Since $\mathbb{R}$ is separable, $\mathbb{D}$ is countable. □

2-10.10. **Theorem** Let X, Y be metric spaces. If both X, Y are connected, then so is the product space $X \times Y$.

Proof. Take any point $(a, b) \in X \times Y$. The map $f : X \to X \times Y$ given by $f(x) = (x, b), \forall\ x \in X$ is continuous. Hence the image $f(X) = X \times \{b\}$ is a connected subset of $X \times Y$. Similarly, $\{a\} \times Y$ is a connected subset of $X \times Y$. Let $K(a) = [X \times \{b\}] \cup [\{a\} \times Y]$. Since $(a, b) \in [X \times \{b\}] \cap [\{a\} \times Y]$, $K(a)$ is connected. Since $X \times \{b\} \subset K(a), \forall\ a \in X$, the set $\bigcup_{a \in X} K(a) = X \times Y$ is connected. This completes the proof. □

2-10.11. **Exercise** Let X, Y be metric spaces and $f : X \to Y$ a continuous map. Prove that if X is connected then the graph $\{[x, f(x)] : x \in X\}$ of f is a connected subset of the product space $X \times Y$.

2-99. **References and Further Readings** : Dugundji-66, Thron and Hocking.

Chapter 3

Banach Spaces

3-1 Uniform Convergence

3-1.1. Functional analysis is supposed to analyze certain classes of functions. A lot of typical examples will be given soon so that you have an intuitive idea of what is going on.

3-1.2. Let X be a non-empty set. Let f, g be functions on X and α, β scalars in $\mathbb{K}$. Then the pointwise operations are defined as follow:
(a) equality: $f = g$ if $f(x) = g(x), \forall\, x \in X$;
(b) linear combination: $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \forall\, x \in X$;
(c) product: $(f \cdot g)(x) = f(x)g(x), \forall x \in X$;
(d) absolute-value: $|f|(x) = |f(x)|, \forall x \in X$;
(e) complex conjugate: $f^-(x) = [f(x)]^-, \forall x \in X$;
(f) real and imaginary parts: $\mathrm{Re}(f) = \frac{1}{2}(f + f^-)$; $\mathrm{Im}(f) = \frac{1}{2i}(f - f^-)$.
For all *real* functions f, g on X define
(g) order: $f \leq g$ if $f(x) \leq g(x), \forall x \in X$;
(h) maximum: $(f \vee g)(x) = f(x) \vee g(x) = \max\{f(x), g(x)\}, \forall\, x \in X$;
(i) minimum: $(f \wedge g)(x) = f(x) \wedge g(x) = \min\{f(x), g(x)\}, \forall\, x \in X$;
(j) positive and negative parts: $f_+ = f \vee 0$; $f_- = (-f) \vee 0$.
Pointwise addition and scalar multiplication of vector-valued maps are defined in the similar way. It will be used without further specification.

3-1.3. **Exercise** Prove that for all real functions f, g on a non-empty set X, we have $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$.

3-1.4. **Exercise** Let f, g be real-functions on $\mathbb{R}$ given by $f(x) = |x| - 1$ and $g(x) = 2 - |x - 1|$ for all $x \in \mathbb{R}$. Sketch $f + g$, $|f|$, $f \vee g$, $f \wedge g$, f_+ and f_-.

3-1.5. **Exercise** Prove that the set $\mathbb{F}(X)$ of all functions from X into $\mathbb{K}$ forms a vector space. Most vector spaces in this book will be derived as vector subspaces of $\mathbb{F}(X)$ by checking closure under addition and scalar multiplication.

3-1.6. Let X, Y be metric spaces and let f_n, g be maps from X into Y. Then $\{f_n\}$ is said to converge to g *uniformly on a subset* M of X if for every $\varepsilon > 0$, there is an integer p such that for all $n \geq p$ and for all $x \in M$, we have $d(f_n(x), g(x)) \leq \varepsilon$. In symbols, write $f_n \to g$ uniformly on M. When $M = X$, we may drop M in our notation. Note that the functions f_n, g need not be bounded.

3-1.7. **Theorem** Uniform limits of continuous maps are continuous. More precisely, let X, Y be metric spaces and f_n, g be maps from X into Y. Suppose that all f_n are continuous. If $f_n \to g$ uniformly on X, then g is continuous on X.

Proof. Let $a \in X$ and $\varepsilon > 0$ be given. Since $f_n \to g$ uniformly, there is p such that for all $n \geq p$ and for all $x \in X$, we have $d[f_n(x), g(x)] \leq \varepsilon/3$. Now f_p is continuous at $a \in X$. There is $\delta > 0$ such that $d(x, a) \leq \delta$ implies $d[f_p(x), f_p(a)] \leq \varepsilon/3$. Now suppose $d(x, a) \leq \delta$. Then we obtain

$$d[g(x), g(a)] \leq d[g(x), f_p(x)] + d[f_p(x), f_p(a)] + d[f_p(a), g(a)] \leq \varepsilon.$$

Therefore g is continuous at $a \in X$. Since $a \in X$ is arbitrary, g is continuous on X. □

3-1.8. Let X be a set and Y a metric space. Let f_n, g be maps from X into Y. Then $\{f_n\}$ is said to converge to g *pointwise* if for every $x \in X$, we have $f_n(x) \to g(x)$ in Y. Clearly uniform convergence implies pointwise convergence.

3-1.9. **Exercise** Let $f_n(x) = nx$ if $0 \leq x \leq \frac{1}{n}$ and $f_n(x) = 0$ if $x > \frac{1}{n}$. Show that $\{f_n\}$ converges pointwise on $[0, 1]$. Prove that the sequence $\{f_n\}$ does not converge uniformly on $[0, 1]$.

3-2 Bounded Continuous Functions

3-2.1. Let E be a normed space. Then E is a metric space under the distance $d(x, y) = \|x - y\|$. If E is complete, it is called a *Banach space*. It has been shown that all $\mathbb{K}_1^n$, $\mathbb{K}_2^n$ and $\mathbb{K}_\infty^n$ are examples of Banach spaces but they are all finite dimensional. We shall begin to study function spaces which are usually infinite dimensional. Chapter 13 on Hilbert spaces should be read concurrently.

3-2.2. **Example** Let X be a non-empty set. The *sup-norm* of a function $f : X \to \mathbb{K}$ is defined by $\|f\|_\infty = \sup_{x \in X} |f(x)|$ which may be ∞. Obviously, it is finite iff the function f is bounded on X. Let $B(X)$ denote the set of all bounded functions on X. It is an exercise to show that $B(X)$ is a vector subspace of $\mathbb{F}(X)$ and the sup-norm is a norm on $B(X)$. Convergence in $B(X)$ is the same as uniform convergence on X. Note that $B(X)$ is a generalization of $\mathbb{K}_\infty^n$ when $X = \{1, 2, \cdots, n\}$.

3-2.3. **Exercise** Let $X = (0, \infty)$ and $f(x) = e^x \cos x$, $g(x) = e^{-x} \sin x$ for $x \in X$. Find the sup-norms of f, g and determine whether they belong to $B(X)$.

3-2.4. **Theorem** The vector space $B(X)$ of all bounded functions on a non-empty set X forms a Banach space under the sup-norm.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $B(X)$. Then for every $\varepsilon > 0$ there is an integer k such that $\|f_m - f_n\|_\infty \leq \varepsilon, \quad \forall\, m, n \geq k$. For each $x \in X$, since $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \leq \varepsilon$, the sequence $\{f_n(x)\}$ is Cauchy in the complete metric space $\mathbb{K}$ and hence it converges to some $f(x) \in \mathbb{K}$. Now $f : X \to \mathbb{K}$ is a well-defined function. Because

$$|f_m(x)| \leq |f_m(x) - f_k(x)| + |f_k(x)| \leq \varepsilon + \|f_k\|_\infty,$$

setting $m \to \infty$ we have $|f(x)| \leq \varepsilon + \|f_k\|_\infty$ for all $x \in X$. Therefore $f \in B(X)$. Letting $m \to \infty$ in $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \leq \varepsilon$, we have for all $n \geq k$ $|f(x) - f_n(x)| \leq \varepsilon$. Since $x \in X$ is arbitrary, we get $\|f - f_n\|_\infty \leq \varepsilon, \quad \forall\, n \geq k$. This proves $f_n \to f$ in $B(X)$. Consequently $B(X)$ is a Banach space. □

3-2.5. **Exercise** Let X be a metric space and E a normed space. Let $f, g : X \to E$ be continuous maps and $\lambda : X \to \mathbb{K}$ a continuous function. Prove that the sum $f + g$, the product λf and the absolute-value $|\lambda|$ are continuous on X. Furthermore, prove that if λ is non-zero on X then its *reciprocal* defined by $(\frac{1}{\lambda})(x) = \frac{1}{\lambda(x)}$ is also continuous on X.

3-2.6. **Example** Let X be a metric space. Then the set $BC(X)$ of all bounded continuous functions on X is a closed vector subspace of the Banach space $B(X)$. Hence $BC(X)$ is also a Banach space under the sup-norm. Note that $BC(X)$ is a generalization of $\mathbb{K}_\infty^n$ when $X = \{1, 2, \cdots, n\}$ is equipped with the discrete metric.

3-2.7. **Example** Let X be a compact metric space. Since every continuous functions on X is bounded, $BC(X)$ is identical with the vector space $C(X)$ of all continuous functions on X.

3-2.8. Let $C[a,b]$ be the vector space of all continuous functions on the compact set $[a,b]$ where $a < b$. Of course, $C_\infty[a,b]$ forms a Banach space. On the other hand, for each $f \in C[a,b]$ define

$$\|f\|_1 = \int_a^b |f(x)|dx \qquad \text{and} \qquad \|f\|_2 = \sqrt{\int_a^b |f(x)|^2 dx}\ .$$

It is routine to verify that both are norms on $C[a,b]$. Write $C_1[a,b]$ and $C_2[a,b]$ to indicate explicitly which norm is used on the same vector space $C[a,b]$. Whenever nothing is mentioned, $C_\infty[a,b]$ will be assumed.

3-2.9. **Example** The normed space $C_1[-1,1]$ is not complete.

Proof. We shall construct a Cauchy but divergent sequence in $C_1[-1,1]$. Define

$$f_n(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0, \\ nx, & \text{if } 0 \le x \le \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

All f_n are continuous by Glue-Theorem. For all $m, n \ge 3$, since

$$\|f_m - f_n\|_1 \le \int_0^{\frac{1}{m}+\frac{1}{n}} |f_m(x) - f_n(x)|dx \le \frac{2}{m} + \frac{2}{n},$$

the sequence $\{f_n\}$ is Cauchy in $C_1[-1,1]$. Suppose to the contrary that $\{f_n\}$ converges to some continuous function g in $C_1[-1,1]$. Then the estimation

$$\int_{-1}^0 |g(x)|dx = \int_{-1}^0 |f_n(x) - g(x)|dx \le \|f_n - g\|_1 \to 0$$

together with the continuity of g ensures that $g(x) = 0$ for all $x \le 0$. On the other hand, take any $0 < a < 1$. Observe that for all $n > \frac{1}{a}$,

$$\int_a^1 |1 - g(x)|dx = \int_a^1 |f_n(x) - g(x)|dx \le \|f_n - g\|_1 \to 0.$$

The continuity of g gives $g(x) = 1$ for all $a \le x \le 1$. Since $a > 0$ is arbitrary, we have $g(x) = 1$ for all $x > 0$. This contradiction to the continuity of g shows that $C_1[-1,1]$ cannot be complete. □

3-2.10. **Exercise** Show that $C[-1,1]$ is infinite dimensional. Note that $C_\infty[-1,1]$ is a Banach space while $C_1[-1,1]$ is not.

3-2.11. Let E be a normed space. An infinite series $\sum_{n=1}^\infty x_n$ in E is said to *converge* to a sum $s \in E$ if the sequence of partial sums $s_n = x_1 + x_2 + \cdots + x_n$ converges to s. It is said to converge ***absolutely*** if the series $\sum_{n=1}^\infty \|x_n\|$ converges in $\mathbb{R}$. The following results will be needed in next section.

3-2.12. **Theorem** Let $\sum_{n=1}^{\infty} x_n$ be an infinite series in a Banach space E. If the given series is absolutely convergent, then it converges in E. Furthermore, we have $\| \sum_{n=1}^{\infty} x_n \| \leq \sum_{n=1}^{\infty} \|x_n\|$.

Proof. Let $s_k = \sum_{n=1}^{k} x_n$ denote the partial sums. Suppose $\varepsilon > 0$ is given. Since $\sum_{n=1}^{\infty} \|x_n\|$ converges, there is an integer p such that $\sum_{n=p}^{\infty} \|x_n\| \leq \varepsilon$. Now for all $k \geq p$ and all positive integer i, we have

$$\|s_{k+i} - s_k\| = \|x_{k+1} + x_{k+2} + \cdots + x_{k+i}\| \leq \|x_{k+1}\| + \|x_{k+2}\| + \cdots + \|x_{k+i}\| \leq \varepsilon.$$

Hence $\{s_k : k \geq 1\}$ is a Cauchy sequence in the Banach space E. It converges to some limit, say $s \in E$. Consequently the series $\sum_{n=1}^{\infty} x_n$ also converges to s. Since $\|s_k\| \leq \sum_{n=1}^{k} \|x_n\| \leq \sum_{n=1}^{\infty} \|x_n\|$, letting $k \to \infty$, we obtain $\|s\| \leq \sum_{n=1}^{\infty} \|x_n\|$. □

3-2.13. **Exercise** Prove that if a normed space is *not* complete then there exists an absolutely convergent series which is divergent.

3-2.14. **Exercise** Prove that if the infinite series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent in a normed space, then for all scalars α, β, the series $\sum_{n=1}^{\infty}(\alpha x_n + \beta y_n)$ converges to $\alpha \sum_{n=1}^{\infty} x_n + \beta \sum_{n=1}^{\infty} y_n$.

3-2.15. **Exercise** Prove that if a series $\sum_{n=1}^{\infty} x_n$ is convergent in E, then $x_n \to 0$ as $n \to \infty$. Also prove that the set $\{\|x_n\| : n \geq 1\}$ is bounded. This simple result will be needed to establish a formula for spectral radius.

3-2.16. **Exercise** Extend the results of §§3-2.4,6,7 to functions from X into a Banach space.

3-3 Sequence Spaces

3-3.1. Let ℓ denote the set of all sequences $x = (x_1, x_2, x_3, \cdots)$ of scalars $x_n \in \mathbb{K}$. It can be considered as functions on the set of integers $1, 2, 3, \cdots$ and hence ℓ forms a vector space. For each $x \in \ell$, define

$$\|x\|_1 = |x_1| + |x_2| + |x_3| + \cdots; \qquad \|x\|_\infty = \sup\{|x_1|, |x_2|, |x_3|, \cdots\};$$

and $$\|x\|_p = \{|x_1|^p + |x_2|^p + |x_3|^p + \cdots\}^{1/p}, \text{ for } 1 < p < \infty.$$

Note that these numbers may be infinity. For $1 \leq p \leq \infty$, let ℓ_p denote the set of $x \in \ell$ satisfying $\|x\|_p < \infty$. As a special case of last section, we have the following

3-3.2. **Theorem** ℓ_∞ is a Banach space.

3-3.3. For $1 < p < \infty$, let q be defined uniquely by the equation $\frac{1}{p} + \frac{1}{q} = 1$. In order to unify the notation, define $\frac{1}{\infty} = 0$. If $p = 1$, let $q = \infty$. If $p = \infty$, let $q = 1$. Consequently $\frac{1}{p} + \frac{1}{q} = 1$ holds for all $1 \le p \le \infty$ and p, q are called *conjugate indices*. To show that ℓ_p forms a normed space, we need the following inequalities.

3-3.4. **Lemma** For all $\alpha, \beta \ge 0$ and $1 < p < \infty$, we have $\alpha\beta \le \alpha^p/p + \beta^q/q$.

Proof. If one of α, β is zero then the inequality is obviously true. Assume both α, β are non-zero. Let $f(t) = \alpha^p/p + t^q/q - \alpha t, \quad \forall\, t > 0$. Then $f'(t) = 0$ gives a root $r = \alpha^{1/(q-1)}$. Since $f''(t) > 0$ we have $f(t) \ge f(r) = 0$, for all $t > 0$. Now $f(\beta) \ge 0$ gives the required inequality. □

3-3.5. **Holder's Inequality** For $1 < p < \infty$, if $x \in \ell_p$ and $y \in \ell_q$ then we have $xy \in \ell_1$ where $xy = (x_1y_1, x_2y_2, x_3y_3, \cdots)$. Furthermore, we have the inequality:

$$\|xy\|_1 \le \|x\|_p \|y\|_q.$$

Proof. If $\|x\|_p = 0$ then $x = 0$ or $xy = 0$ and hence $xy \in \ell_1$. In this case, both sides of the inequality are zero. Similarly the result holds for $\|y\|_q = 0$. Now assume $\|x\|_p \ne 0$ and $\|y\|_q \ne 0$. Observe that

$$\sum_{n=1}^{\infty} \frac{|x_n y_n|}{\|x\|_p \|y\|_q} \le \sum_{n=1}^{\infty} \left(\frac{1}{p} \frac{|x_n|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_n|^q}{\|y\|_q^q} \right)$$

$$= \frac{1}{p\|x\|_p^p} \sum_{n=1}^{\infty} |x_n|^p + \frac{1}{q\|y\|_q^q} \sum_{n=1}^{\infty} |y_n|^q = \frac{1}{p} + \frac{1}{q} = 1,$$

i.e. $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \|y\|_q$. Hence the series $\sum_{n=1}^{\infty} x_n y_n$ is absolutely convergent in $\mathbb{K}$ and $\|xy\|_1 \le \sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \|y\|_q$. This completes the proof. □

3-3.6. **Minkowski's Inequality** For $1 < p < \infty$, if x, y are in ℓ_p then so is $x + y$. Furthermore, we have $\|x + y\|_p \le \|x\|_p + \|y\|_p$.

Proof. Since

$$\sum_{j=1}^{\infty} |x_j + y_j|^p \le \sum_{j=1}^{\infty} \left(|x_j| + |y_j| \right)^p$$

$$\le \sum_{j=1}^{\infty} \left(2|x_j| \right)^p + \sum_{j=1}^{\infty} \left(2|y_j| \right)^p \le 2^p \left(\|x\|_p^p + \|y\|_p^p \right) < \infty,$$

we have $x + y \in \ell_p$. Suppose $\|x + y\|_p \ne 0$, otherwise the required inequality holds already. It follows from Holder's Inequality that

$$\|x + y\|_p^p = \sum_{j=1}^{\infty} |x_j + y_j|^p = \sum_{j=1}^{\infty} |x_j + y_j|\ |x_j + y_j|^{p-1}$$

$$\leq \sum_{j=1}^{\infty} |x_j|\ |x_j + y_j|^{p-1} + \sum_{j=1}^{\infty} |y_j|\ |x_j + y_j|^{p-1}$$
$$\leq \left[\left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} \right] \left(\sum_{j=1}^{\infty} |x_j + y_j|^{(p-1)q}\right)^{1/q}$$
$$= (\|x\|_p + \|y\|_p)\|x + y\|_p^{p-1}.$$

Dividing by $\|x + y\|_p^{p-1}$, the result follows. □

3-3.7. **Theorem** For $1 < p < \infty$, ℓ_p forms a Banach space.

Proof. It is an exercise to verify that ℓ_p is a normed space. To prove the completeness, let $\{x_n\}$ be a Cauchy sequence in ℓ_p. For every $\varepsilon > 0$ there is an integer k such that for all $m, n \geq k$ we have

$$\|x^m - x^n\| \leq \varepsilon. \qquad \#1$$

Write $x^n = (x_1^n, x_2^n, x_3^n, \cdots)$. Since $|x_j^n - x_j^m| \leq \|x^m - x^n\| \leq \varepsilon$, the sequence $\{x_j^n : n \geq 1\}$ is Cauchy in the complete metric space $\mathbb{K}$. Suppose $x_j^n \to x_j$ as $n \to \infty$ and define $x = (x_1, x_2, x_3, \cdots)$. It follows from #1 that for every integer $r \geq 1$, we have $\sum_{j=1}^{r} |x_j^m - x_j^n|^p \leq \varepsilon^p$. Letting $m \to \infty$, we obtain

$$\sum_{j=1}^{r} |x_j - x_j^n|^p \leq \varepsilon^p. \qquad \#2$$

Observe that

$$\left(\sum_{j=1}^{r} |x_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{r} |x_j - x_j^k|^p\right)^{1/p} + \left(\sum_{j=1}^{r} |x_j^k|^p\right)^{1/p} \leq \varepsilon + \|x^k\|_p < \infty.$$

Since r is arbitrary, we have $x \in \ell_p$. Letting $r \to \infty$ in #2, we get $\|x - x^n\| \leq \varepsilon$ for all $n \geq k$. Therefore $x^n \to x$ in ℓ_p. Consequently, ℓ_p is a Banach space. □

3-3.8. **Exercise** Prove that for every $x \in \ell_1$ and $y \in \ell_\infty$, we have $xy \in \ell_1$ and $\|xy\|_1 \leq \|x\|_1 \|y\|_\infty$.

3-3.9. **Exercise** Prove that ℓ_1 is a Banach space.

3-3.10. A sequence in ℓ is said to be ***finite*** if it has only a finite number of non-zero terms. The set of all finite sequences in ℓ is denoted by $\mathcal{F}$. A sequence $x = (x_1, x_2, x_3, \cdots)$ is said to be ***null*** if $x_n \to 0$ in $\mathbb{K}$ as $n \to \infty$. The set of all null sequences is denoted by c_0.

3-3.11. **Exercise** Show that $\mathcal{F} \subset \ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$. Also prove by counter examples that $\mathcal{F} \neq \ell_1 \neq \ell_2 \neq c_0 \neq \ell_\infty$. Note that all these spaces are infinite dimensional.

3-3.12. **Exercise** Prove that c_0 is a closed vector subspace of ℓ_∞. We shall use the sup-norm on c_0 unless other norm is specified explicitly. Show that c_0 is also a Banach space.

3-3.13. **Exercise** Prove that the set $\mathcal{F}$ of all finite sequence is not a closed subset of ℓ_p for each $1 \leq p \leq \infty$. We shall write $\mathcal{F}_p$ to indicate the norm used on $\mathcal{F}$. Prove that $\mathcal{F}_p$ is not a Banach space for any $1 \leq p \leq \infty$. If nothing is mentioned to $\mathcal{F}$, then $\mathcal{F}_\infty$ will be assumed.

3-3.14. **Exercise** Let $e_n = (\delta_{1n}, \delta_{2n}, \delta_{3n}, \cdots)$ where $\delta_{jn} = 1$ if $j = n$ and $\delta_{jn} = 0$ otherwise. Choose vectors $x_n = e_n/2^n$ in $\mathcal{F}_1$. Show that the series $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$, but the series $\sum_{n=1}^{\infty} x_n$ diverges in $\mathcal{F}_1$. Consequently, $\mathcal{F}_1$ cannot be a Banach space.

3-3.15. **Exercise** For $1 < p < \infty$, prove that $\mathbb{K}^n$ is a Banach space under the norm given by $\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$ for every $x \in \mathbb{K}^n$. To indicate this norm explicitly, write $\mathbb{K}_p^n$.

3-3.16. **Exercise** Let $1 < p < \infty$. Consider the vector space $C[a, b]$ of all continuous functions on $[a, b]$ where $a < b$ in $\mathbb{R}$. Prove that $C[a, b]$ is a normed space under $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$ for every $f \in C[a, b]$. Write $C_p[a, b]$ to indicate this norm explicitly. Prove or disprove that $C_p[a, b]$ is a Banach space.

3-4 Continuous Linear Maps

3-4.1. Continuous linear maps is useful because of its simplicity. Actually differential calculus is to approximate a nonlinear map locally by a linear map plus a constant. Standard examples of continuous linear maps will be given in next section.

3-4.2. Let E, F be normed spaces. The *norm* of a linear map $f : E \to F$ is defined by $\|f\| = \sup\{\|f(x)\| : x \in E, \|x\| \leq 1\}$. Note that f need not be continuous and its norm may be infinity. Since $\|f(0)\| = \|0\| = 0$, we have $\|f\| \geq 0$.

3-4.3. **Theorem** For every linear map $f : E \to F$, the following statements are equivalent.
(a) f is continuous at the origin of E.
(b) f is continuous on E.
(c) The norm of f is finite, i.e. $\|f\| < \infty$.
(d) There is $0 \leq \lambda < \infty$ such that $\|f(x)\| \leq \lambda \|x\|$ for all $x \in E$.
Furthermore, if f is continuous then we have $\|f(x)\| \leq \|f\| \, \|x\|, \forall\, x \in E$ and

$$\|f\| = \sup_{\|x\|\leq 1} \|f(x)\| = \sup_{\|x\|=1} \|f(x)\| = \inf\{\lambda \geq 0 : \|f(x) \leq \lambda\|x\|, \forall\, x \in E\}.$$

Proof. Let $\alpha = \sup_{\|x\|=1} \|f(x)\|$ and $\beta = \inf\{\lambda \geq 0 : \|f(x)\| \leq \lambda\|x\|, \forall\, x \in E\}$.
$(a \Rightarrow b)$ Let $x_n \to y$ be a convergent sequence in E. Then $x_n - y \to 0$ in E. Since f is continuous at $0 \in E$, we have $f(x_n - y) \to f(0)$ in F. The linearity of f gives $f(x_n) - f(y) \to 0$ in F, i.e. $f(x_n) \to f(y)$ in F. Therefore f is continuous on E.
$(b \Rightarrow c)$ Suppose to the contrary that $\|f\| = \infty$. Then for each n, there is $x_n \in E$ such that $\|x_n\| \leq 1$ and $\|f(x_n)\| \geq n$. Since $\left\|\frac{x_n}{n}\right\| = \frac{\|x_n\|}{n} \leq \frac{1}{n} \to 0$, but at the same time $\left\|f\left(\frac{x_n}{n}\right)\right\| = \left\|\frac{f(x_n)}{n}\right\| = \frac{\|f(x_n)\|}{n} \geq 1$, f is discontinuous at the origin.
$(c \Rightarrow d)$ Since the unit sphere is a subset of the closed unit ball in E, we have $\alpha \leq \|f\| < \infty$. Take any $x \in E$. We claim $\|f(x) \leq \alpha\|x\|$. In fact, if $x = 0$, then $\|f(x)\| = 0 \leq \alpha\|x\|$. Suppose $x \neq 0$. Then $\|x\| \neq 0$. Since $\left\|\frac{x}{\|x\|}\right\| = 1$, we obtain $\left\|f\left(\frac{x}{\|x\|}\right)\right\| \leq \alpha$, i.e. $\|f(x)\| \leq \alpha\|x\|$. By definition of β, we have $\beta \leq \alpha$.
$(d \Rightarrow a)$ Let $x_n \to 0$ in E. Then $\|f(x_n)\| \leq \lambda\|x_n\| \to 0$. Therefore f is continuous at $0 \in E$. Furthermore, to prove $\|f\| \leq \beta$, take any $\lambda \geq 0$ such that $\|f(x)\| \leq \lambda\|x\|$ for all $x \in E$. In particular, for $\|y\| \leq 1$ in E, we have $\|f(y)\| \leq \lambda\|y\| \leq \lambda$. Taking supremum over $\|y\| \leq 1$, we get $\|f\| \leq \lambda$. Taking infimum over λ, we obtain $\|f\| \leq \beta$. Therefore $\|f\| = \alpha = \beta$. Since $\|f(x)\| \leq \alpha\|x\|$ for all $x \in E$, we have $\|f(x)\| \leq \|f\|\ \|x\|$ for all $x \in E$. □

3-4.4. **Theorem** Let E, F, G be normed spaces. If $f : E \to F$ and $g : F \to G$ are continuous linear maps, then we have $\|gf\| \leq \|g\|\|f\|$. Furthermore if $f_n \to f \in L(E, F)$ and $g_n \to f \in L(F, G)$, then $g_n f_n \to gf \in L(E, G)$.

Proof. It follows immediately by taking supremum over $\|x\| \leq 1$ in E in the calculation: $\|(gf)(x)\| = \|g[f(x)]\| \leq \|g\|\ \|f(x)\| \leq \|g\|\ \|f\|\ \|x\| \leq \|g\|\ \|f\|$. The proof of last statement is left as an exercise. □

3-4.5. **Exercise** Let E, F be normed spaces and $f : E \to F$ a linear map. Prove that the following statements are equivalent.
(a) f is continuous on E.
(b) For every null sequence $x_n \to 0$ in E, the set $\{f(x_n)\}$ is bounded in F.
(c) f is bounded on some open ball of E.

This is why continuous linear maps between normed spaces are also called *bounded* linear maps.

3-4.6. Let E, F be normed spaces. The set of all continuous linear maps from E into F is denoted by $L(E, F)$ and the set of algebraic linear maps which need not be continuous by $L^*(E, F)$. A linear map from E into the scalar field $\mathbb{K}$ is called a *linear form*. The set of all linear forms on E is called the *algebraic dual* of E and it is denoted by E^*. The set of all *continuous* linear forms on E is called the *topological dual* or simply the dual space of E and it is denoted by E'.

3-4.7. **Theorem** Let f be a linear form on a normed space E. Then f is continuous on E iff the kernel $f^{-1}(0)$ of f is closed in E.

Proof. If f is continuous, then the kernel as the inverse image of the close set $\{0\}$ in $\mathbb{K}$ is closed. Conversely, it is trivial for $f = 0$. So, assume that there is $a \in E$ such that $f(a) \neq 0$. Then a belongs to the open set $E \setminus f^{-1}(0)$. There is a ball $\mathbb{B}(a, r) \subset E \setminus f^{-1}(0)$. Take any $\|x\| \leq 1$ in E. Suppose to the contrary that $\|f\| > \dfrac{|f(a)|}{r}$. Let $y = -\dfrac{f(a)}{f(x)}x$. Then we have $\|y\| = \dfrac{|f(a)|}{|f(x)|}\|x\| < r$. Hence $a + y \in \mathbb{B}(a, r)$ and so $a + r \notin f^{-1}(0)$, i.e. $f(a + y) \neq 0$. On the other hand, we also have $f(a + y) = f(a) + f(y) = f(a) - \dfrac{f(a)}{f(x)}f(x) = 0$. This contradiction shows $\|f(x)\| \leq \dfrac{|f(a)|}{r}$. Taking supremum over $\|x\| \leq 1$ in E, we obtain $\|f\| \leq \dfrac{|f(a)|}{r} < \infty$. Consequently, f is continuous. □

3-4.8. **Exercise** Prove that if a linear form f on a normed space E is discontinuous, then its kernel is dense in E.

3-4.9. **Lemma** The set $L(E, F)$ of all continuous linear maps forms a normed space.

Proof. Let f, g belong to $L(E, F)$. Take any $\|x\| \leq 1$ in E. Observe that

$$\begin{aligned} \|(f+g)x\| &= \|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\| \\ &\leq \|f\|\,\|x\| + \|g\|\,\|x\| \leq \|f\| + \|g\|. \end{aligned}$$

Taking supremum over $\|x\| \leq 1$, we obtain $\|f + g\| \leq \|f\| + \|g\| < \infty$. Hence $f + g \in L(E, F)$ and the triangular inequality holds. Next, let $\lambda \in \mathbb{K}$ be a given scalar. For any $\|x\| \leq 1$ in E, observe

$$\|(\lambda f)x\| = \|\lambda[f(x)]\| = |\lambda|\;\|f(x)\| \leq |\lambda|\|f\|\;\|x\| \leq |\lambda|\;\|f\|.$$

Hence $\|\lambda f\| \leq |\lambda|\ \|f\| < \infty$. Therefore λf also belongs to $L(E, F)$. Consequently, $L(E, F)$ is a vector space. If $\lambda = 0$, we deduce $\|\lambda f\| = 0 = |\lambda|\ \|f\|$. In particular, $\|O\| = 0$ in $L(E, F)$. Suppose $\lambda \neq 0$. Replacing λ by λ^{-1} and f by λf, we get $\|\lambda^{-1}(\lambda f)\| \leq |\lambda^{-1}|\|\lambda f\|$, i.e. $|\lambda|\ \|f\| \leq \|\lambda f\|$. Combining with the last inequality, we obtain $|\lambda|\ \|f\| = \|\lambda f\|$. Clearly $\|f\| \geq 0$ for all $f \in L(E, F)$. Now assume $\|f\| = 0$. Then for all $x \in E$, we have $\|f(x)\| \leq \|f\|\ \|x\| = 0$, i.e. $f(x) = 0$. Hence $f = 0$. We conclude that $L(E, F)$ is a normed space. □

3-4.10. **Theorem** If F is a Banach space, then so is $L(E, F)$.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L(E, F)$. We want to show that $\{f_n\}$ converges to some $g \in L(E, F)$. Let $\varepsilon > 0$ be given. There is an integer p such that for all $m\| \geq p$, we have $\|f_m - f_n\| \leq \varepsilon$. Take any $x \in E$. Since

$$\|f_m(x) - f_n(x)\| \leq \|f_m - f_n\|\ \|x\| \leq \varepsilon\|x\|, \qquad \#1$$

the sequence $\{f_n(x) : n \geq 1\}$ is a Cauchy sequence in the complete metric space F. Define $g(x) = \lim_{n\to\infty} f_n(x)$. Since $x \in E$ is arbitrary, we have defined a map g from E into F. Letting $n \to \infty$ in the equation:

$$f_n(\alpha x + \beta y) = \alpha f_n(x) + \beta f_n(y), \forall\ x, y \in E,\ \forall\ \alpha, \beta \in \mathbb{K},$$

we have $g(\alpha x + \beta y) = \alpha g_n x + \beta g_n y$, i.e. g is a linear map. For $m \to \infty$ in #1,

$$\|g(x) - f_n(x)\| \leq \varepsilon\|x\| \leq \varepsilon, \forall\ n \geq p, \forall\ \|x\| \leq 1 \text{ in } E. \qquad \#2$$

In particular, when $n = p$ we get $\|g(x)\| \leq \|f_p(x)\| + \|g(x) - f_p(x)\| \leq \|f_p\| + \varepsilon$, i.e. $\|g\| \leq \|f_p\| + \varepsilon < \infty$. Hence g is continuous. Taking supremum over $\|x\| \leq 1$ in #2, we have $\|g - f_n\| \leq \varepsilon, \forall\ n \geq p$. Therefore $f_n \to g$ in $L(E, F)$. Consequently, every Cauchy sequence in $L(E, F)$ is convergent. This completes the proof. □

3-4.11. **Exercise** Prove that the topological dual of a normed space is a Banach space.

3-4.12. **Exercise** Let E be a normed space and F a Banach space. Let M be a vector subspace of E. Prove that every continuous linear map $f : M \to F$ is uniformly continuous. Hence or otherwise prove that f can be extended uniquely to a continuous linear map on the closure of M.

3-4.13. **Exercise** Let X be a compact metric space and E a Banach space. Clearly the set $C(X, E)$ is a vector space under the pointwise operations. Show that $C(X, E)$ is a Banach space under the sup-norm $\|f\| = \sup\{\|f(x)\| : x \in X\}$. To emphasize the sup-norm, write $C_\infty(X, E)$ instead of $C(X, E)$.

3-5 Examples of Continuous Linear Maps

3-5.1. **Example** Let $A : \mathbb{K}_p^n \to \mathbb{K}_p^m$ be a linear map where $1 \le p \le \infty$. Since $\mathbb{K}^n, \mathbb{K}^m$ are column vectors, A is identified as an $m \times n$ matrix $A = [a_{ij}]$. The norm of A is defined by the general formula $\|A\|_p = \sup_{\|x\|_p \le 1} \|Ax\|_p$. Then we have the following explicit expressions:

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}|, \qquad \text{and} \qquad \|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}|.$$

Proof. Let $\alpha = \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}|$. Take any $\|x\|_\infty \le 1$ in $\mathbb{K}^n$ and write $y = Ax$. Then we get

$$\|Ax\|_\infty = \max_{1 \le i \le m} |y_i| = \max_{1 \le i \le m} \left|\sum_{j=1}^n a_{ij} x_j\right|$$
$$\le \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}| \max_{1 \le k \le n} |x_k| \le \max_{1 \le i \le m} \sum_{j=1}^n |a_{ij}| \, \|x\|_\infty \le \alpha.$$

Taking supremum over $\|x\|_\infty \le 1$, we have $\|A\|_\infty \le \alpha$. Pick an index k satisfying $\sum_{j=1}^n |a_{kj}| = \alpha$. Choose $x_j \in \mathbb{K}$ such that $a_{kj} x_j = |a_{kj}|$ and $|x_j| = 1$. Then $x = (x_1, x_2, \cdots, x_n) \in \mathbb{K}^n$ satisfies $\|x\|_\infty = 1$. Observe that

$$\|Ax\|_\infty = \max_{1 \le i \le m} \left|\sum_{j=1}^n a_{ij} x_j\right| \ge \left|\sum_{j=1}^n a_{kj} x_j\right| = \sum_{j=1}^n |a_{kj}| = \alpha.$$

This proves the first formula. For the second formula, let $\beta = \max_{1 \le j \le n} \sum_{i=1}^m |a_{ij}|$. Take any $\|x\|_1 \le 1$ in $\mathbb{K}^n$ and write $y = Ax$. Then

$$\|Ax\|_1 = \sum_{i=1}^m |y_i| = \sum_{i=1}^m \left|\sum_{j=1}^n a_{ij} x_j\right| \le \sum_{j=1}^n \left(\sum_{i=1}^m |a_{ij}|\right) |x_j| \le \sum_{j=1}^n \beta |x_j| \le \beta.$$

Taking supremum over $\|x\|_1 \le 1$, we have $\|A\|_1 \le \beta$. Pick an index k satisfying $\sum_{i=1}^m |a_{ik}| = \beta$. Define $x_j = 1$ if $j = k$ and $x_j = 0$ for all $j \ne k$. Then $\|x\|_1 = 1$ and the following equality completes the proof:

$$\|Ax\|_1 = \sum_{i=1}^m \left|\sum_{j=1}^n a_{ij} x_j\right| = \sum_{i=1}^m |a_{ik}| = \beta. \qquad \square$$

3-5.2. Let E, F be normed spaces. A linear map $f : E \to F$ is said to be *isometric* if $\|f(x)\| = \|x\|$ for all $x \in E$. Clearly every isometry is a continuous injection. Two normed spaces are considered to be identical if they are isometrically isomorphic. For convenience to work in sequence spaces, let $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j}, \cdots)$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Therefore e_j is the sequence with zero terms except the j-th coordinate which is one. This notation will be used without further specification.

3-5.3. **Exercise** Prove that it is impossible to find any isomorphism from $\mathbb{R}_1^2$ onto $\mathbb{R}_2^2$ which preserves their norms.

3-5.4. **Example** The dual space of c_0 is ℓ_1.

Proof. Take any $a \in \ell_1$. Then for each $x \in c_0$, since

$$\sum_{n=1}^{\infty} |a_n x_n| \leq \sum_{n=1}^{\infty} |a_n| \|x\|_\infty \leq \|a\|_1 \|x\|_\infty,$$

the infinite series $< a, x > = \sum_{n=1}^{\infty} a_n x_n$ converges absolutely. Define $f(x) = < a, x >$ for all $x \in c_0$. Clearly f is a linear form on c_0. Also $|f(x)| = | < a, x > | \leq \|a\|_1 \|x\|_\infty$. Hence $\|f\| \leq \|a\|_1$. Therefore f is a continuous linear form on c_0. Define $T(a) = f$. Obviously, $T : \ell_1 \to c_0'$ is a linear map. To show that T is surjective, let f be a continuous linear form on c_0. Define $a_j = f(e_j)$ and $a = (a_1, a_2, a_3, \cdots)$. Choose $y_j = 0$ if $a_j = 0$ and $y_j = |a_j|/a_j$ otherwise. Then $y^n = \sum_{j=1}^{n} y_j e_j \in c_0$. Since $|y_j| \leq 1$, we have $\|y^n\| \leq 1$. For arbitrary n,

$$\sum_{j=1}^{n} |a_j| = \sum_{j=1}^{n} a_j y_j = f(y^n) = |f(y^n)| \leq \|f\| \ \|y^n\|_\infty \leq \|f\|$$

gives $a \in \ell_1$ and $\|a\|_1 \leq \|f\|$. Suppose $x \in c_0$ is given. Let $x^n = \sum_{j=1}^{n} x_j e_j$. Then $x^n \to x$ in c_0 and hence

$$f(x) = \lim_{n\to\infty} f(x^n) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j f(e_j) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j a_j = < a, x >.$$

Therefore $T : \ell_1 \to c_0'$ is surjective and isometric. This completes the proof. □

3-5.5. **Example** The dual space of ℓ_1 is ℓ_∞.

Proof. Take any $a \in \ell_\infty$. Then for each $x \in \ell_1$, since

$$\sum_{n=1}^{\infty} |a_n x_n| \leq \sum_{n=1}^{\infty} \|a_n\|_\infty |x_n| \leq \|a\|_\infty \|x\|_1,$$

the infinite series $< a, x > = \sum_{n=1}^{\infty} a_n x_n$ converges absolutely. Define $f(x) = < a, x >$ for all $x \in \ell_1$. Clearly f is a linear form on ℓ_1. Also $|f(x)| = | < a, x > | \leq \|a\|_1 \|x\|_\infty$. Hence $\|f\| \leq \|a\|_1$. Therefore f is a continuous linear form on ℓ_1. Define $T(a) = f$. Obviously, $T : \ell_\infty \to \ell_1'$ is a linear map. To show that T is surjective, let f be a continuous linear form on ℓ_1. Define $a_j = f(e_j)$ and $a = (a_1, a_2, a_3, \cdots)$. Then $|a_j| = |f(e_j)| \leq \|f\| \ \|e_j\|_1 \leq \|f\|$. Hence $a \in \ell_\infty$ and $\|a\|_\infty \leq \|f\|$. Suppose $x \in \ell_1$ is given. Let $x^n = \sum_{j=1}^{n} x_j e_j$. Then $x^n \to x$ in ℓ_1 and hence

$$f(x) = \lim_{n\to\infty} f(x^n) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j f(e_j) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j a_j = < a, x >.$$

Therefore $T : \ell_\infty \to \ell_1'$ is surjective and isometric. This completes the proof.□

3-5.6. **Exercise** Prove that the dual space of ℓ_p is ℓ_q where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3-5.7. **Exercise** For every sequence $x = (x_1, x_2, x_3, \cdots)$, let $\varphi(x) = x_j$ denote the projection onto the j-th coordinate of x. Prove that φ is continuous on ℓ_p for all $1 \le p \le \infty$ and also c_0 respectively.

3-5.8. **Exercise** For every sequence $x = (x_1, x_2, \cdots) \in \ell$, the *left shift* of x is defined by $LS(s) = (x_2, x_3, \cdots)$ and the *right shift* by $RS(s) = (0, x_1, x_2, \cdots)$. Show that both shifts are linear maps on the vector space ℓ of all sequences. The restrictions of shifts to all vector subspaces of ℓ will be denoted by the same notations LS and RS.

3-5.9. **Exercise** For ℓ_p and c_0, find the norms of both shifts. Are they injective? Are they surjective?

3-5.10. **Exercise** For every function $f \in C[-1, 1]$, let $\varphi(f) = f(0)$ denote the evaluation map. Prove or disprove that φ is continuous on $C_1[-1, 1], C_\infty[-1, 1]$ respectively.

3-5.11. **Exercise** For every $f \in C_\infty[-1, 1]$, let $\varphi(f) = \int_{-1}^{0} f(x)dx - \int_0^1 f(x)dx$. Show that φ is a continuous linear form on $C_\infty[-1, 1]$ and find its norm.

3-5.12. **Exercise** Let $f, g, h : C_\infty[0, 1] \to C_\infty[0, 1]$ be linear maps defined by $(fx)(t) = t \int_0^1 x(s)ds$; $(gx)(t) = tx(t)$ and $(hx)(t) = \int_0^t x(s)ds$ respectively. Find the norms $\|f\|$, $\|g\|$, $\|h\|$, $\|fg\|$ and $\|gf\|$. Is it true that $fg = gf$?

3-5.13. **Exercise** Let E denote the Banach space $BC_\infty[0, \infty)$. For each $x \in E$, let $f(x)(t) = \frac{1}{t} \int_0^t x(s)ds$. Show that $f(x) \in E$ and that $f : E \to E$ is a continuous linear map. Find the norm of f.

3-6 Finite Dimensional Normed Spaces

3-6.1. In this section, we shall prove that all normed spaces of the same finite dimension share the same topological properties under any algebraic isomorphisms. Furthermore, every infinite dimensional normed space cannot be locally compact. This may be a reason why abstract harmonic analysis on infinite dimensional Banach spaces and their spheres still require a lot of attention before its theory becomes more mature.

3-6.2. Let E, F be normed spaces. Then an algebraic isomorphism $f : E \to F$ is called a *topological isomorphism* if both f, f^{-1} are continuous.

3-6.3. **Exercise** Show that the identity map from $C_\infty[0,1]$ onto $C_1[0,1]$ is a continuous isomorphism but not a topological isomorphism.

3-6.4. **Lemma** Let E, F be normed spaces and $f : E \to F$ a topological isomorphism. Then E is complete iff F is.

Proof. It is a simple manipulation of Cauchy sequences. □

3-6.5. **Theorem** Every algebraic isomorphism from a finite dimensional normed space onto a normed space is a topological isomorphism.

Proof. We shall prove a particular case first. Let $b_1, b_2, \cdots, b_m$ be a basis of a normed space F and let $h : \mathbb{K}^m \to F$ be given by

$$h(\alpha_1, \alpha_2, \cdots, \alpha_m) = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_m b_m.$$

Then $h : \mathbb{K}^m \to F$ is an algebraic isomorphism. Suppose

$$(\alpha_1^n, \alpha_2^n, \cdots, \alpha_m^n) \to (0, 0, \cdots, 0),$$

in $\mathbb{K}^m$ as $n \to \infty$. Then $\alpha_i^n \to 0$ for each i-th coordinate. Hence

$$h(\alpha_1^n, \alpha_2^n, \cdots, \alpha_m^n) = \alpha_1^n b_1 + \alpha_2^n b_2 + \cdots + \alpha_m^n b_m \to 0 \text{ in } F.$$

Therefore h is continuous on $\mathbb{K}^m$. Next for $m = 1$, since $h^{-1} : F \to \mathbb{K}$ is a linear form of which the kernel $\{0\}$ is a closed subset of F, it is continuous. Inductively, let $p_j : \mathbb{K}^m \to \mathbb{K}$ denote the projection of $\mathbb{K}^m$ onto the j-th coordinate. Then $p_j h^{-1}$ is a linear form on F. Its kernel is the vector subspace spanned by $\{b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_m\}$ of which the dimension is $m - 1$. By induction, it is topologically isomorphic to $\mathbb{K}^{m-1}$. Thus it is complete and thus closed in F. Therefore the linear form $p_j h^{-1}$ is continuous. Now suppose $x_n \to 0$ in F. Then $\alpha_j^n = p_j h^{-1}(x_n) \to 0$ as $n \to \infty$. Hence the inverse image $h^{-1}(x_n) = (\alpha_1^n, \alpha_2^n, \cdots, \alpha_m^n) \to (0, 0, \cdots, 0)$ in $\mathbb{K}^m$. Therefore h^{-1} is also continuous. We have proved that $h : \mathbb{K}^m \to F$ is a topological isomorphism. In general, let $f : E \to F$ be an algebraic isomorphism. Then $g = f^{-1}h : \mathbb{K}^m \to E$ is an algebraic isomorphism. Hence it is a topological isomorphism. Consequently, $f = hg^{-1}$ is also a topological isomorphism. □

3-6.6. **Exercise** Prove that every finite dimensional normed space is a Banach space.

3-6.7. **Exercise** Prove that every finite dimensional vector subspace of a normed space is closed.

3-6.8. **Exercise** Prove that every linear form on a finite dimensional normed space is continuous.

3-6.9. **Exercise** Prove that a subset in a finite dimensional normed space E is compact iff it is closed and bounded. Also prove that a subset of E is precompact iff it is bounded.

3-6.10. **Lemma** Let M be a closed vector subspace of a normed space E. If $M \neq E$ then for every $\varepsilon > 0$, there is $x \in E$ such that $\|x\| = 1$ and $d(x, M) \geq 1 - \varepsilon$ where $d(x, M)$ denotes the distance from x to M.

Proof. Since $M \neq E$, take any $a \in E \backslash M$. Let $\lambda = d(a, M)$ be the distance from a to M. Because M is closed, we have $\lambda > 0$. Since $t/(\lambda + t)$ is continuous at $t = 0$, there is $\delta > 0$ such that $\delta/(\lambda + \delta) < \varepsilon$. Pick $b \in M$ so that $\|a - b\| \leq \lambda + \delta$. Since $a \notin M$, $a - b \neq 0$. Let $x = (a - b)/\|a - b\|$. Then $\|x\| = 1$. Take any $m \in M$. Observe

$$\|x - m\| = \left\| \frac{a - b}{\|a - b\|} - m \right\| = \frac{\|a - (b + \|a - b\|m)\|}{\|a - b\|} \geq \frac{\lambda}{\lambda + \delta} = 1 - \frac{\delta}{\lambda + \delta} \geq 1 - \varepsilon.$$

Taking infimum over $m \in M$, we have $d(x, M) \geq 1 - \varepsilon$. □

3-6.11. **Characterization Theorem of Finite Dimension** Let E be a normed space. Then the following statements are equivalent.

(a) E is finite dimensional.

(b) The closed unit ball $\mathbb{B} = \{x \in E : \|x\| \leq 1\}$ is compact.

(c) Every bounded sequence has a convergent subsequence.

(d) The unit sphere $\mathbb{S} = \{x \in E : \|x\| = 1\}$ is compact.

Proof. ($a \Rightarrow b$) Let $f : E \to \mathbb{K}^n$ be an algebraic isomorphism from E onto some $\mathbb{K}^m$. Then it is also a topological isomorphism. Now the set $A = \{y \in \mathbb{K}^m : \|y\| \leq \|f\|\}$ is closed and bounded in $\mathbb{K}^m$ and hence A must be compact. Since f^{-1} is continuous, $f^{-1}(A)$ is compact. As a closed subset of $f^{-1}(A)$, the unit ball $\mathbb{B}$ of E is compact.

($b \Rightarrow c$) Let $\{x_n\}$ be a bounded sequence in E. There is $\lambda > 0$ such that $\|x_n\| \leq \lambda$ for all n. Since $x \to \lambda x$ is continuous, the set $\{x \in E : \|x\| \leq \lambda\}$ is compact. Therefore $\{x_n\}$ has a convergent subsequence.

($c \Rightarrow d$) Let $\{x_n\}$ be a sequence on the unit sphere $\mathbb{S}$. Then it is a bounded sequence. It has a convergent subsequence, say $y_n \to a$. Letting $n \to \infty$ in $\|y_n\| = 1$, we have $\|a\| = 1$. Therefore $\mathbb{S}$ is compact.

($d \Rightarrow a$) Suppose to the contrary that E is infinite dimensional. Take any $\|x_1\| = 1$. Let M_1 be the vector subspace spanned by $\{x_1\}$. Since M_1 is finite dimensional, it is closed and $M_1 \neq E$. There is $\|x_1\| = 1$ in E such that $\|x_2 - x_1\| \geq \frac{1}{2}$. By induction, suppose $x_1, x_2, \cdots, x_{n-1}$ are chosen such that

$\|x_i\| = 1$ and $\|x_i - x_j\| \geq \frac{1}{2}$ for all $i \neq j$. Let M_n be the vector subspace spanned by $\{x_1, x_2, \cdots, x_{n-1}\}$. Since M_{n-1} is finite dimensional, it is closed and $M_{n-1} \neq E$. There is $\|x_n\| = 1$ in E such that $\|x_n - y\| \geq \frac{1}{2}, \forall\, y \in M_{n-1}$. In particular, $\|x_n - x_j\| \geq \frac{1}{2}, \forall\, j < n$. Now $\{x_n\}$ is a sequence on the unit sphere $\mathcal{S}$. It has a convergent subsequence, say $\{y_n\}$. Then $\{y_n\}$ is Cauchy. There is an integer p such that for all $m, n \geq p$, $\|y_m - y_n\| \leq \frac{1}{3}$. Now $\frac{1}{2} \leq \|y_p - y_{p+1}\| \leq \frac{1}{3}$ is a contradiction. Therefore E must be finite dimensional. $\square$

3-7 Infinite Dimensional Compact Sets

3-7.1. In next two chapters, compactness will play an import role in nonlinear analysis. Ascoli's Theorem provides a very useful criterion for compactness in function spaces. We shall also study the compact sets in some sequence spaces.

3-7.2. Let X be a *compact* metric space and Y a *complete* metric space. The set of all continuous maps from X into Y will be denoted by $C(X, Y)$. For all $f, g \in C(X, Y)$, let $d(f, g) = \sup_{x \in X} d[f(x), g(x)]$. Since the function $x \to d[f(x), g(x)]$ on the compact space X is continuous, it is bounded above. Therefore $d(f, g) \in \mathbb{R}$ is well defined. Clearly, when $Y = \mathbb{K}$ we have $C(X, Y) = C_\infty(X)$.

3-7.3. **Theorem** The space $C(X, Y)$ is a complete metric space.

Proof. It is routine to show that $d(f, g)$ is a metric on $C(X, Y)$. To prove the completeness, let $\{f_n\}$ be a Cauchy sequence in $C(X, Y)$. Let $\varepsilon > 0$ be given. There is an integer p such that for all $m, n \geq p$, we have $d(f_m, f_n) \leq \varepsilon$. Take any $x \in X$. Since $d[f_m(x), f_n(x)] \leq d(f_m, f_n) \leq \varepsilon$ for all $m, n \geq p$, $\{f_n(x)\}$ is a Cauchy sequence in the complete metric space Y. Write $g(x) = \lim_{n\to\infty} f_n(x)$. Hence a map $g : X \to Y$ has been defined. Since $d(u, v)$ is a continuous function of u, letting $m \to \infty$ in $d[f_m(x), f_n(x)] \leq \varepsilon$, we have $d[g(x), f_n(x)] \leq \varepsilon$ for all $x \in X$ and $n \geq p$. As the uniform limit of continuous maps, g is continuous. Taking supremum over $x \in X$ in the last inequality, we have $d(g, f_n) \leq \varepsilon, \forall\, n \geq p$, i.e. $f_n \to g$ in $C(X, Y)$. This completes the proof. $\square$

3-7.4. Let H be a subset of $C(X, Y)$. Then H is said to be *equicontinuous at* $a \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $d(x, a) \leq \delta$ in X and for every $h \in H$ we have $d[h(x), h(a)] \leq \varepsilon$. The set H is said to be *equicontinuous on* X if it is equicontinuous at every point of X. For every $x \in X$, write $H(x) = \{h(x) : h \in H\}$.

3-7.5. **Lemma** For every $x \in X$, the *evaluation map* $\varphi_x : C(X,Y) \to Y$ given by $\varphi_x(f) = f(x)$ is uniformly continuous.

Proof. It follows from $d[\varphi_x(f), \varphi_x(g)] = d[f(x), g(x)] \leq d(f,g)$. □

3-7.6. **Ascoli's Theorem** Let X be a compact metric space and Y a complete metric space. A subset H of $C(X,Y)$ is relatively compact iff the following two conditions hold:

(a) For every $x \in X$, the set $H(x)$ is relatively compact in Y.

(b) The set H is equicontinuous on X.

Proof. ($\Rightarrow a$) Fix $x \in X$. Since H is relatively compact, it is precompact. The uniform continuity of the evaluation map φ_x ensures that $\varphi_x(H) = H(x)$ is precompact. By completeness of Y, $H(x)$ is relatively compact.

($\Rightarrow b$) Fix any $a \in X$ and any $\varepsilon > 0$. Since H is relatively compact, it is precompact. There is a finite subset J of H such that $H \subset \bigcup_{g\in J} \mathbb{B}(g,\varepsilon)$. There is $\delta > 0$ such that for all $d(x,a) \leq \delta$ in X and for all $g \in J$ we have $d[g(x), g(a)] \leq \varepsilon$. Now take any $f \in H$. Choose $g \in J$ such that $d(f,g) \leq \varepsilon$. For any $d(x,a) \leq \delta$ in X, observe

$$d[f(x), f(a)] \leq d[f(x), g(x)] + d[g(x), g(a)] + d[g(a), f(a)]$$
$$\leq d(f,g) + d[g(x), g(a)] + d(g,f) \leq 3\varepsilon.$$

Therefore H is equicontinuous at a. Since $a \in X$ is arbitrary, H is equicontinuous on X.

($\Leftarrow$) Since $C(X,Y)$ is complete, it suffice to show that H is precompact. Let $\varepsilon > 0$ be given. Take any $a \in X$. Since H is equicontinuous at a, there is $\delta_a > 0$ such that for every $d(x,a) \leq \delta_a$ in X we have $d[h(x), h(a)] \leq \varepsilon, \forall\, h \in H$. By compactness of X, there is a finite subset $a_1, a_2, \cdots, a_m$ of X such that $X = \bigcup_{i=1}^m \mathbb{B}(a_i, \delta_i)$ where $\delta_i = \delta_{a_i}$. Now the product $\Pi_{i=1}^m H(a_i)$ of precompact sets is precompact. Its subset $\{(h(a_1), h(a_2), \cdots, h(a_m)) : h \in H\}$ is also precompact. There is a finite subset $g_1, g_2, \cdots, g_n$ of H such that for each $h \in H$ there is j such that $d[h(a_i), g_j(a_i)] \leq \varepsilon \forall\, i$. We claim that $H \subset \bigcup_{j=1}^n \mathbb{B}(g_j, 4\varepsilon)$. Indeed let $h \in H$ be given. Choose g_j as above. Take any $x \in X$. There is i such that $d(x, a_i) \leq \delta_i$. Hence

$$d[h(x), g_j(x)] \leq d[h(x), h(a_i)] + d[h(a_i), g_j(a_i)] + d[g_j(a_i), g_j(x)] \leq 3\varepsilon.$$

Since $x \in X$ is arbitrary, it follows that $d(h, g_j) \leq 3\varepsilon < 4\varepsilon$. This proves that H is precompact. □

3-7.7. **Corollary** A subset H of $C_\infty(X)$ is relatively compact iff the following two conditions hold:
(a) H is *uniformly bounded*, i.e. there is $\lambda > 0$ such that for all $h \in H$ and $x \in X$, we have $|h(x)| \le \lambda$.
(b) The set H is equicontinuous on X.

Proof. Assume that H is relatively compact in $C(X)$. Then it is precompact. For $\varepsilon = 1$, there are $g_1, g_2, \cdots, g_n \in H$ such that for every $h \in H$ there is j satisfying $d(h, g_j) \le 1$. Since all g_i are continuous on the compact space X, the set $\bigcup_{i=1}^n g_i(X)$ is bounded in $\mathbb{K}$. There is $\alpha > 0$ such that for each i and each $x \in X$, we have $|g_i(x)| \le \alpha$. Now take any $h \in H$ and any $x \in X$. Choose j with $d(h, g_j) \le 1$. Observe $|h(x)| \le |h(x) - g_j(x)| + |g_j(x)| \le d(h, g_j) + |g_j(x)| \le 1 + \alpha$. Now part (a) follows by letting $\lambda = 1 + \alpha$. We leave the rest of the proof as an exercise. □

3-7.8. **Exercise** Let X be a finite set equipped with the discrete metric. Identify $C_\infty(X)$ with $\mathbb{K}_\infty^n$. Explain why Ascoli's Theorem is a generalization of the fact that a subset of $\mathbb{K}_\infty^n$ is compact iff it is closed and bounded.

3-7.9. **Example** Prove that for $1 \le p < \infty$, a subset X of ℓ_p is compact iff the following conditions hold:
(a) X is closed and bounded.
(b) For every $\varepsilon > 0$, there is an integer m such that $\sum_{j=m}^\infty |x_j|^p \le \varepsilon, \forall\, x \in X$.

Proof. Assume that X is compact. Certainly it is closed and bounded. Let $\varepsilon > 0$ be given. Since X is compact, it is precompact. There is a finite subset A of X such that $X \subset \bigcup_{a \in A} \mathbb{B}(a, \frac{1}{2}\varepsilon)$. Since A is a finite subset of ℓ_p, there is an integer m such that for all $a \in A$, we have $\left(\sum_{j=m}^\infty |a_j|^p\right)^{1/p} \le \frac{1}{2}\varepsilon$. Now take any $x \in X$. There is $a \in A$ such that $x \in \mathbb{B}(a, \frac{1}{2}\varepsilon)$, i.e. $\left(\sum_{j=1}^\infty |x_j - a_j|^p\right)^{1/p} \le \frac{1}{2}\varepsilon$. Hence we have

$$\left(\sum_{j=m}^\infty |x_j|^p\right)^{1/p} \le \left(\sum_{j=m}^\infty |x_j - a_j|^p\right)^{1/p} + \left(\sum_{j=m}^\infty |a_j|^p\right)^{1/p}$$
$$\le \left(\sum_{j=1}^\infty |x_j - a_j|^p\right)^{1/p} + \left(\sum_{j=m}^\infty |a_j|^p\right)^{1/p} \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Conversely assume that both conditions hold for a subset X of ℓ_p. We claim that X is precompact. Let $\varepsilon > 0$ be given. It follows from (b) that there is an integer m such that for all $x \in X$ we have $\left(\sum_{j=m}^\infty |x_j - a|^p\right)^{1/p} \le \frac{1}{4}\varepsilon$. Define $f(x) = (x_1, x_2, \cdots, x_m)$ for every $x \in \ell_p$. Since $\|f(x)\|_p \le \|x\|_p$, the map $f : \ell_p \to \mathbb{K}_p^m$ is linear and continuous. In particular, the set $f(X)$ is bounded in $\mathbb{K}_p^m$ and hence it is precompact. There is a finite subset A of X such that for every $x \in X$ there is $a \in A$ such that $\|f(x) - f(a)\|_p \le \frac{1}{4}\varepsilon$. Because

$$\begin{aligned}\|x-a\|_p &= \left(\sum_{j=1}^{\infty} |x_j - a_j|^p\right)^{1/p} \\ &\le \left(\sum_{j=1}^{m} |x_j - a_j|^p\right)^{1/p} + \left(\sum_{j=m+1}^{\infty} |x_j|^p\right)^{1/p} + \left(\sum_{j=m+1}^{\infty} |a_j|^p\right)^{1/p} \\ &\le \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon < \varepsilon,\end{aligned}$$

we have $X \subset \bigcup_{a\in A} \mathbb{B}(a,\varepsilon)$. Therefore X is precompact in ℓ_p. Since X is closed in the Banach space ℓ_p, it is also complete. Consequently, X is compact. □

3-7.10. **Exercise** Prove that a subset X of c_0 is compact iff the following conditions hold:
(a) X is closed and bounded.
(b) For every $\varepsilon > 0$, there is an integer m such that for all $x \in X$ and all $n \ge m$, we have $|x_n| \le \varepsilon$.

3-8 Approximation in Function Spaces

3-8.1. We shall study the approximation of a continuous function by smaller classes. Originally, Weierstrass's Theorem states that continuous functions on closed bounded intervals can be uniformly approximated by trigonometric polynomials and the proof involved a lot of classical analysis. Stone's Theorem generalized it to abstract space with completely different proofs.

3-8.2. Let X be a compact metric space. Let $C(X)$ be the set of all continuous *complex* functions on X and $C^r(X)$ all continuous *real* functions on X. Sup-norm will be used throughout this section. Sufficient conditions will be given to ensure that certain vector subspaces are dense in the function space $C_\infty(X)$.

3-8.3. A vector space E of functions on X is called a *vector lattice* if E is closed under absolute values, i.e. $|f| \in E$ whenever $f \in E$. Every real vector lattice is obviously closed under formations of maxima, minima, positive and negative parts. A family E of functions on X is said to *separate points* of X if for all distinct points $x \neq y$ in X there is f in E such that $f(x) \neq f(y)$.

3-8.4. **Approximation Theorem in Real Lattice Form** Let $C^r_\infty(X)$ be the set of all real continuous functions on a compact metric space X. If E is a *vector lattice* of functions in $C^r_\infty(X)$ which contains all real constant functions and separates points of X, then E is dense in $C^r_\infty(X)$.

Proof. Let f be a given function in $C^r_\infty(X)$ and $\varepsilon > 0$ be given. For every pair of distinct points $a \neq b$ in X, since E separates points, there is p_{ab} in E such that $p_{ab}(a) \neq p_{ab}(b)$. For every x in X define

$$h_{ab}(x) = f(a) + \frac{[f(b) - f(a)][p_{ab}(x) - p_{ab}(a)]}{p_{ab}(b) - p_{ab}(a)}.$$

Also for $a = b$, define $h_{ab}(x) = f(a), \forall\, x \in X$. Then clearly $h_{ab}(a) = f(a)$ and $h_{ab}(b) = f(b)$. Since E is a vector space containing all constant functions we have h_{ab} in E. Define $V_b = \{x \in X : h_{ab}(x) < f(x)+\varepsilon\}$. Then V_b is an open set containing b. By compactness of X, there is a finite subset B of X such that $X \subset \bigcup_{b\in B} V_b$. Define $g_a = \wedge\{h_{ab} : b \in B\}$. Since E is a vector lattice each g_a is in E. Clearly we have $g_a(a) = f(a)$ and $g_a(x) \leq f(x) + \varepsilon, \forall x \in X$. Next define $W_a = \{x \in X : g_a(x) > f(x) - \varepsilon\}$. Then W_a is an open set containing a. Since X is compact there is a finite subset A of X such that $X \subset \bigcup_{a\in A} W_a$. Define $g = \vee\{g_a : a \in A\}$. Then g is in the vector lattice E. Clearly we have $f(x) - \varepsilon \leq g(x) \leq f(x) + \varepsilon, \forall x \in X$; i.e. $|f(x) - g(x)| \leq \varepsilon$. Taking supremum for $x \in X$, we have $\|f - g\|_\infty \leq \varepsilon$. This completes the proof. □

3-8.5. A vector space of functions on X is called an *algebra* if it is closed under (pointwise) products. It is easy to prove that for all f, g in $C(X)$, we have $\|f \cdot g\|_\infty \leq \|f\|_\infty \|g\|_\infty$. As a result, multiplication is jointly continuous. The following Lemma is stated for real but it is also true for complex case.

3-8.6. **Lemma** The closure of an algebra E of functions in $C^r_\infty(X)$ is again an algebra.

Proof. Take two functions f, g in the closure of E. Then $f_n \to f$ and $g_n \to g$ for some $f_n, g_n \in E$. Since multiplication is continuous, $f_n \cdot g_n \to f \cdot g$. Hence $f \cdot g$ is a closure point of E. Therefore the closure is closed under pointwise multiplications. Similarly, it is also closed under linear combinations. □

3-8.7. **Dini's Theorem** Let $\{f_n\}$ be a sequence of continuous functions on a metric space X with compact supports, i.e. all supp(f_n) are compact. Suppose for each $x \in X$, we have $0 \leq f_{n+1}(x) \leq f_n(x)$ for all n. If $f_n \to 0$ pointwise, then $f_n \to 0$ uniformly.

Proof. Let $\varepsilon > 0$ be given. Define $A_n = \{x \in X : f_n(x) \geq \varepsilon\}$ for each n. Then A_n is the inverse image of the closed subset $[\varepsilon, \infty)$ in $\mathbb{R}$ under the continuous function f. Hence A_n is a closed subset of the compact set supp(f_n). Therefore each A_n is a compact set. Since $f_n \to 0$ pointwise, we have $\bigcap_{n=1}^\infty A_n = \emptyset$. By compactness, there is some integer p satisfying $\bigcap_{n=1}^p A_n = \emptyset$.

Since $0 \leq f_{n+1}(x) \leq f_n(x)$ for all X, we obtain $A_{n+1} \subset A_n$ for each n. Therefore $A_p = \bigcap_{n=1}^{p} A_n = \emptyset$. Hence for every $x \in X$, we have $x \notin A_p$, i.e. $0 \leq f_p(x) \leq \varepsilon$. Consequently, for all $n \geq p$, we get $0 \leq f_n(x) \leq f_p(x) \leq \varepsilon$ for all $x \in X$. This proves $f_n \to 0$ uniformly. □

3-8.8. **Exercise** Let f_n be functions on $\mathbb{R}$ given by $f(x) = 0$ for $x \leq n$; $f(x) = 1$ for $x \geq n+1$ and $f(x) = x - n$ for $n < x < n+1$. Show that $\{f_n\}$ is a sequence of continuous functions convergent pointwise to zero monotonically but the convergence is not uniform.

3-8.9. **Lemma** There is a sequence of polynomials $\{p_n\}$ without constant term such that $0 \leq p_n(t) \uparrow \sqrt{t}$ uniformly on $[0, 1]$.

Proof. For each t in $[0, 1]$ define $p_0(t) = 0$ and $p_{n+1}(t) = p_n(t) + (1/2)[t - p_n^2(t)]$. It can be proved inductively that $0 \leq p_{n-1}(t) \leq p_n(t) \leq \sqrt{t}$ for all $t \in [0, 1]$. Since $p_n(0) = 0$, all p_n have no constant term. Now for each t, the sequence $\{p_n(t)\}$ of real numbers is increasing, bounded above by $\sqrt{t}$ and hence it is convergent. Letting $n \to \infty$ in its recursive formula we have $p_n(t) \to \sqrt{t}$. Clearly every p_n is a polynomial. Applying Dini's Theorem to $\sqrt{t} - p_n(t)$ for $t \in [0, 1]$, it follows that $p_n(t) \uparrow \sqrt{t}$ uniformly on $[0, 1]$. □

3-8.10. **Approximation Theorem in Real Algebra Form** Let $C_\infty^r(X)$ be the set of all real continuous functions on a compact metric space X. If E is an *algebra* of functions in $C_\infty^r(X)$ which contains all real constant functions and separates points of X, then E is dense in $C_\infty^r(X)$.

Proof. We claim that the closure G of E in $C_\infty^r(X)$ is a vector lattice. In fact, let f be a function in G. Then $|f|$ is also a continuous function on X and hence it is in $C_\infty^r(X)$. Since X is compact, there is a real number $\alpha > |f(x)|$ for all $x \in X$. Let p_n be a sequence of polynomials defined by last lemma. Then the function $p_n(f^2/\alpha^2)$ is also in the algebra G. since $p_n(f^2/\alpha^2) \to \sqrt{f^2/\alpha^2} = |f|/\alpha$ uniformly on X as $n \to \infty$, $|f|/\alpha$ is a closure point of G in $C_\infty^r(X)$. Because G is closed, we have $|f|/\alpha$ in G. Consequently $|f|$ is in G. Therefore G is a vector lattice. Since $E \subset G, G$ contains all real constant functions and separates points of X. Therefore G is dense in $C_\infty^r(X)$, i.e. $\overline{E} = G = \overline{G} = C_\infty^r(X)$. Consequently E is also dense in $C_\infty^r(X)$. □

3-8.11. A subset E of $C(X)$ is said to be *self-conjugate* if for every f in E, its complex conjugate f^- also belongs to E.

3-8.12. **Approximation Theorem in Complex Form** Let $C(X)$ be the set of all complex continuous functions on a compact metric space X. Let E be a complex vector subspace of $C_\infty(X)$ which contains all complex constant functions, separates points of X and is self-conjugate. If E is either a vector lattice or an algebra then E is dense in $C_\infty(X)$.

Proof. Let E^r be the set of all real functions in E. For two distinct points $a \neq b$ in X, there is g in E such that $g(a) \neq g(b)$ and hence either $\text{Re}[g(a)] \neq \text{Re}[g(b)]$ or $\text{Im}[g(a)] \neq \text{Im}[g(b)]$. Since E is self-conjugate, both $\text{Re}(g)$ and $\text{Im}(g)$ are in E^r. Therefore E^r separates points of X. Clearly E^r contains all real constant functions. Also E^r is either a real vector lattice or a real algebra. Hence E^r is dense in $C^r_\infty(X)$. For every function f in $C_\infty(X)$, both $\text{Re}(f)$ and $\text{Im}(f)$ can be approximated by functions in E^r uniformly on X. Since E is a complex vector space, f can be approximated by functions in E uniformly on X. □

3-8.13. **Exercise** Prove that every continuous function f on a bounded closed subset of $\mathbb{K}^n$ can be uniformly approximated by polynomials in n variables. Furthermore, if f is a real-valued function, we may choose real polynomials to approximate f.

3-8.14. **Exercise** Prove that for every continuous function f on the unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$ and for every $\varepsilon > 0$, there is a trigonometric polynomial of the form $g(z) = \sum_{k=-n}^{n} c_k z^k$ where $c_k \in \mathbb{C}$ such that $|f(z) - g(z)| \leq \varepsilon$ for every $z \in T$.

3-8.15. **Exercise** Let X, Y be compact metric spaces. Prove that for every continuous function f on the product space $X \times Y$ and for every $\varepsilon > 0$ there are continuous functions $g_1, g_2, \cdots, g_n$ on X and $h_1, h_2, \cdots, h_n$ on Y such that for all $x \in X$ and $y \in Y$, we have $|f(x,y) - \sum_{j=1}^{n} g_j(x) h_j(y)| \leq \varepsilon$. Furthermore if f is real-valued, we may choose all g_j and h_j to be real-valued. If f is positive (≥ 0), we may choose all g_j and h_j to be positive.

3-8.16. **Exercise** Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk of $\mathbb{C}$ and E the vector space of polynomials in z with complex coefficients. Show that E is not self-conjugate and that the function z^- cannot be uniformly approximated by functions in E otherwise z^- would be analytic on the open disk $\{z \in \mathbb{C} : |z| < 1\}$.

3-99. **References and Further Readings** : Fan-92, Khan, Glicksberg, Starkloff, Gamelin-89, Cotter, Spears and Mullins.

Chapter 4

Simplicial Complexes

4-1 Geometrically Independent Sets

4-1.1. One way to approximate an unknown function f on a region of a two dimensional plane is to triangulate the region, take samples of f at the vertices of triangles and then estimate $f(x)$ by linear extension $g(x)$ of the samples over the interior of each triangle. To see whether $g(x) = 0$ has any solution on a triangle, it is merely a simple matter of solving a system of linear equations. This would give an approximate solution to the equation $f(x) = 0$. The method seems to be very simple and does not require any information about the function f. We shall formalize the idea into simplicial approximations in §4-8. It forms the foundation of simplicial homology which is beyond our scope. We include enough material for motivation of numerical analysis along this direction. The only links of this chapter to the subsequent development are §4-2 on convex sets and §4-10.8 on intersection of closed subsets of the unit sphere. These statements can be understood without any reference to other concepts introduced in this chapter.

4-1.2. The conditions whether three points are not collinear, four points not coplanar, etc. are important to ensure that triangles, tetrahedra, etc. are non-degenerate. The corresponding concept in higher dimensional spaces are geometrically independent sets. The property of being geometrically independent is invariant under small perturbation and can be obtained through small perturbation.

4-1.3. Let $a_0, a_1, \cdots, a_k$ be points in $\mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is called an *affine combination* of $a_0, a_1, \cdots, a_k$ if there are $\alpha_0, \alpha_1, \cdots, \alpha_k \in \mathbb{R}$ such that

$$x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$$

and

$$\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1.$$

The set of all affine combinations of $a_0, a_1, \cdots, a_k$ is called the *affine hull*. The points $a_0, a_1, \cdots, a_k$ are said to be *geometrically independent*, or *in general* *position* if the equations in $\alpha_j \in \mathbb{R}$ given by

$$\alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k = 0, \quad \text{in } \mathbb{R}^n$$

and

$$\alpha_0 + \alpha_1 + \cdots + \alpha_k = 0, \quad \text{in } \mathbb{R}$$

have only the trivial solution: $\alpha_0 = \alpha_1 = \cdots = \alpha_k = 0$. Note that points in $\mathbb{R}^n$ are of column vectors.

4-1.4. **Theorem** Let $a_0, a_1, \cdots, a_k$ be points in $\mathbb{R}^n$. Then the following statements are equivalent.

(a) $a_0, a_1, \cdots, a_k$ are geometrically independent.

(b) $a_1 - a_0, a_2 - a_0, \cdots, a_k - a_0$ are linearly independent.

(c) For every x in the affine hull of $a_0, a_1, \cdots, a_k$, there exists a *unique* sequence $\alpha_0, \alpha_1, \cdots, \alpha_k \in \mathbb{R}$ such that $x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. In this case, $\alpha_0, \alpha_1, \cdots, \alpha_k$ are called the *barycentric coordinates* of x relative to $a_0, a_1, \cdots, a_k$.

(d) The matrix $\begin{bmatrix} 1 & 1 & 1 & & 1 \\ a_0 & a_1 & a_2 & & a_k \end{bmatrix}$ has rank $k+1$.

(e) The above matrix has a square submatrix of order $k+1$ with non-zero determinant.

Proof. The equivalence $(a \Leftrightarrow b \Leftrightarrow c)$ follows immediately from simple algebraic manipulation. The equivalence $(d \Leftrightarrow e)$ is a standard fact in Linear Algebra. To show $(b \Leftrightarrow d)$, observe that the rank of a matrix is invariant under elementary column operations, i.e.

$$\begin{aligned} &rank \begin{bmatrix} 1 & 1 & 1 & & 1 \\ a_0 & a_1 & a_2 & & a_k \end{bmatrix} \\ &= rank \begin{bmatrix} 1 & 0 & 0 & & 0 \\ a_0 & a_1 - a_0 & a_2 - a_0 & & a_k - a_0 \end{bmatrix} \\ &= 1 + rank[a_1 - a_0 \quad a_2 - a_0 \qquad a_k - a_0]. \end{aligned}$$

Therefore (d) is equivalent to

$$rank[a_1 - a_0 \quad a_2 - a_0 \qquad a_k - a_0] = k$$

which means all columns are linearly independent, i.e. (b). □

4-1.5. **Corollary** Subsets of geometrically independent sets are geometrically independent.

4-1.6. **Corollary** If the points $a_0, a_1, \cdots, a_k$ are geometrically independent in $\mathbb{R}^n$ then we have $k \leq n$. Furthermore there are $a_{k+1}, a_{k+2}, \cdots, a_n \in \mathbb{R}^n$ such that $a_0, \cdots, a_k, a_{k+1}, \cdots, a_n$ are geometrically independent in $\mathbb{R}^n$.

4-1.7. **Exercise** Show that the points $(1,0)$, $(0,1)$ and $(2,2)$ are geometrically independent in $\mathbb{R}^2$. Find the barycentric coordinates of $(1,1)$ relative to the above triangle.

4-1.8. **Exercise** Let $mat(m,n)$ be the vector space of all $m \times n$ matrices. Show that it is a Banach space under the norm given by $\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$ Write $mat(n)$ instead of $mat(m,n)$ when $m = n$. Show that the determinant $det(A)$ is a continuous function of square matrices $A \in mat(n)$.

4-1.9. **Theorem** If $a_0, a_1, \cdots, a_k$ are geometrically independent in $\mathbb{R}^n$ then there is $\delta > 0$ such that whenever $\|x_j - a_j\| \leq \delta$ for all $0 \leq j \leq k$, the points $x_0, x_1, \cdots, x_k$ are also geometrically independent.

Proof. Since $a_0, \cdots, a_k$ are geometrically independent, the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & & 1 \\ a_0 & a_1 & a_2 & \cdots & a_k \end{bmatrix}$ is $k+1$. There is a square submatrix A of order $k+1$ with non-zero determinant. Since the determinant function is continuous in its entries, there is $\delta > 0$ such that whenever $\|x_j - a_j\| \leq \delta$ for all $0 \leq j \leq k$, the corresponding submatrix obtained from $\begin{bmatrix} 1 & 1 & 1 & & 1 \\ x_0 & x_1 & x_2 & & x_k \end{bmatrix}$ is of order $k+1$ and with non-zero determinant. Therefore $x_0, x_1, \cdots, x_k$ are also geometrically independent. □

4-1.10. **Theorem** Let $a_0, a_1, \cdots, a_k$ be points in $\mathbb{R}^n$ where $k \geq n$. If every subset of $a_0, a_1, \cdots, a_k$ consisting of $n+1$ points is geometrically independent, then there is $\delta > 0$ such that whenever $\|x_j - a_j\| \leq \delta$ for all $0 \leq j \leq k$, every subset of $x_0, x_1, \cdots, x_k$ consisting of $n+1$ points is also geometrically independent.

Proof. Let $H = \{b_0, b_1, \cdots, b_n\}$ be any subset of $a_0, a_1, \cdots, a_k$ consisting of $n+1$ points. Then there is $\delta_H > 0$ such that whenever $\|x_i - b_i\| \leq \delta$ for all $0 \leq i \leq n$, the points $x_0, x_1, \cdots, x_n$ are geometrically independent. Let δ be the minimum among all δ_H when H varies over all subsets of $a_0, a_1, \cdots, a_k$ consisting of $n+1$ points. Then clearly δ satisfies all requirements of the theorem. □

4-1.11. **Theorem** Let $a_0, a_1, \cdots, a_k$ be points of $\mathbb{R}^n$ where $k \leq n$. Then for every $\varepsilon > 0$ there are $x_0, x_1, \cdots, x_k$ in $\mathbb{R}^n$ such that

(a) $\|x_j - a_j\| \leq \varepsilon$ for all $0 \leq j \leq k$; and

(b) $x_0, x_1, \cdots, x_k$ are geometrically independent.

Proof. Let $f(t)$ be the determinant of the upper-left square submatrix of order

$k+1$ of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ (1-t)a_0 + te_0 & (1-t)a_1 + te_1 & \cdots & (1-t)a_k + te_k \end{bmatrix}$$

where $e_0 = 0$ is the origin and $\{e_1, e_2, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Thus $f(t)$ is a polynomial in t. Since $f(1) = 1$, f is not identically zero. Hence f has only a finite number of zeros. There is $t > 0$ such that $f(t) \neq 0$ and $t\|e_j - a_j\| \leq \varepsilon$ for all $0 \leq j \leq k$. Then the points $x_j = (1-t)a_j + te_j$ for $0 \leq j \leq k$ satisfy both required conditions. □

4-1.12. **Theorem** Let $a_0, a_1, \cdots, a_k$ be given points of $\mathbb{R}^n$ where $k \geq n$. Then for every $\varepsilon > 0$ there are $x_0, x_1, \cdots, x_k$ in $\mathbb{R}^n$ such that

(a) $\|x_j - a_j\| \leq \varepsilon$ for all $0 \leq j \leq k$; and

(b) every subset of $x_0, x_1, \cdots, x_k$ consisting of $n+1$ points is geometrically independent.

Proof. As a matter of convenience, a j-set is defined as a set consisting of exactly j distinct elements. For $k = n$, it has been proved in last theorem. Assume $k > n$. Inductively, there are $b_1, b_2, \cdots, b_k$ in $\mathbb{R}^n$ such that $\|b_j - a_j\| \leq \frac{1}{2}\varepsilon$ for all $1 \leq j \leq k$ and every $(n+1)$-subset of $b_1, b_2, \cdots, b_k$ is geometrically independent. There is $0 < \delta < \frac{1}{4}\varepsilon$ such that whenever $\|x_j - b_j\| \leq \delta$ for all $1 \leq j \leq k$, every $(n+1)$-subset of $x_1, x_2, \cdots, x_k$ is geometrically independent. Let $H_1, H_2, \cdots, H_p$ be a list of all n-subsets of the indices $1, 2, \cdots, k$. Define $\delta_0 = \delta$, $x_0^0 = a_0$ and $x_j^0 = b_j$ for each $1 \leq j \leq k$. We shall construct $\delta_i, x_0^i, x_1^i, x_2^i, \cdots, x_k^i$ by induction on $i \leq p$. Suppose $i < p$. Write $H_{i+1} = \{h(1), h(2), \cdots, h(n)\}$. Then there is a geometrically independent set $\{y_0, y_1, \cdots, y_n\}$ such that $\|y_0 - x_0^i\| \leq \frac{1}{2}\delta_i$ and $\|y_j - x_{h(j)}^i\| \leq \frac{1}{2}\delta_i$ for all $1 \leq j \leq n$. Define $x_0^{i+1} = y_0$, $x_{h(j)}^{i+1} = y_j$ for all $1 \leq j \leq n$ and $x_m^{i+1} = x_m^i$ if $m \notin H_{i+1}$. Choose $0 < \delta_{i+1} < \frac{1}{2}\delta_i$ such that whenever $\|x_j - y_j\| \leq \delta_{i+1}$ for all $0 \leq j \leq n$, the set $\{x_1, x_2, \cdots, x_n\}$ is geometrically independent. By induction, $x_0^p, x_1^p, x_2^p, \cdots, x_k^p$ have been constructed. Let $x_j = x_j^p$ for all $0 \leq j \leq k$. Observe that

$$\begin{aligned} &\|x_j^p - x_j^0\| \leq \|x_j^p - x_j^{p-1}\| + \|x_j^{p-1} - x_j^{p-2}\| + \cdots + \|x_j^1 - x_j^0\| \\ &\leq \tfrac{1}{2}\delta_{p-1} + \tfrac{1}{2}\delta_{p-2} + \cdots + \tfrac{1}{2}\delta_0 \leq \delta. \end{aligned}$$

For $j = 0$ we have $\|x_0 - a_0\| \leq \delta < \varepsilon$. On the other hand, for $1 \leq j \leq k$, we get $\|x_j - b_j\| \leq \|x_j - x_j^0\| \leq \delta$. Therefore every $(n+1)$-subset of $x_1, x_2, \cdots, x_k$ is geometrically independent and hence (b) is partly satisfied. Because

$$\|x_j - a_j\| \leq \|x_j - b_j\| + \|b_j - a_j\| \leq \delta + \tfrac{1}{2}\varepsilon \leq \varepsilon$$

for all $1 \le j \le k$, part (a) is satisfied. Finally, consider any H_{i+1}. Since

$$\|x_{h(j)} - y_j\| \le \|x^p_{h(j)} - x^{p-1}_{h(j)}\| + \|x^{p-1}_{h(j)} - x^{p-2}_{h(j)}\| + \cdots + \|x^{i+1}_{h(j)} - x^i_{h(j)}\|$$
$$\le \tfrac{1}{2}\delta_{p-1} + \tfrac{1}{2}\delta_{p-2} + \cdots + \tfrac{1}{2}\delta_i \le \delta_i,$$

$x_0, x_{h(1)}, x_{h(2)}, \cdots, x_{h(n)}$ are geometrically independent. □

4-2 Convex Sets in Normed Spaces

4-2.1. Let E be a vector space. The *line segment* between two points a, b of E is defined as the set $[a,b] = \{\alpha a + \beta b : \alpha, \beta \ge 0, \text{ and } \alpha + \beta = 1\}$ and the *open line segment* by $(a,b) = \{\alpha a + \beta b : \alpha, \beta > 0, \text{ and } \alpha + \beta = 1\}$. A subset X of E is said to be *convex* if X contains $[a,b]$ whenever $a, b \in X$. Since the concept of convex sets will be used in various parts of functional analysis, this is why it has to be introduced in the general framework rather than in $\mathbb{R}^n$ only.

4-2.2. **Exercise** Let A, B be convex subsets of E and let α, β be scalars in $\mathbb{K}$. Prove that the linear combination $\alpha A + \beta B = \{\alpha x + \beta y : x \in A, y \in B\}$ is also convex. In particular, the *translate* $A + b$ of a convex set A by any $b \in E$ is convex.

4-2.3. **Exercise** Let V be a convex subset of E. Prove that for all $\alpha, \beta \ge 0$, we have $\alpha V + \beta V = (\alpha + \beta)V$.

4-2.4. A point x is called a *convex combination* of points $a_0, a_1, \cdots, a_k$ in E if there are $\alpha_j \ge 0$ such that $x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. The set of all convex combinations of points in a subset M of E is called the *convex hull* of M and it will be denoted by $co(M)$.

4-2.5. **Exercise** Prove that a convex set is closed under formation of convex combinations. Prove that the convex hull of a set M is the smallest convex set containing M.

4-2.6. **Theorem** The convex hull of a finite set in a normed space E is compact.

Proof. Let $V = \{a_0, a_1, \cdots, a_k\}$ be a finite subset of E. The map $f : \mathbb{R}^{k+1} \to E$ defined by $f(\alpha_0, \alpha_1, \cdots, \alpha_k) = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$ is continuous. Since the set $K = \{(\alpha_0, \alpha_1, \cdots, \alpha_k) : \text{ all } \alpha_j \ge 0 \text{ and } \alpha_0 + \alpha_1 + \cdots + \alpha_k = 1\}$ is closed bounded in $\mathbb{R}^{k+1}$, it is compact. Consequently its continuous image $f(K) = co(V)$ is also compact. □

4-2.7. **Exercise** Prove that the closure of a convex set in a normed space E is convex. Prove that the closure of the convex hull of a subset M of E is the smallest closed convex set containing M. It is called the *closed convex hull* of M and is denoted by $\overline{co}(M)$.

4-2.8. The following interesting result about convex sets with non-empty interior may be skipped without discontinuity for this and next chapters.

4-2.9. **Theorem** Let A be a convex set in a normed space E. Suppose that A has at least one interior point.

(a) For every interior point $a \in A$ and for every closure point $b \in A$, the open line segment $(a, b) = \{(1-t)a + tb : 0 < t < 1\}$ is contained in the interior A^o.

(b) A^o is a convex set.

(c) $A^- = A^{o-}$ where A^- denote the closure of A.

(d) $A^o = A^{-o}$.

Proof. (a) Let $u = (1-t)a + tb$ where $0 < t < 1$. Let

$$V = \{y \in E : u = (1-t)x + ty, x \in A^o\}.$$

Then V is the inverse image of the open set A^o under the continuous map $y \to x = (u - ty)/(1-t)$. Hence V is open in E. Since $a \in A^o$, we have $b \in V$. There exists $r > 0$ such that $\mathbb{B}(b, r) \subset V$. Since b is a closure point of A, there is $d \in A \cap \mathbb{B}(b, r)$. Let $W = \{v \in E : v = (1-t)x + td, x \in A^o\}$. Then W is the inverse image of the open set A^o under the continuous map $v \to x = (v - td)/(1-t)$. Hence W is open in E. Since $d \in V$, there is $c \in A^o$ such that $u = (1-t)c + td$. Thus $u \in W$. Since x, d are in the convex set A, W is an open subset of A and consequently $W \subset A^o$. Therefore $u \in A^o$. This proves (a).

(b) It follows immediately from (a).

(c) Since $A^o \subset A$, we have $A^{o-} \subset A^-$. Conversely let $x \in A^-$ and $a \in A^o$. The points $\frac{1}{n}a + (1 - \frac{1}{n})x$ are in A^o for all $n \geq 2$. Letting $n \to \infty$, we have $x \in A^{o-}$. Thus $A^- = A^{o-}$. This proves (c).

(d) Let $x \in A^{-o}$. There exists $r > 0$ such that $\mathbb{B}(x, r) \subset A^-$. Clearly $x \in A^-$. By (c), we have $x \in A^{o-}$. There is $y \in \mathbb{B}(x, r) \cap A^o$. Thus we have

$$2x - y = x + (x - y) \in \mathbb{B}(x, r) \subset A^-.$$

Since $y \in A^o$, we have by (a), $x = \frac{1}{2}y + \frac{1}{2}(2x - y) \in A^o$. Therefore $A^{o-} \subset A^o$. The reversed inequality is obvious. □

4-2.10. **Exercise** Let $A = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1, 0 \le y < 1\}$. Find A^{-o} and A^{o-}. Let $B = A \cap (\mathbb{Q} \times \mathbb{Q})$ where $\mathbb{Q}$ denotes the set of all rational numbers. Find B^{-o} and B^{o-}.

4-3 Simplexes

4-3.1. Most of the following results also hold for Banach spaces. Since we never need the maximum generality in this book, we restrict ourselves to $\mathbb{R}^n$ only. In this section, we shall study the generalization of triangles and tetrahedra.

4-3.2. Consider the space $\mathbb{R}_2^n$ with the usual norm. If the points $a_0, a_1, \cdots, a_k$ are geometrically independent then their convex hull A is called a *simplex* or a k-simplex with *vertices* $a_0, a_1, \cdots, a_k$. In this case, k is called the *dimension* of the simplex A. The set of all vertices of A will be denoted by $ver(A)$. Clearly every simplex is compact.

4-3.3. Let A be a simplex with vertices $a_0, a_1, \cdots, a_k$ in $\mathbb{R}^n$. Suppose we have $x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$; all $\alpha_j \ge 0$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. Then x is called a *geometric interior point* of A if all $\alpha_j > 0$ and a *geometric boundary point* if at least one $\alpha_j = 0$. The set of all geometric interior points is called the *geometric interior* and is denoted by $gin(A)$. The set of all geometric boundary points is called the *geometric boundary* and is denoted by $gbd(A)$. Clearly $gin(A), gbd(A)$ form a partition of A. The geometric interior of A is also called an *open simplex* with vertices $a_0, a_1, \cdots, a_k$ but we seldom use open simplexes in this book.

4-3.4. **Theorem** Let A be a simplex in $\mathbb{R}^n$.
(a) If $x \in gin(A)$ then for every $y \in A$ and for every $0 < t < 1$, then the point $z = (1 - t)x + ty$ is in the geometric interior of A.
(b) The closure of $gin(A)$ in $\mathbb{R}^n$ is A.

Proof. Let $a_0, a_1, \cdots, a_k$ be vertices of A. Write

$$x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k, \quad \alpha_j > 0, \quad \alpha_0 + \alpha_1 + \cdots + \alpha_k = 1;$$

$$y = \beta_0 a_0 + \beta_1 a_1 + \cdots + \beta_k a_k, \quad \beta_j \ge 0, \quad \beta_0 + \beta_1 + \cdots + \beta_k = 1.$$

Since $\sum_{j=0}^k (1-t)\alpha_j + t\beta_j = 1$ and $(1-t)\alpha_j + t\beta_j > 0$, $z = \sum_{j=0}^k [(1-t)\alpha_j + t\beta_j] a_j$ is a geometric interior point of A. To prove (b), take any $y \in A$. For each integer $m > 1$ let

$$z_m = \frac{1}{m} x + \left(1 - \frac{1}{m}\right) y$$

where $x = \sum_{j=0}^{k} \frac{a_j}{k+1}$ is a geometric interior point of A. Hence all $z_m \in gin(A)$ and $z_m \to y$ as $m \to \infty$. Therefore y is a closure point of $gin(A)$. Consequently $A \subset \overline{gin(A)}$. The reversed inequality follows from $\overline{gin(A)} \subset \overline{A} = A$. □

4-3.5. Let A be a k-simplex in $\mathbb{R}^n$. Let W be a subset of $ver(A)$. Since W is geometrically independent, the convex hull $B = co(W)$ is also a simplex in $\mathbb{R}^n$. In this case, B is called a *face* or a t-face of A where t is the dimension of B. If $t = k - 1$ then B is also called a *facet* of A.

4-3.6. **Theorem** Let A be a simplex in $\mathbb{R}^n$.
(a) $gbd(A)$ is a compact set.
(b) $gin(A)$ is open in A although it may not be open in $\mathbb{R}^n$.

Proof. Every facet of A is itself a simplex and hence it is compact. The geometric boundary is the union of $k+1$ facets and thus it is also compact. To prove (b), since $gbd(A)$ is closed in A, its complement $gin(A)$ is open in A. □

4-3.7. **Exercise** Show that the convex hull of $(1,1,1)$, $(1,0,0)$, $(0,1,0)$, $(1,1,2)$ and $(2,2,0)$ is a simplex in $\mathbb{R}^3$. Identify its vertices, geometrical interior and boundary.

4-4 Affine Maps

4-4.1. Affine maps carries straight lines to straight lines. Hence they were called linear maps which has become a standard terminology taken up by linear algebra. The domains of affine maps can be translates of vector subspaces. However it is sufficient for us to restrict ourselves to convex sets in order to lay the foundation for next chapter.

4-4.2. Let X, Y be a *convex* subset of $\mathbb{R}^n$. A map $f : X \to \mathbb{R}^m$ is said to be *affine* if f preserves the convex combinations, i.e. for all $a, b \in X$ and for all $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$ we have $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$. An affine bijection from X onto Y is called an *affine homeomorphism*.

4-4.3. **Exercise** Let f be an affine map from a convex subset X of $\mathbb{R}^n$ into $\mathbb{R}^m$. Then $f(X)$ is convex. Furthermore for all a_j in X and for all $\alpha_j \geq 0$ satisfying $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$ we have

$$f(\alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k) = \alpha_0 f(a_0) + \alpha_1 f(a_1) + \cdots + \alpha_k f(a_k).$$

4-4.4. **Exercise** Let X, Y be convex sets of $\mathbb{R}^n, \mathbb{R}^m$ respectively. If $f : X \to Y$ is a bijective affine map then the inverse map $f^{-1} : Y \to X$ is also an affine map.

4-4.5. **Exercise** Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that the map $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(x) = g(x) + b$ where $b \in \mathbb{R}^m$ is an affine map.

4-4.6. **Theorem** Let A be a simplex in $\mathbb{R}^n$. Then for each affine map $f : A \to \mathbb{R}^m$ there exist a linear map $g : \mathbb{R}^n \to \mathbb{R}^m$ and a point $b \in \mathbb{R}^m$ such that $f(x) = g(x) + b$ for all $x \in A$.

Proof. The vertices of A, say $a_0, a_1, \cdots, a_k$ are geometrically independent. Hence $a_1 - a_0, a_2 - a_0, \cdots, a_k - a_0$ are linearly independent. There is a linear map $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(a_j - a_0) = f(a_j) - f(a_0)$ for all $1 \leq j \leq k$. Define $b = f(a_0) - g(a_0)$. Take any $x \in A$. Write $x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$ where $\{\alpha_j\}$ are the barycentric coordinates of x. Then all $\alpha_j \geq 0$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. Since f is affine we have

$$f(x) = \sum_{j=0}^{k} \alpha_j f(a_j) = \sum_{j=0}^{k} \alpha_j [g(a_j - a_0) + f(a_0)]$$

$$= \sum_{j=0}^{k} \alpha_j g(a_j) - \sum_{j=0}^{k} \alpha_j g(a_0) + \sum_{j=0}^{k} \alpha_j f(a_0)$$

$$= g\Big(\sum_{j=0}^{k} \alpha_j a_j\Big) - g(a_0) + f(a_0) = g(x) + b.$$ □

4-4.7. **Corollary** Every affine map on a simplex is continuous.

Proof. The linear map g on $\mathbb{R}^n$ in last theorem is continuous. □

4-4.8. **Theorem** Let A be a k-simplex with vertices $a_0, a_1, \cdots, a_k$ in $\mathbb{R}^n$. Then for every given points $b_0, b_1, \cdots, b_k$ in $\mathbb{R}^m$ there is a unique affine map $f : A \to \mathbb{R}^m$ such that $f(a_j) = b_j$ for all $0 \leq j \leq k$.

Proof. Since $a_1 - a_0, a_2 - a_0, \cdots, a_k - a_0$ are linearly independent, there is a linear map $g : \mathbb{R}^n \to \mathbb{R}^m$ such that $g(a_j - a_0) = b_j - b_0$, for all $1 \leq j \leq k$. For each $x \in A$ define $f(x) = g(x) + b_0 - g(a_0)$. Then clearly f is one of the required affine maps. The uniqueness is left as an exercise. □

4-4.9. **Exercise** Let A, B be k-simplexes with vertices $\{a_0, a_1, \cdots, a_k\}$ and $\{b_0, b_1, \cdots, b_k\}$ in $\mathbb{R}^n, \mathbb{R}^m$ respectively. Prove that there is a unique affine homeomorphism $f : A \to B$ such that $f(a_j) = b_j$ for all $0 \leq j \leq k$.

4-4.10. Let X be a non-empty subset of $\mathbb{R}^n$. Then $z \in X$ is called an *extreme point* if whenever $z = (1 - t)x + ty$ where $x, y \in X$ and $0 < t < 1$ we have $x = y = z$.

4-4.11. **Lemma** Let A be a simplex and $z \in A$ be a given point. Then z is a vertex of A iff z is an extreme point of A.

Proof. Let $a_0, a_1, \cdots, a_k \in \mathbb{R}^n$ be vertices of A. To show that $z = a_0$ is an extreme point of A, assume $z = (1-t)x + ty$ where $x, y \in A$ and $0 < t < 1$. Write

$$x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k, \quad \alpha_j \geq 0, \quad \alpha_0 + \alpha_1 + \cdots + \alpha_k = 1;$$
$$y = \beta_0 a_0 + \beta_1 a_1 + \cdots + \beta_k a_k, \quad \beta_j \geq 0, \quad \beta_0 + \beta_1 + \cdots + \beta_k = 1.$$

Substitution gives $a_0 = \sum_{j=0}^{k}[(1-t)\alpha_j + t\beta_j]a_j$. Since $(1-t)\alpha_j + t\beta_j \geq 0$ and their sum is one, by uniqueness of barycentric coordinates we have $1 = (1-t)\alpha_0 + t\beta_0$ which implies $\alpha_0 = \beta_0 = 1$. Hence $\alpha_j = \beta_j = 0$ for all $j > 0$, i.e. $x = y = z = a_0$. Conversely assume that $z \in A$ is not a vertex. Write $z = \sum_{j=0}^{k} \rho_j a_j$, all $\rho_j \geq 0$ and $\sum_{j=0}^{k} \rho_j = 1$. Since $z \in A$ is *not* a vertex, at least two $\rho_j > 0$. Without loss of generality, let $\rho_0, \rho_1 > 0$. Define $x = (\rho_0 + \rho_1)a_0 + \sum_{j=2}^{k} \rho_j a_j$; $y = (\rho_0 + \rho_1)a_1 + \sum_{j=2}^{k} \rho_j a_j$ and $t = \rho_1/(\rho_0 + \rho_1)$. Then $z = (1-t)x + ty$; $0 < t < 1$; $x \neq y$ and $x, y \in A$. Therefore z cannot be an extreme point of A. □

4-4.12. **Theorem** Let A, B be simplexes in $\mathbb{R}^n, \mathbb{R}^m$ respectively. If $f : A \to B$ is an affine homeomorphism then f is a bijection from the vertices of A onto the vertices of B. In particular, both A and B have the same dimension.

Proof. Let $ver(A), ver(B)$ be the sets of vertices of A, B respectively. Let $c \in ver(A)$. We claim that $z = f(c)$ is an extreme point of B. In fact, assume $z = (1-t)x + ty$ where $x, y \in B$ and $0 < t < 1$. Let $a, b \in A$ satisfy $f(a) = x$ and $f(b) = y$. Since f is affine, $f(c) = (1-t)f(a) + tf(b) = f[(1-t)a + tb]$, i.e. $c = (1-t)a + tb$. Because c is an extreme point of A, we have $a = b = c$, that is, $x = y = z$. Therefore z is an extreme point of B, i.e. $f(c) \in ver(B)$. Consequently $f[ver(A)] \subset ver(B)$. Since f is injective, we have $\dim(A) \leq \dim(B)$. Since f^{-1} is also an affine homeomorphism, we obtain $\dim(B) \leq \dim(A)$. Therefore $f[ver(A)] = ver(B)$. □

4-4.13. **Exercise** Let A, B be simplexes in $\mathbb{R}^n, \mathbb{R}^m$ respectively. Prove that if $f : A \to B$ is an affine homeomorphism then we have $f[ver(A)] = ver(B)$, $f[gin(A)] = gin(B)$ and $f[gbd(A)] = gbd(B)$.

4-5 Simplicial Complexes

4-5.1. Consider the unit sphere in $\mathbb{R}^3_2$. We can cut it into upper and lower hemispheres. Each hemisphere is just like a piece of rubber which can be turned into a square via homeomorphisms. The diagonals would cut the squares into triangles. This is an example of reducing a surface into triangles. In opposite direction, we can start with triangles, glue them up properly to form piecewise rectilinear surface and finally twist it into surfaces of more general shape without sharp edges and corners. In higher dimensions, we use simplexes instead of triangles.

4-5.2. Let A, B be simplexes in $\mathbb{R}^n$. Write $A \leq B$ if A is a face of B. Write $A < B$ if $A \leq B$ and $A \neq B$. In this case, A is called a *proper face* of B. Two simplexes A, B are said to be *properly situated* if $A \cap B$ is either empty or it is a common face of A and B. A non-empty finite family K of simplexes in $\mathbb{R}^n$ is called a *simplicial complex* if the following conditions hold:
(a) every face of a simplex in K belongs to K;
(b) every pair of simplexes in K is properly situated.

4-5.3. **Exercise** The family K of all faces of a given simplex A in $\mathbb{R}^n$ is a simplicial complex.

4-5.4. Let K be a simplicial complex in $\mathbb{R}^n$. One dimensional simplexes are called *edges* and 0-simplexes called *vertices* of K. The set of all vertices of K is denoted by $ver(K)$. The *dimension* of K is defined as the maximum dimension among all simplexes in K. For every integer $m \geq 0$, the family $K(m)$ of all simplexes $A \in K$ with $\dim(A) \leq m$ is called a *skeleton* or an m-skeleton. A subset L of K is called a *subcomplex* of K if L itself forms a simplicial complex.

4-5.5. **Exercise** Every skeleton of a simplicial complex is a subcomplex.

4-5.6. Let K be a simplicial complex in $\mathbb{R}^n$. The union of all simplexes in K is called the *underlying space* of K and it is denoted by $|K|$. A subset of $\mathbb{R}^n$ is called a *polyhedron* if it is the underlying space of some simplicial complex. A simplicial complex K is called a *triangulation* of a polyhedron P if $|K| = P$. Clearly every polyhedron is compact since it is a finite union of compact sets of its simplexes.

4-5.7. **Theorem** Let K be a simplicial complex in $\mathbb{R}^n$. Then every point $x \in |K|$ is a geometric interior point of a unique simplex A in K.

Proof. To prove the existence, let $x \in |K|$. Then $x \in A$ for some $A \in K$. Write $x = \alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k$ where $a_0, a_1, \cdots, a_k$ are vertices of A, all $\alpha_j \geq 0$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1$. Without loss of generality assume $\alpha_j > 0$ for all $0 \leq j \leq t$ and $\alpha_j = 0$ for all $t < j \leq k$. Let B be the simplex with vertices

$a_0, a_1, \cdots, a_t$. Then B is a face of A. Hence $B \in K$ and $x \in gin(B)$. For the uniqueness, let A, B be simplexes of K such that x is a geometric interior point of both A and B. Then $x \in A \cap B$. Since $A \cap B$ is a common face of A, B, let $ver(A \cap B) = \{c_0, c_1, \cdots, c_t\}$, $ver(A) = \{c_0, c_1, \cdots, c_t, a_{t+1}, \cdots, a_u\}$ and $ver(B) = \{c_0, c_1, \cdots, c_t, b_{t+1}, \cdots, b_v\}$ where u, v are integers $\geq t$. From $x \in A \cap B$, write $x = \delta_0 c_0 + \delta_1 c_1 + \cdots + \delta_t c_t$, all $\delta_j \geq 0$ and $\sum_{j=0}^{t} \delta_j = 1$. Since $x \in gin(A)$, write $x = \sum_{j=0}^{t} \alpha_j c_j + \sum_{j=t+1}^{u} \alpha_j a_j$, all $\alpha_j > 0$ and $\sum_{j=0}^{u} \alpha_j = 1$. By uniqueness of barycentric coordinates relative to the vertices of A, we have $u = t$, i.e. $ver(A \cap B) = ver(A)$ or $A \cap B = A$. Similarly $A \cap B = B$. Consequently $A = B$. □

4-5.8. Let K be a simplicial complex in $\mathbb{R}^n$. Then the unique simplex $A \in K$ satisfying $x \in gin(A)$ is called the *carrier simplex* of x and it is denoted by $car(x)$. For every vertex a of K, the union of those open simplexes $gin(A)$ of K satisfying $a \in A$ is called the *star* of a and it is denoted by $star(a)$.

4-5.9. **Exercise** Consider the xy-plane. Let K consist of two triangles $(0,0)$, $(0,2)$, $(1, \pm 2)$ and one edge $(0,2)$, $(0,4)$ together with all their faces. Identify its underlying space. Describe all skeletons. Find the carrier simplexes of $(0,3)$ and $(1,1)$. Find the star of each vertex.

4-5.10. **Theorem** Let a be a vertex of a simplicial complex K in $\mathbb{R}^n$ and x a point in the underlying space $|K|$.
(a) $x \in star(a)$ iff $a \in car(x)$.
(b) The family $\{star(a) : a \in ver(K)\}$ is an open cover of $|K|$.

Proof. (a) $x \in star(a)$ iff there is $A \in K$ such that $a \in A$ and $x \in gin(A)$, i.e. $A = car(x)$ iff $a \in car(x)$.
(b) Let L be the family of simplexes $A \in K$ such that $a \notin A$. Then $x \notin star(a)$ iff $a \notin car(x)$ iff $car(x) \in L$. Hence $star(a) = |K| \setminus |L|$. Since $|L|$ is compact, it is closed in $|K|$ and therefore its complement $star(a)$ is open in $|K|$. Finally take any $x \in |K|$. Let a be a vertex of $car(x)$. Since $x \in car(x)$, we have $a \in car(x)$, i.e. $x \in star(a)$. Therefore $\{star(a) : a \in ver(K)\}$ covers $|K|$. □

4-5.11. **Theorem** Let K be a simplicial complex in $\mathbb{R}^n$. If f is a map from the set $ver(K)$ of vertices of K into $\mathbb{R}^m$, then f can be extended uniquely to a continuous map g on the underlying space $|K|$ into $\mathbb{R}^m$ such that the restriction $g|A$ onto every simplex $A \in K$ is an affine map.

Proof. Since an affine map on a simplex is completely determined by the values at the vertices, the uniqueness follows immediately. To prove the existence, for each simplex $A \in K$, there is an affine map f_A on A which agrees with f on the vertices of A. Let A, B be simplexes of K and let $C = A \cap B$. Then C is a common face of A, B. Since an affine map is uniquely determined by values at the vertices, we have $f_A|C = f_B|C = f_C$. Therefore there is a map $g : |K| \to \mathbb{R}^m$ such that $g|A = f_A$ for each $A \in K$. Since every f_A is continuous, so is g on $|K|$ by Glue Theorem. □

4-5.12. **Exercise** Let $|K|$ be a simplicial complex in $\mathbb{R}^n$ and $J = [0, 1]$ the unit interval. Let h be a map from $|K| \times J$ into $\mathbb{R}^m$. Prove that if for each simplex $A \in K$ the restriction $h|(A \times J)$ is continuous, then h is continuous on $|K| \times J$.

4-5.13. **Exercise** Prove that for all $x, y \in |K|$ if $x \in car(y)$, then we have $car(x) \subset car(y)$.

4-6 Small Simplexes

4-6.1. Let A be a simplex with vertices $\{a_0, a_1, \cdots, a_k\}$. Then the *barycenter* of A is defined as the point given by $bar(A) = \dfrac{1}{k+1}(a_0 + a_1 + \cdots + a_k)$. When A is a triangle, the barycenter is the centroid.

4-6.2. **Lemma** If $A_0 < A_1 < A_2 < \cdots < A_k$ be simplexes, then the set $\{bar(A_i) : 0 \le i \le k\}$ is geometrically independent.

Proof. If $k = 0$, it is obvious. Suppose $k > 0$, $\sum_{i=0}^k \alpha_i \ bar(A_i) = 0$ and $\sum_{i=0}^k \alpha_i = 0$. Suppose to the contrary that $\alpha_k \neq 0$. Then both sets $P = \{i : \alpha_i > 0\}$ and $Q = \{i : \alpha_i < 0\}$ are non-empty. We may assume $k \in Q$ otherwise replacing all α_j by $-\alpha_j$. Since $\sum_{i \in P} \alpha_i = \sum_{i \in Q}(-\alpha_i) > 0$, we have a convex combination

$$x = \frac{\sum_{i \in P} \alpha_i bar(A_i)}{\sum_{i \in P} \alpha_i} = \frac{\sum_{i \in Q}(-\alpha_i) bar(A_i)}{\sum_{i \in Q}(-\alpha_i)}.$$

Since $A_0 < A_1 < A_2 < \cdots < A_k$, we have $x \in A_k$ and there is $b \in ver(A_k) \setminus A_{k-1}$. Expanding each $bar(A_i)$ in terms of its vertices, let α, β be the coefficients of b for the expression on the left and right hand sides respectively. Then $\alpha = 0$ and

$$\beta = \frac{-\alpha_k}{(1 + \dim A_k) \sum_{i \in Q}(-\alpha_i)} > 0.$$

This contradiction to the uniqueness of barycentric coordinates of x relative to A_k shows that $\alpha_k = 0$. Now the proof is completed by induction on k. □

4-6.3. Let A be a simplex. Then a subset B of A is called a *small simplex* of A if there are faces $A_0 \leq A_1 \leq \cdots \leq A_k \leq A$ such that $B = co\{bar(A_j) : 0 \leq j \leq k\}$. Note that if $A_p = A_{p+1}$ then $B = co\{bar(A_j) : 0 \leq j \leq k, j \neq p\}$. Therefore we may assume $A_0 < A_1 < \cdots < A_p \leq A$ by dropping all duplicated ones. As a result of last lemma, every small simplex is a simplex. It is also obvious that every small simplex of A is a subset of A.

4-6.4. **Exercise** Let $A \leq B$ be given simplexes. Prove that if S is small simplex of A, then S is a small simplex of B.

4-6.5. **Lemma** Every simplex A is the union of all its small simplexes.

Proof. Let $a_0, a_1, \cdots, a_k$ be the vertices of A. Assume $k > 0$ and $x \in A$. Let C be the carrier simplex of x. If $C < A$ then by induction, x belongs to some small simplex of C which is also a small simplex of A. Next consider $C = A$. Assume that $x \neq bar(A)$ otherwise the lemma is proved. Intuitively, we shall extend the line segment from $bar(A)$ to x until it cut a facet at y and the lemma follows since x is between $bar(A)$ and y. Analytically, write $x = \sum_{j=0}^k \alpha_j a_j$ where all $\alpha_j > 0$ and $\sum_{j=0}^k \alpha_j = 1$. After renaming the vertices if necessary, we may assume $\alpha_k \leq \alpha_j$ for all $0 \leq j \leq k$. Then we have

$$x = \sum_{j=0}^{k} \alpha_j a_j - \alpha_k \sum_{j=0}^{k} a_j + \alpha_k (k+1) bar(A) = \sum_{j=0}^{k-1} (\alpha_j - \alpha_k) a_j + \alpha_k (k+1) bar(A).$$

Observe the coefficients:

$$\sum_{j=0}^{k-1} (\alpha_j - \alpha_k) + \alpha_k (k+1) = \sum_{j=0}^{k} (\alpha_j - \alpha_k) + \alpha_k (k+1) = \sum_{j=0}^{k} a_j = 1.$$

Suppose to the contrary that $\sum_{j=0}^{k-1} (\alpha_j - \alpha_k) = 0$, i.e. $\sum_{j=0}^{k-1} \alpha_j - k\alpha_k = 0$, or $(1 - \alpha_k) - k\alpha_k = 0$, that is $\alpha_k = \frac{1}{k+1}$. Since $\alpha_j - \alpha_k \geq 0$ for all j, we have $\alpha_j = \frac{1}{k+1}$ for all j, i.e. $x = bar(A)$ contradicting $x \neq bar(A_p)$. Therefore we obtain $\sum_{j=0}^{k-1} (\alpha_j - \alpha_k) > 0$. Now define

$$y = \frac{\sum_{j=0}^{k-1} (\alpha_j - \alpha_k) a_j}{\sum_{j=0}^{k-1} (\alpha_j - \alpha_k)}$$

and $A_{k-1} = co\{a_0, a_1, \cdots, a_{k-1}\}$. Then $y \in A_{k-1}$. By induction, there are $A_0 \leq A_1 \leq \cdots \leq A_{k-2} \leq A_{k-1}$ such that $y \in co\{bar(A_j) : 0 \leq j \leq k-2\}$. Consequently $x = \left[\sum_{j=0}^{k-1} (\alpha_j - \alpha_k)\right] y + (k+1)\alpha_k bar(A)$ belongs to

$$co\{bar(A_j) : A_0 \leq A_1 \leq \cdots \leq A_{k-2} \leq A_{k-1} \leq A\}$$

which is a small simplex of A. □

4-6.6. **Lemma** Let $A \leq B$ be given simplexes and S a small simplex of B. If $A \cap S$ is non-empty, then it is a small simplex of A.

Proof. As a small simplex of B, $S = co\{bar(B_j) : 0 \leq j \leq k\}$ where $B_0 \leq B_1 \leq \cdots \leq B_k \leq B$. Let $P = \{j : ver(B_j) \subset A\}$ and Q the complement of P. Take any $x \in A \cap S$. Write $x = \sum_{j=0}^{k} \alpha_j bar(B_j)$ where all $\alpha_j \geq 0$ satisfy $\sum_{j=0}^{k} \alpha_j = 1$. Consider any $j \in Q$. At least one vertex of B_j does not belong to A. Since $x \in A$, it is an affine combination of vertices of A only, i.e. $\alpha_j = 0$. Therefore $x = \sum_{j \in P} \alpha_j bar(B_j)$. Since $A \cap S \neq \emptyset$, we have $P \neq \emptyset$. Let p be the maximum among all integers in P and $D = co\{bar(B_j) : 0 \leq j \leq p\}$. Then $B_0 \leq B_1 \leq \cdots \leq B_p \leq A$ and $x = \sum_{j=0}^{k} \alpha_j bar(B_j)$, i.e. $x \in D$. Hence $A \cap S \subset D$. Clearly $D \subset A \cap S$. Hence $A \cap S = D$ is a small simplex of A. □

4-6.7. **Lemma** Every two small simplexes S, T of a given simplex A are properly situated. If $S \cap T$ is non-empty, then $S \cap T$ is a small simplex of A.

Proof. It is left as an exercise. □

4-6.8. **Exercise** Sketch a tetrahedron D in $\mathbb{R}^3$ of which the vertices are labelled by 0123. Let A be the small simplex of D defined by the sequence of barycenters of faces labelled by 0, 01, 012, 0123 and B by 2, 012, 0123 respectively. Identify the vertices of $D, A, B, A \cap B$.

4-7 Barycentric Subdivisions

4-7.1. There are many ways to break simplexes of simplicial complexes into smaller pieces which remain to be properly situated. Barycentric subdivisions are chosen because of its theoretical simplicity and its popularity in simplicial homology even though it is known by now that it has poor volume-mesh ratio.

4-7.2. Let K, L be simplicial complexes in $\mathbb{R}^n$. Then L is called a *subdivision* of K if

(a) every simplex of L is contained in some simplex of K;

(b) every simplex of K is a finite union of simplexes of L.

4-7.3. **Exercise** Prove that if L is a subdivision of K, then both K, L have the same underlying space.

4-7.4. **Exercise** Prove that if L is a subdivision of K and if K is a subdivision of M then L is a subdivision of M.

4-7.5. **Theorem** Let K be a simplicial complex in $\mathbb{R}^n$. Then the family $sd(K)$ of all small simplexes of every simplex in K is a subdivision of K.

Proof. Let S, T be small simplexes of given simplexes A, B in K respectively. Suppose $S \cap T \neq \emptyset$. Since A, B are properly situated, $C = A \cap B$ is a simplex in K. Hence both $S \cap C$ and $T \cap C$ are small simplexes of C. Thus $S \cap T = (S \cap C) \cap (T \cap C)$ is a small simplex of C. Therefore $S \cap T$ is also in $sd(K)$. Since S, T are properly situated, $sd(K)$ is a simplicial complex. It is obvious to verify that $sd(K)$ is a subdivision of K. □

4-7.6. Let K be a simplicial complex in $\mathbb{R}^n$. Then $sd(K)$ is called the *barycentric subdivision* of K. Clearly both K and $sd(K)$ have the same dimension. In general, let $sd^0(K) = K$ and $sd^m(K) = sd[sd^{m-1}(K)]$ for all $m \geq 1$. In this case, $sd^m(K)$ is called the m-th *barycentric subdivision* of K. The *mesh* of a simplicial complex K is the maximum diameter among all the simplexes in K.

4-7.7. **Lemma** For every non-empty bounded set M in $\mathbb{R}^n$, both M and its convex hull have the same diameter.

Proof. Let $N = co(M)$. Since $M \subset N$, we have $diam(M) \leq diam(N)$. Conversely, take any $x, y \in N$. Write $x = \sum_{i=0}^{s} \alpha_i a_i$ and $y = \sum_{j=0}^{t} \beta_j b_j$ where $a_i, b_j \in M$, $\alpha_i, \beta_j \geq 0$ and $\sum_{i=0}^{s} \alpha_i = 1 = \sum_{j=0}^{t} \beta_j$. Then we have

$$\|a_i - y\| = \|a_i - \sum_{j=0}^{t} \beta_j b_j\| \leq \sum_{j=0}^{t} \beta_j \|a_i - b_j\| \leq diam(M)$$

and
$$\|x - y\| \leq \sum_{i=0}^{s} \alpha_i \|a_i - y\| \leq diam(M).$$

Taking supremum over all $x, y \in N$, we have $diam(N) \leq diam(M)$. □

4-7.8. **Exercise** Prove that the diameter of a simplex is the maximum length of its edges.

4-7.9. **Theorem** For every simplicial complex K, we have

$$mesh(sd^m K) \leq \left(\frac{p}{p+1}\right)^m mesh(K)$$

where p denotes the dimension of K.

Proof. Let A be a simplex in K with vertices $a_0, a_1, \cdots, a_q$. A small simplex of A is of the form: $B = co\{bar(A_j) : 0 \leq j \leq k\}$ where

$$A_0 < A_1 < \cdots < A_k \le A.$$

Any two vertices of B can be expressed by

$$x = \frac{1}{r+1}\sum_{j=0}^{r} a_j \qquad \text{and} \qquad y = \frac{1}{r+s+1}\sum_{j=0}^{r+s} a_j.$$

Routine calculation gives $\quad x - y = \dfrac{s}{r+s+1}(u-v)$

where $\quad u = \dfrac{1}{r+1}\sum_{j=0}^{r} a_j \quad$ and $\quad v = \dfrac{1}{s}\sum_{j=r+1}^{r+s} a_j.$

As averages of vertices of A, both u, v belong to A. Hence

$$\|x-y\| \le \frac{s}{r+s+1} diam(A) \le \frac{r+s}{r+s+1} diam(A) \le \frac{p}{p+1} diam(A).$$

It follows that

$$diam(B) \le \frac{p}{p+1} diam(A) \le \frac{p}{p+1} mesh(K).$$

Since $B \in sd(K)$ is arbitrary, we have

$$mesh[sd(K)] \le \frac{p}{p+1}\ mesh(K).$$

The general case follows by induction on m. □

4-7.10. **Exercise** Let K be a simplicial complex. Prove that for every $\varepsilon > 0$ there is an integer r such that for all $m \ge r$ we have $mesh(sd^m K) \le \varepsilon$.

4-8 Simplicial Approximations

4-8.1. Let K, L be simplicial complexes in $\mathbb{R}^n, \mathbb{R}^m$ respectively. A map $f : |K| \to |L|$ is called a *simplicial map* or a *piecewise linear map* if for every simplex $A \in K$, the restriction $f|A$ is an affine map *onto* a simplex of L.

4-8.2. **Theorem** A given map $g : ver(K) \to ver(L)$ can be extended to a simplicial map f from K into L iff for each simplex $A \in K$, the set $g[ver(A)]$ consists of vertices of a simplex in L.

Proof. ($\Rightarrow$) Let A be a simplex in K. Then $B = f(A)$ is a simplex in L. Since for each vertex v of A the singleton $\{v\}$ is a simplex in K, $\{f(v)\}$ is a simplex in L and $\{f(v)\} = \{f(v)\} \cap B$ is a face of B, i.e. $g(v) = f(v)$ is a vertex of B.
($\Leftarrow$) Let f be the unique extension of g such that $f|A$ is an affine map for every $A \in K$. Take any $A \in K$. There is $B \in L$ such that $f[ver(A)] = ver(B)$. Hence $f(A) = B$. Therefore f is a simplicial map. □

4-8.3. **Exercise** Prove that composites of simplicial maps are simplicial maps.

4-8.4. **Exercise** Prove that for every simplicial complex K there is a simplicial map f from $sd(K)$ into K such that for every simplex A of K, we have $f[bar(A)] \in ver(A)$.

4-8.5. **Theorem** If f is a simplicial map from K into L, then there is a unique simplicial map $sd(f)$ from $sd(K)$ into $sd(L)$ such that for every simplex $A \in K$ we have $sd(f)[bar(A)] = bar[f(A)]$.

Proof. For each $A \in K$, $f(A)$ is a simplex in L and hence $bar[f(A)]$ is a vertex of $sd(L)$. Define $g[bar(A)] = bar[f(A)]$. When A runs over all simplexes of K, we define a map $g : ver[sd(K)] \to ver[sd(L)]$. Suppose $A_0 \leq A_1 \leq \cdots \leq A_m$ in K. Then $f(A_0) \leq f(A_1) \leq \cdots \leq f(A_m)$ is a chain of simplexes in L. Therefore the vertices of a small simplex of K are carried to vertices of a small simplex in L. Therefore g can be extended uniquely to a simplicial map $sd(f) : sd(K) \to sd(L)$. □

4-8.6. Let K, L be simplicial complexes in $\mathbb{R}^n, \mathbb{R}^m$ respectively and let $f : |K| \to |L|$ be a given map. Then a simplicial map $g : K \to L$ is called a *simplicial approximation* to f if for every $x \in |K|$ we have $g(x) \in car f(x)$.

4-8.7. **Theorem** Let $f : |K| \to |L|$ be a given map and $g : K \to L$ a simplicial map. Then the following statements are equivalent.

(a) g is a simplicial approximation to f.

(b) For every vertex a of K, we have $f[star(a)] \subset star[g(a)]$.

(c) For each $x \in |K|$ and each $B \in L$, if $f(x) \in B$ then $g(x) \in B$.

Proof. $(a \Rightarrow b)$ Assume that $g(x) \in car[f(x)]$ for all $x \in |K|$. Take any $x \in star(a)$ where a is a vertex of K. Then $a \in car(x)$. Let $\{a, a_1, a_2, \cdots, a_k\}$ be the vertices of $car(x)$. Then we have $x = \alpha a + \sum_{j=1}^{k} \alpha_j a_j$, all $\alpha, \alpha_j > 0$ and $\alpha + \sum_{j=1}^{k} \alpha_j = 1$. Thus $g(x) = \alpha g(a) + \sum_{j=1}^{k} \alpha_j g(a_j)$, i.e. $g(a)$ is a vertex of $car[g(x)]$. Now $g(x) \in car[f(x)]$, i.e. $car[g(x)] \subset car[f(x)]$, or $g(a) \in car[f(x)]$, that is $f(x) \in star[g(a)]$. Because $x \in star(a)$ is arbitrary, we obtain $f[star(a)] \subset star[g(a)]$.

$(b \Rightarrow a)$ Assume $f[star(a)] \subset star[g(a)]$ for every vertex a of K. Take any $x \in |K|$. Let $\{a_0, a_1, \cdots, a_k\}$ be the vertices of the carrier simplex of x. Then $x = \sum_{j=0}^{k} \alpha_j a_j$ where all $\alpha_j > 0$ and $\sum_{j=0}^{k} \alpha_j = 1$. Since each $a_j \in car(x)$, we have $x \in star(a_j)$, or $f(x) \in f[star(a_j)] \subset star[g(a_j)]$, i.e. $g(a_j) \in car[f(x)]$. Therefore $g(x) = \sum_{j=0}^{k} \alpha_j g(a_j) \in car[f(x)]$.

$(a \Leftrightarrow c)$ It follows immediately from the definition of carrier simplexes. □

4-8.8. **Exercise** Let K, M, N be simplicial complexes. If $\varphi : K \to M$ and $\psi : M \to N$ are simplicial approximations of given maps $f : |K| \to |M|$ and $g : |M| \to |N|$ respectively, then the composite map $\psi\varphi$ is a simplicial approximation of the composite map gf.

4-8.9. **Exercise** Show that if g is a simplicial map from $sd(K)$ into K such that for every simplex A of K we have $g[bar(A)] \in ver(A)$, then g is a simplicial approximation to the identity map.

4-8.10. **Lemma** Let S be a set of vertices of a simplicial complex K. Then the convex hull $co(S)$ of S is a simplex of K iff the intersection $\bigcap\{star(a) : a \in S\}$ is non-empty. Note that the elements of S need not be all distinct.

Proof. ($\Rightarrow$) Take any geometric interior point x of $co(S)$. Then $car(x) = co(S)$. Hence for all $a \in S$, we have $a \in car(x)$, i.e. $x \in star(a)$. Therefore we obtain $x \in \cap\{star(a) : a \in S\}$.
($\Leftarrow$) Suppose $x \in \cap\{star(a) : a \in S\}$. Then for all $a \in S$, we have $x \in star(a)$, i.e. $a \in car(x)$. Hence $co(S)$ is a face of $car(x)$. Therefore $co(S)$ is in K. □

4-8.11. **Theorem** Let f be a given map from $|K|$ into $|L|$. If for every vertex $a \in K$ there is a vertex $b \in L$ such that $f[star(a)] \subset star(b)$, then f admits the simplicial approximation g defined by $g(a) = b$ for every $a \in ver(K)$.

Proof. For each $a \in ver(K)$ define $g(a) = b$. Now suppose that A is a simplex of K. Then there is $x \in \cap\{star(a) : a \in ver(A)\}$. For each vertex a of A, we have $x \in star(a)$, or $f(x) \subset f[star(a)] \subset star[g(a)]$. Therefore $f(x) \in \cap\{star[g(a)] : a \in ver(A)\}$. Thus $co\{g(a) : a \in ver(A)\}$ is a simplex in L. Consequently g can be extended to a simplicial map from K to L. Since $f[star(a)] \subset star[g(a)]$ for each vertex a of K, g is a simplicial approximation to f. □

4-9 Existence of Simplicial Approximations

4-9.1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Choose any subdivision $a = x_0 < x_1 < \cdots < x_k = b$. Then the broken line obtained by joining consecutive points $(x_j, f(x_j))$ is a simplicial approximation of f. We get better approximation when the mesh of the subdivision becomes smaller. To extend it to finite dimensional spaces, we need a tool called Lebesgue number.

4-9.2. **Theorem** Let $\{D_j : 1 \leq j \leq m\}$ be an open cover of a compact metric space X. Then there is $\delta > 0$ such that for every $x \in X$ we have

$\mathbb{B}(x, \delta) \subset D_j$ for some j. In this case, δ is called a *Lebesgue number* of the open cover $\{D_j : 1 \leq j \leq m\}$.

Proof. Let $E_j = X \setminus D_j$. For every $(x_1, x_2, \cdots, x_m) \in \Pi_{j=1}^m E_j$, define

$$f(x_1, x_2, \cdots, x_m) = \max_{1 \leq i,j \leq m} d(x_i, x_j).$$

Then f is a continuous real-valued function on the compact set $\Pi_{j=1}^m E_j$. There exist $a_j \in E_j$ such that $f(a_1, a_2, \cdots, a_m) \leq f(x_1, x_2, \cdots, x_m)$ for all $x_j \in E_j$. Let $\delta = \frac{1}{3} f(a_1, a_2, \cdots, a_m)$. We have $\delta > 0$ otherwise $d(a_i, a_j) = 0$ for all i, j and hence $a_1 = a_2 = \cdots = a_m \in \bigcap_{j=1}^m E_j$, i.e. $a_1 \notin D_j$ for all j which is a contradiction. We claim that δ is what we want. Suppose to the contrary that there is $y \in X$ such that for every $1 \leq j \leq m$, $\mathbb{B}(y, \delta) \not\subset D_j$, i.e. there is $x_j \in E_j \cap \mathbb{B}(y, \delta)$. Hence $d(x_i, x_j) \leq d(x_i, y) + d(y, x_j) \leq 2\delta$ for all i, j. Therefore $3\delta \leq f(x_1, x_2, \cdots, x_m) \leq 2\delta$ which is a contradiction to complete the proof. □

4-9.3. **Theorem** Let K, L be simplicial complexes of $\mathbb{R}^n, \mathbb{R}^m$ respectively. Let r be a Lebesgue number for the open cover $\{star(b) : b \in ver(L)\}$ of the compact space $|L|$. Then there is $\delta > 0$ such that for every subdivision N of K with $mesh(N) < \delta$ and for all continuous maps $f, g : |K| \to |L|$ satisfying $\|f(x) - g(x)\| < r$ for each $x \in |K|$, both f, g admit a common simplicial approximation from N into L.

Proof. To each vertex b of L, let $V(b) = f^{-1}[star(b)] \cap g^{-1}[star(b)]$. To show that $\{V(b) : b \in ver(L)\}$ is a cover of $|K|$, take any $x \in |K|$. By the choice of r, we get $\mathbb{B}[f(x), r] \subset star(b)$ for some $b \in ver(L)$. From $\|f(x) - g(x)\| < r$ we have $f(x), g(x) \in \mathbb{B}[f(x), r] \subset star(b)$, i.e. $x \in f^{-1}[star(b)] \cap g^{-1}[star(b)] = V(b)$. Therefore $\{V(b) : b \in ver(L)\}$ is a cover of $|K|$. The continuity of f, g ensures that all $V(b)$ are open in $|K|$. Let $\delta > 0$ be a Lebesgue number of the open cover $\{V(b) : b \in ver(L)\}$ for the compact space $|K|$ and let N be a subdivision of K with $mesh(N) < \delta$. Now take any vertex a of N. Then $\mathbb{B}(a, \delta) \subset V(b)$ for some vertex b of L. Suppose $x \in star(a)$. Then $a \in car(x)$. Hence we get $\|x - a\| \leq mesh(N) < \delta$, i.e. $x \in \mathbb{B}(a, \delta)$, or $x \in V(b)$. Therefore $star(a) \subset V(b)$, i.e. $f[star(a)] \subset star(b)$ and $g[star(a)] \subset star(b)$. Consequently f, g admit the common simplicial approximation defined by $h(a) = b$ for each vertex a of K. □

4-9.4. **Corollary** Let K, L be simplicial complexes of $\mathbb{R}^n, \mathbb{R}^m$ respectively and let $f : |K| \to |L|$ be a continuous map. Then there is $\delta > 0$ such that for every subdivision N of K with $mesh(N) < \delta$, the map f admits a simplicial approximation from N into L.

4-9.5. Let g, h be simplicial maps from K into L. Then g, h are said to be *contiguous* if for every simplex A of K, both $g(A), h(A)$ are faces of a common simplex of L.

4-9.6. **Theorem** Every two simplicial approximations g, h of a given map f from K into L are contiguous.

Proof. Take any simplex $A \in K$. Choose any $x \in \cap\{star(a) : a \in ver(A)\}$. Then we have $f(x) \in f[star(a)] \subset star[g(a)]$, i.e.

$$f(x) \in \left(\bigcap_{a \in ver(A)} star[g(a)]\right) \cap \left(\bigcap_{a \in ver(A)} star[h(a)]\right).$$

Consequently the vertices $g(a), h(a)$ for all $a \in ver(A)$ span a simplex $B \in L$. Hence $g(A), h(A)$ are faces of a common simplex $B \in L$. □

4-9.7. **Exercise** Let K, L be simplicial complexes in $\mathbb{R}^n, \mathbb{R}^m$ respectively and let $h : |K| \times [0,1] \to |L|$ be a continuous maps. Write $h_t(x) = h(x,t)$ for all $(x,t) \in |K| \times [0,1]$. Show that there is $\delta > 0$ such that for every subdivision N of K with $mesh(N) < \delta$, there are simplicial maps $g_1, g_2, \cdots, g_k$ from N into L such that g_1, g_k are simplicial approximations of h_0, h_1 respectively and g_{j-1}, g_j are contiguous for each $2 \le j \le k$.

4-10 A Combinatorial Lemma with Application

4-10.1. The main result of this section is §4-10.8 on intersections of closed subsets of unit spheres. It forms the foundation of topological method of next chapter.

4-10.2. Throughout this section, we shall work on $\mathbb{R}_1^{n+1}$ with norm given by $\|x\| = |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$ for every $x = (x_1, x_2, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. Let $\{e_1, e_2, \cdots, e_n, e_{n+1}\}$ denote the standard basis of $\mathbb{R}^{n+1}$. The family of all simplexes of the form $co\{\pm e_1, \pm e_2, \cdots, \pm e_n, \pm e_{n+1}\}$ together with all their faces is called the *basic triangulation* of the unit sphere $S^n = \{x \in \mathbb{R}_1^{n+1} : \|x\| = 1\}$ of $\mathbb{R}_1^{n+1}$. Let K be an iterated barycentric subdivision of the basic triangulation. Then K is a triangulation of S^n which is symmetrical with respect to the origin. For each vertex $v \in K$, let $h(v)$ be an integer, called *label*, selected from $\pm 1, \pm 2, \cdots, \pm m$ such that the following conditions hold:

(a) $h(u) + h(v) \neq 0$ for every adjacent vertices of each simplex in K, i.e. h is a simplicial map from K into the basic triangulation of S^m.

(b) $h(-v) = -h(v)$ for every vertex $v \in K$, i.e. the antipodal condition.

4-10.3. A simplex of K is said to be *positive* if its labels can be listed as

$$+i(1), -i(2), +i(3), -i(4), \cdots, (-1)^n i(n), (-1)^{n+1} i(n+1)$$

where $1 \leq i(1) < i(2) < \cdots < i(n) < i(n+1) \leq m$. Note that this positive simplex has exactly on positive facet labelled by

$$+i(1), -i(2), +i(3), -i(4), \cdots, (-1)^n i(n).$$

A simplex of K is said to be *negative* if its labels can be listed as

$$-i(1), +i(2), -i(3), +i(4), \cdots, (-1)^{n-1} i(n), (-1)^n i(n+1)$$

where $1 \leq i(1) < i(2) < \cdots < i(n) < i(n+1) \leq m$. Note that this negative simplex has exactly on positive facet labelled by

$$+i(2), -i(3), +i(4), \cdots, (-1)^{n-1} i(n), (-1)^n i(n+1).$$

A simplex is said to be *neutral* if it is neither positive nor negative.

4-10.4. **Lemma** A neutral simplex has either none or exactly two positive facets.

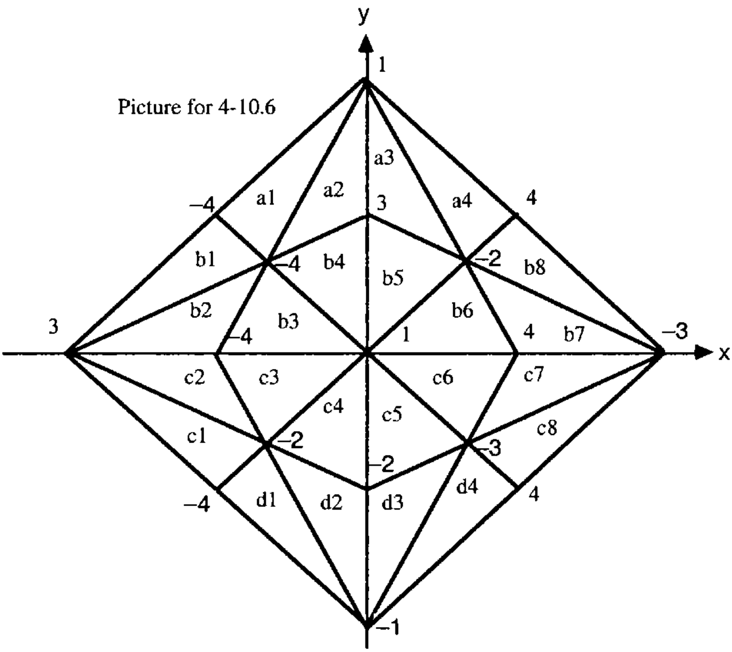
Proof. Assume that a neutral simplex A has one positive facet B labelled by

$$+i(1), -i(2), +i(3), -i(4), \cdots, (-1)^n i(n)$$

where $1 \leq i(1) < i(2) < \cdots < i(n)$. Let j be the label of the only vertex v of A which does not belong to B. We shall exhaust all cases as follow. Suppose $|j| < i(1)$. Since A is neutral, j must be positive. Hence A has exactly one more positive facet by replacing the first vertex by v. Suppose $i(p) \leq |j| < i(p+1)$. If $i(p)$ and j have the same sign, then A has exactly one other positive facet by replacing the vertex of B labelled by $(-1)^p i(p)$ with v. If $i(p)$ and j have opposite signs, then A has exactly one other positive facet by replacing the vertex of B labelled by $(-1)^{p+1} i(p+1)$ with v. Suppose $i(p) < |j| \leq i(p+1)$. If $i(p+1)$ and j have the same sign, then A has exactly one other positive facet by replacing the vertex of B labelled by $(-1)^{p+1} i(p+1)$ with v. If $i(p+1)$ and j have opposite signs, then A has exactly one other positive facet by replacing the vertex of B labelled by $(-1)^p i(p)$ with v. Suppose $|j| > i(n)$. Since A is neutral, j must have the sign $(-1)^n$. Hence A has exactly one more positive facet by replacing the last vertex of B by v.□

4-10.5. **Combinatorial Lemma** The total number of positive n-simplexes in S^n is odd. In particular, it is necessary to have $m \geq n+1$.

Proof. For convenience, the *parity* of an integer k is defined to be 1 if k is odd and to be 0 if k is even. Since the labels on S^n satisfy the antipodal condition, every positive n-simplex in the lower hemisphere corresponds to exactly one negative n-simplex in the upper hemisphere. Therefore the total number β_n of positive n-simplexes of S^n is the number of positive and negative n-simplexes in the upper hemisphere only. Now consider each n-simplex A of the upper hemisphere. Each positive or negative A gives exactly one positive facet and each neutral A gives either none or two positive facets. Hence β_n has the same parity of the total number of positive facets F of all n-simplexes in the upper hemisphere. If F is not on the boundary of the upper hemisphere, it is the facet of exactly two n-simplexes in the upper hemisphere. Therefore the total number β_n of positive n-simplexes has the same parity of the total number β_{n-1} of positive $(n-1)$-simplexes on the equator S^{n-1} of S^n. Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. By induction, β_n has the same parity of the total number β_0 of 0-simplexes of S^0 which consists of exactly two vertices. The antipodal condition shows that exactly one vertex of S^0 has positive label. Therefore the parity of β_n is one, i.e. β_n is an odd integer. □



4-10.6. **Example** The above picture is the top view of the upper hemisphere in $\mathbb{R}^3_1$ with a single barycentric subdivision $sd(S^2)$ of the basic triangulation. The triangles $a4$ is positive because its labels are $1, -2, 4$ which gives one positive facet with labels $1, -2$. The triangle $c1$ is negative with labels $-2, 3, -4$ which gives one positive facet with labels $3, -4$. The neutral triangles $b4$ gives two positive facets labelled by $1, -4$ and $3, -4$. The neutral triangle $d3$ has no positive facet at all.

4-10.7. **Exercise** Find the total number of positive and negative triangles of the upper hemisphere of the above graph. Supposing that the labelling rule satisfies the antipodal condition, sketch a graph to show all labels on the lower hemisphere. Find the total number of positive line segments of the unit sphere S^1 on the xy-plane.

4-10.8. **Theorem** Let $M_1, M_2, \cdots, M_n, M_{n+1}$ be closed subsets of the n-sphere $S^n = \{x \in \mathbb{R}_1^{n+1} : \|x\| = 1\}$. For each j, let $-M_j = \{-x : x \in M_j\}$. Suppose $M_j \cap (-M_j) = \emptyset$ for every j. If $S^n \subset \bigcup_{j=1}^{n+1}(M_j \cup -M_j)$, then the intersection $\bigcap_{j=1}^{n+1} M_j$ is non-empty.

Proof. Let d_j denote the distance between the sets $M_j, -M_j$. Because $M_j \cap (-M_j) = \emptyset$, the sets M_j, $-M_j$ are disjoint compact sets and hence $d_j > 0$. Suppose to the contrary that $\cap_{j=1}^{n+1} M_j = \emptyset$. Then $\{S^n \setminus M_j : 1 \leq j \leq n+1\}$ is an open cover of the compact space S^n. Let $r > 0$ be the Lebesgue number for this open cover. Let K be an iterated barycentric subdivision of S^n so that $mesh(K) < \min\{r, d_1, d_2, \cdots, d_{n+1}\}$. For each vertex $v \in K$, let $j > 0$ be the smallest integer such that $v \in M_j \cup -M_j$. Define $h(v) = j(-1)^{j+1}$ if $v \in M_j$ and $h(v) = j(-1)^j$ otherwise. It is easy to verify that $h(-v) = -h(v)$. Next, take any adjacent vertices u, v of a simplex in K. Suppose to the contrary that if $h(u) + h(v) = 0$ then one of them belongs to M_j while another to $-M_j$ and consequently we get a contradiction: $\|u - v\| \geq d_j > mesh(K) \geq \|u - v\|$. Therefore $h(u) + h(v) \neq 0$. It follows from the combinatorial lemma that there is at least one positive simplex A in K. Arrange the vertices $v_1, v_2, \cdots, v_{n+1}$ of A such that their labels are listed as: $1, -2, 3, -4, \cdots, (-1)^n(n+1)$, that is $h(v_j) = j(-1)^{j+1}$, or $v_j \in M_j$ for every j. Since $diam(A) \leq mesh(K) < r$, we have $A \subset S^n \setminus M_j$ for some j. In particular, $v_j \notin M_j$. This contradiction establishes the proof. □

4-99. **References and Further Readings** : Fan-60,90,99, Dugundji-82, Wolsey, Foster, Hoang, Seki, Pontryagin, Shashkin, Steenrod, Kearfott, Kulpa, Mara, Todd, Talman, Eaves, Atiyah and Bytheway.

Chapter 5

Topological Fixed Points

5-1 Antipodal Maps

5-1.1. One dimensional intermediate value theorem says that if a continuous function $f : [a, b] \to \mathbb{R}$ satisfies $f(a)f(b) < 0$ then the equation $f(x) = 0$ has at least one solution. Clearly the function $\varphi(x) = \frac{1}{2}(b-a)x + \frac{1}{2}(a+b)$ is a homeomorphism from $[-1, 1]$ onto $[a, b]$. Hence $f(x) = 0$ has a solution in $[a, b]$ iff $f\varphi(x) = 0$ has a solution. Therefore the domain of f can be standardized to $[-1, 1]$ which is the one dimensional closed unit ball. Furthermore, working with the y-axis, we may standardize f to satisfy $f(-1) = -f(1)$. This motivates the study of antipodal maps in this section.

5-1.2. Let E be a normed space. We shall work on the closed unit ball B and the unit sphere S of E. Any pair $x, -x$ of points in S are called *antipodal points*. For any subset M of S, its *antipodal set* is defined as $-M = \{x \in E : -x \in S\}$. Clearly $-M$ is a subset of S. A map f from B or S into E is said to be *antipodal* if $f(-x) = -f(x)$ for all $x \in S$. We start with the special normed space $\mathbb{R}_1^{n+1}$ of which the unit sphere is denoted by S^n. If no specific norm on $\mathbb{R}^n$ is mentioned explicitly, $\mathbb{R}_2^n$ will be assumed.

5-1.3. **Lemma** Let $M_1, M_2, \cdots, M_n$ be n-closed subsets of the sphere S^n. If they together with their antipodal sets cover S^n, then at least one of them contains a pair of antipodal points.

Proof. Define $M_{n+1} = -M_n$. Suppose to the contrary that none of $M_1, \cdots, M_n$ contains a pair of antipodal points. By §4-10.8, we have $\bigcap_{j=1}^{n+1} M_j \neq \emptyset$, i.e. $M_n \cap -M_n = M_n \cap M_{n+1} \neq \emptyset$ which is a contradiction. □

5-1.4. **Borsuk-Ulam Antipodal Theorem** For every continuous map $f : S^n \to \mathbb{R}_1^n$, there is $x \in S^n$ such that $f(-x) = f(x)$.

Proof. Suppose to the contrary that for every $x \in S^n$, we have $f(-x) \neq f(x)$, i.e. $f_j(x) - f_j(-x) \neq 0$ for some $1 \leq j \leq n$ where $f_1, f_2, \cdots, f_n$ are the coordinate functions of f. By compactness of S^n, we obtain

$$\lambda = \inf_{x \in S^n} \max_{1 \le j \le n} |f_j(x) - f_j(-x)| > 0.$$

Define $M_j = \{x \in S^n : f_j(x) - f_j(-x) \ge \frac{1}{2}\lambda\}$. Then for each $x \in S^n$, we get $\max_{1 \le j \le n} |f_j(x) - f_j(-x)| \ge \lambda > \frac{1}{2}\lambda$. Hence $|f_j(x) - f_j(-x)| > \frac{1}{2}\lambda$ for some j, that is $x \in M_j$ or $x \in -M_j$. Therefore $\{\pm M_j : 1 \le j \le n\}$ is a closed cover of S^n. Accordingly, at least one M_k contains a pair of antipodal points $x, -x$ of S^n. Now the contradiction $f_k(x) - f_k(-x) \ge \frac{1}{2}\lambda$ and $f_k(-x) - f_k(x) \ge \frac{1}{2}\lambda$ completes the proof. □

5-1.5. Base on Borsuk-Ulam Theorem, at any time you can always find two antipodal points of the earth with the same temperature and atmospheric pressure. No matter how you squeeze a basketball onto a plane, at least a pair of antipodal points will sit on top of each other.

5-1.6. **Intermediate Value Theorem for $\mathbb{R}^n_1$** If f is a continuous antipodal map on the closed unit ball B of $\mathbb{R}^n_1$ into $\mathbb{R}^n_1$, then the equation $f(x) = 0$ has at least one solution.

Proof. For every $y = (y_1, \cdots, y_n, y_{n+1})$ in $\mathbb{R}^{n+1}$, let $x = (y_1, \cdots, y_n)$. Define $g : S^n \to \mathbb{R}^n$ by

$$g(y) = \begin{cases} f(x), & \text{if } y_{n+1} \ge 0; \\ -f(-x), & \text{if } y_{n+1} \le 0. \end{cases}$$

Since f is antipodal, g is well-defined on the equator S^{n-1}. It is continuous on the upper and lower hemisphere respectively. It follows from Glue-Theorem that g is continuous on S^n. By Borsuk-Ulam Theorem, we have $g(y) = g(-y)$ for some $y \in S^n$. Thus $f(x) = -f(x)$ if $y_{n+1} \ge 0$ and $f(-x) = -f(-x)$ otherwise. In both cases, the given equation $f(x) = 0$ has a solution. □

5-1.7. As a stepping stone to generalize these results to infinite dimensional cases, it is necessary to free the restrictions of dependence on the specific norm of $\mathbb{R}^n_1$.

5-1.8. Let E, F be normed spaces. If $g : E \to F$ a topological isomorphism, then the *scaling homeomorphism* of g is the map $h : E \to F$ defined by

$$h(x) = \begin{cases} \dfrac{g(x)}{\|g(x)\|}\|x\|, & \text{if } x \ne 0, \\ 0, & \text{if } x = 0. \end{cases}$$

5-1.9. **Lemma** (a) h is a homeomorphism.
(b) h^{-1} is the scaling homeomorphism of g^{-1}.

(c) $\|h(x)\| = \|x\|$ and $h(\lambda x) = \lambda h(x)$ for all $x \in E$ and all $\lambda \in \mathbb{K}$.

(d) h carries the unit sphere of E onto the unit sphere of F and the closed unit ball onto the closed unit ball.

Proof. Let $k : F \to E$ be the scaling homeomorphism of g^{-1} given by

$$k(y) = \begin{cases} \dfrac{g^{-1}(y)}{\|g^{-1}(y)\|}\|y\|, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases}$$

By simple substitution, kh is the identity map of E and hk the identity map of F. Hence h is bijective and $h^{-1} = k$. Part (c) follows from definition and (d) is a consequence of (c). Clearly h is continuous at every $a \neq 0$ in E. By (c), h is also continuous at $a = 0$. By symmetry, k is continuous too. Therefore h is a homeomorphism. □

5-1.10. **Example** For each vector (z, w) in $\mathbb{C}^2$, write $z = x + iy$ and $w = u + iv$ where x, y, u, v are all real. Restricting to real scalar multiplication only, $\mathbb{C}^2$ is a real vector space with basis $(1,0)$, $(i,0)$, $(0,1)$ and $(0,i)$. The map $f(v,w) = (x,y,u,v)$ is a real algebraic isomorphism from $\mathbb{C}^2$ onto $\mathbb{R}^4$ and therefore it is a topological isomorphism from $\mathbb{C}^2_2$ onto $\mathbb{R}^4_p$ for any $1 \le p \le \infty$.

5-1.11. Since the scalar multiplication of this chapter does not come into effect explicitly, we shall assume that all normed spaces are over the real field.

5-1.12. **Finite Dimensional Antipodal Theorem** Let E be a *finite dimensional* normed space and let f be a continuous antipodal map from the unit sphere S of E into a vector subspace M of E. If $M \neq E$, then there is some $x \in S$ such that $f(x) = f(-x)$.

Proof. Take any $b_0 \in E \setminus M$. Since E is finite dimensional, so is M. Let $b_1, \cdots, b_k$ be a basis for M. Then $b_0, b_1, \cdots, b_k$ are linearly independent. Extend it to a basis $b_0, b_1, \cdots, b_k, b_{k+1}, \cdots, b_n$ for E. Let $e_1, e_2, \cdots, e_n, e_{n+1}$ be the standard basis for $\mathbb{R}^{n+1}$. Then there is an algebraic isomorphism $g : \mathbb{R}^{n+1} \to E$ such that $g(e_{n+1}) = b_0$ and $g(e_j) = b_j$ for all $1 \le j \le n$. Let $h : \mathbb{R}^{n+1}_1 \to E$ be the scaling homeomorphism of the topological isomorphism $g : \mathbb{R}^{n+1}_1 \to E$. Then the map $\varphi = h^{-1} f h : S^n \to R^{n+1}$ is continuous. Because $h(-x) = -h(x)$, φ is an antipodal map. Take any $a \in S^n$. Since $fh(a)$ is a linear combination of $b_1, b_2, \cdots, b_k$, $\varphi(a) = h^{-1} f h(a)$ is a linear combination of $e_1, e_2, \cdots, e_k$ and hence $\varphi(a)$ belongs to $\mathbb{R}^n$. By Borsuk-Ulam Theorem, $\varphi(-a) = \varphi(a)$ for some $a \in S^n$. Now $x = h(a) \in S$ satisfies $f(-x) = f(x)$. □

5-1.13. **Finite Dimensional Intermediate Value Theorem** Let B be the unit ball of a *finite dimensional* normed space E. If $f : B \to E$ is an antipodal continuous map, then the equation $f(x) = 0$ has at least one solution.

Proof. It is left as an exercise. □

5-1.14. **Finite Dimensional Parallel Vector Theorem** Let $f : B \to E$ be a continuous map such that $f(x) \neq 0$ on B. Then there is a pair of antipodal points $a, -a \in S$ such that $f(a), f(-a)$ are parallel vectors.

Proof. For each $x \in B$, define $g(x) = \dfrac{f(x)}{\|f(x)\|} - \|x\| \dfrac{f(-x)}{\|f(-x)\|}$. Then for every $x \in S$, we have $g(-x) = -g(x)$. Also for $\|x\| < 1$, $g(x) \neq 0$. By intermediate-value theorem, there is $x \in S$ such that $g(x) = 0$, i.e. the vectors $f(x), f(-x)$ are parallel. □

5-2 Retracts and Fixed Points

5-2.1. In this section, we shall remove the restriction to closed unit ball as domains of nonlinear maps and replace it by convex sets. As a result, the notion of being antipodal is no longer available. Consequently intermediate value theorem will take the form of fixed points. Retracts is the tool to achieve this transformation. To derive Brouwer's fixed point theorem, we shall give an explicit formula rather demanding the intuition of readers to accept the continuity of a certain map. Only elementary calculations are used to show that closed convex sets are retracts. Although our result is more restrictive but sufficient for our purpose, it would make the subject more accessible to all undergraduates.

5-2.2. Let M be a subset of a metric space X. A continuous map $f : X \to M$ is called a *retraction* if for every $x \in M$ we have $f(x) = x$. In particular, f must be surjective. In this case, M is also called a *retract* of X. Clearly if M is a retract of X then M is also a retract of every subset of X containing M.

5-2.3. **Lemma** In a finite dimensional normed space, the unit sphere S is not a retract of the closed unit ball B.

Proof. Suppose to the contrary that $f : B \to S$ is a retraction. Since the identity map $f|S$ is antipodal, the equation $f(x) = 0$ has a solution in B which is a contradiction to $f(B) \subset S$. □

5-2.4. **Brouwer's Fixed Point Theorem** Let B_n be the closed unit ball of $\mathbb{R}_2^n$. Then every continuous map from B_n into itself has a fixed point.

Proof. Suppose to the contrary that $f : B_n \to B_n$ is a continuous map such that $f(x) \neq x$ for all $x \in B$. Then extend the line segment from $f(x)$ to x until it meets a point $g(x)$ of the unit sphere S^{n-1}. Because of the inner product of $\mathbb{R}_2^n$, we can actually find an explicit formula $g(x) = (1 + t)x - tf(x)$ where

$$t = \frac{\sqrt{< x, x - f(x) >^2 + \|x - f(x)\|^2(1 - \|x\|^2)} - < x, x - f(x) >}{\|x - f(x)\|^2}.$$

Since t is a continuous function of x, so is g. Because $g|S^{n-1}$ is the identity map, S^{n-1} is a retract of B_n. This contradiction establishes the proof. □

5-2.5. **Parallelogram Law** For all x, y in the inner product space $\mathbb{R}_2^n$, we have $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof. It is left as an exercise. □

5-2.6. **Theorem on Minimum Distance** Let M be a closed convex subset of $\mathbb{R}_2^n$. Then for every $x \in \mathbb{R}_2^n$, there is a unique $y \in M$ such that $\|x - y\|$ is the distance from x to M.

Proof. Let d be the distance from x to M. There is a sequence $\{a_k\}$ in M such that $\|x - a_k\| \leq d + \frac{1}{k}$. By Parallelogram Law, we get

$$2\|x - a_j\|^2 + 2\|x - a_k\|^2 = 4\|x - \tfrac{1}{2}(a_j + a_k)\|^2 + \|a_j - a_k\|^2.$$

Since M is convex, we have $\frac{1}{2}(a_j + a_k) \in M$ and hence $\|x - \frac{1}{2}(a_j + a_k)\| \geq d$. Consequently we obtain

$$\begin{aligned} &4d^2 + \|a_j - a_k\|^2 \leq 4\|x - \tfrac{1}{2}(a_j + a_k)\|^2 + \|a_j - a_k\|^2 \\ &= 2\|x - a_j\|^2 + 2\|x - a_k\|^2 \leq 2(d + \tfrac{1}{j})^2 + 2(d + \tfrac{1}{k})^2, \end{aligned}$$

i.e. $$\|a_j - a_k\|^2 \leq \frac{4d}{j} + \frac{4d}{k} + \frac{2}{j^2} + \frac{2}{k^2}.$$

Therefore $\{a_k\}$ is a Cauchy sequence in a complete set M. Let $y = \lim a_k \in M$. Then we have $d \leq \lim \|x - a_k\| \leq \lim(d + \frac{1}{k}) = d$, i.e. $\|x - y\| = \lim \|x - a_k\| = d$. Finally suppose $a, b \in M$ satisfy $\|x - a\| = d = \|x - b\|$. Then by Parallelogram Law again, we have

$$4d^2 = 2\|x - a\|^2 + 2\|x - b\|^2 = 4\|x - \tfrac{1}{2}(a + b)\|^2 + \|a - b\|^2 \geq 4d^2 + \|a - b\|^2,$$

i.e. $\|a - b\|^2 = 0$, or $a = b$. □

5-2.7. **Retraction Theorem** Every non-empty closed convex subset M of a finite dimensional normed space E is a retract of E.

Proof. Firstly, consider the special case when $E = \mathbb{R}_2^n$. For every $x \in \mathbb{R}^n$, let $f(x)$ be the unique point in M such that $\|x - f(x)\| = d(x, M)$. Clearly $f(x) = x$ for all $x \in M$. To prove that $f : \mathbb{R}_2^n \to M$ is continuous, take any $x, y \in \mathbb{R}_2^n$. Observe the following simple calculation:

$$\|x-y\|^2 - \|f(x) - f(y)\|^2 \qquad \#1$$
$$= < (x-y) - [f(x) - f(y)], (x-y) + [f(x) - f(y)] >$$
$$= < [x - f(x)] - [y - f(y)], [x - f(x)] - [y - f(y)] + 2f(x) - 2f(y) >$$
$$= \|x - f(x) - y + f(y)\|^2$$
$$+2 < x - f(x), f(x) - f(y) > -2 < y - f(y), f(x) - f(y) >. \qquad \#2$$

Choose any $0 < t < 1$. Since M is convex, we have $(1-t)f(x) + tf(y) \in M$. Hence

$$\|x - f(x)\|^2 = d(x, M)^2 \leq \|x - [(1-t)f(x) + tf(y)]\|^2$$
$$= \|x - f(x) + t[f(x) - f(y)]\|^2$$
$$= \|x - f(x)\|^2 + 2t < x - f(x), f(x) - f(y) > +t^2\|f(x) - f(y)\|^2.$$

Thus, $\quad 2 < x - f(x), f(x) - f(y) > +t\|f(x) - f(y)\|^2 \geq 0, \forall\ t \in (0,1),$

i.e. $$< x - f(x), f(x) - f(y) > \ \geq 0.$$

Interchanging x, y, we obtain

$$< y - f(y), f(y) - f(x) > \ \geq 0.$$

Therefore the terms of #2 are positive. Now #1 gives $\|x - y\| \geq \|f(x) - f(y)\|$. Consequently the map f is uniformly continuous on $\mathbb{R}_2^n$. Finally consider an arbitrary finite dimensional normed space E. Let $g : E \to \mathbb{R}_2^n$ be an algebraic isomorphism. Since g is also a topological isomorphism, $g(M)$ is a closed convex subset of $\mathbb{R}_2^n$. Let $f : \mathbb{R}_2^n \to g(M)$ be a retraction. Then it is simple to verify that $g^{-1}fg : E \to M$ is a retraction. □

5-2.8. **Theorem** Every continuous map f from a compact convex subset X of a finite dimensional normed space E into itself has at least one fixed point.

Proof. Firstly consider the special case when $E = \mathbb{R}_2^n$. Since X is bounded, there is $\lambda > \|x\|$ for all $x \in X$. Because the map $h : E \to E$ given by $h(x) = x/\lambda$ is a topological isomorphism, $M = h(X)$ is a closed convex subset of the unit ball B_n of $\mathbb{R}_2^n$. There is a continuous map $g : B_n \to M$ such that $g(x) = x$ for every $x \in M$. Now the continuous map $hfh^{-1}g : B_n \to B_n$ has a fixed point $z = hfh^{-1}g(z)$ in B_n. Since $z \in h(X) = M$, we have $g(z) = z$, i.e. $z = hfh^{-1}(z)$, or $h^{-1}(z) = \lambda z$ is a fixed point of f. For general case, let $\varphi : E \to \mathbb{R}_2^n$ be a topological isomorphism. Then the continuous map $\varphi f \varphi^{-1}$ on the compact convex subset $\varphi(X)$ of $\mathbb{R}_2^n$ has a fixed point $a \in \varphi(X)$. Clearly, $x = \varphi^{-1}(a)$ is a fixed point of f on X. □

5-2.9. **Example** The map $f(z) = -z$ from the closed unit circle of the complex plane onto itself is continuous but has no fixed point. Hence convexity is essential.

5-2.10. **Normal Vector Theorem** Let f be a continuous map from the closed unit ball B of a finite dimensional normed space E into E. If $f(x) \neq 0$ on B, then there exist $a, b \in S$ such that $f(a)$ is an outward normal and $f(b)$ is an inward normal.

Proof. Let $\lambda = \pm 1$ be a constant. Define $f : B \to S$ by $g(x) = \dfrac{\lambda f(x)}{\|f(x)\|}$. Then the continuous map f on the compact convex set B has a fixed point, i.e. $g(a) = a$ for some $a \in B$. Since $\|g(a)\| = 1$, $a \in S$. Therefore $f(a) = \lambda \|f(a)\| a$ is an outward normal if $\lambda = 1$ and an inward normal if $\lambda = -1$. □

5-3 Fixed Points of Compact Maps

5-3.1. Our general approach is to reduce an infinite dimensional problem to finite dimensional case through compactness and then reduce a finite dimensional problem into discrete case solved by combinatorial method. The later one has been done and now we start to look at infinite dimensional normed spaces. The main result of this section is a fixed point theorem §5-3.5.

5-3.2. Let E be a normed space. A map F from a metric space X into E is called a *compact map* if it is continuous and its range $F(X)$ is relatively compact in E. The map F is said to be *finite dimensional* if $F(X)$ is contained in some finite dimensional vector subspace of E.

5-3.3. **Finite Dimensional Approximation Theorem** If $F : X \to E$ is a compact map, then for every $\varepsilon > 0$ there is a compact map G from X into the convex hull of a finite subset of $F(X)$ such that for each $x \in X$ we have $\|F(x) - G(x)\| \leq \varepsilon$. In particular, G is a finite dimensional compact map.

Proof. Since the closure Q of $F(X)$ is compact, there are $y_1, y_2, \cdots, y_k \in F(X)$ such that $Q \subset \bigcup_{i=1}^k \mathbb{B}(y_i, \varepsilon)$. Let $\{\alpha_i : 1 \leq i \leq k\}$ be a partition of unity on Q subordinated to $\{\mathbb{B}(y_i, \varepsilon)\}$. Let $G : X \to E$ be given by $G(x) = \sum_{i=1}^k \alpha_i(x) y_i$.

Then $G(X)$ is contained in $co\{y_1, y_2, \cdots, y_k\}$ which is compact. Since each $\alpha_i : X \to [0,1]$ is continuous, G is continuous. Therefore G is a compact map. Finally for any $x \in X$, if $\|F(x) - y_i\| > \varepsilon$, then $\mu_i(x) = 0$ and so $\mu_i(x)\|F(x) - y_i\| \leq \mu_i(x)\varepsilon$ which is also true for $\|F(x) - y_i\| \leq \varepsilon$. Consequently,

$$\|F(x) - G(x)\| = \left\|\sum_{i=1}^{k} \alpha_i(x)F(x) - \sum_{i=1}^{k} \alpha_i(x)y_i\right\|$$

$$\leq \sum_{i=1}^{k} \alpha_i(x)\|F(x) - y_i\| \leq \sum_{i=1}^{k} \alpha_i(x)\varepsilon = \varepsilon. \qquad \square$$

5-3.4. **Exercise** Formalize a statement from last theorem when F is the identity map on a compact subset X of E.

5-3.5. **Theorem** Let X be a non-empty convex subset of E. Then every continuous map F on X into a compact subset of X has a fixed point.

Proof. For every integer $n \geq 1$, there is a compact map G from X into the convex hull K of a finite subset of $F(X)$ such that $\|F(x) - G(x)\| \leq \frac{1}{n}$ for each $x \in X$. Since K is a finite dimensional compact convex set, the continuous map G has a fixed point $x_n \in K$. Hence $\|F(x_n) - x_n\| = \|F(x_n) - G(x_n)\| \leq \frac{1}{n}$. Let Q be a compact subset of X containing $F(X)$. Then the sequence $F(x_n)$ in Q has a convergent subsequence, say $F(y_n) \to a \in Q$. Then

$$\|a - y_n\| \leq \|a - F(y_n)\| + \|F(y_n) - y_n\| \to 0,$$

i.e. $y_n \to a \in X$. By continuity of F, we have $F(y_n) \to F(a)$. Therefore $F(a) = a$. $\square$

5-3.6. **Exercise** Let X be the closed unit ball of ℓ_1. For every point $x = (x_1, x_2, \cdots)$ in X, define $f(x) = \left(\dfrac{1 - \|x\|}{2}, x_1, x_2, \cdots\right)$. Show that the function $f : X \to X$ is continuous but has no fixed point.

5-4 Compact Fields and their Homotopies

5-4.1. Consider the xy-plane. An equation $f(x) = 0$ works with the x-axis and $F(x) = x$ with the diagonal $y = x$. Most of the graphical presentations use x-axis as reference by sketching the curves of the form $y = f(x)$. On the other hand, if the velocity of certain flux at location x in an infinite dimensional space E is denoted by $-F(x)$, then a vector field can be represented by a localized arrow starting from x ended at $f(x) = x - F(x)$. Mathematically, in order to get sensible results as shown in subsequent sections, all $F(x)$ must be reasonably small, i.e. belong to certain compact set. In physics, the velocity $-F(x)$ of any particle is not supposed to be faster than the speed of light. Perhaps, this may be only a coincidence between mathematics and physics. Note that the choice of negative sign is merely for mathematical convenience to get the symmetry between f and F. Homotopies allow us to reduce a complicate map F at time $t = 0$ to a relatively simple one G at time $t = 1$. Hopefully we can get some information about F from G, e.g. the existence of solutions.

5-4.2. Let F be a compact map from a subset X of a normed space E into E. Clearly $x \in X$ is a fixed point of F iff $x - F(x) = 0$. The map $f : X \to E$ defined by $f(x) = x - F(x)$ on X is called the *compact field* of F. We also write $f = I - F$ where I is the identity map on E. For convenience, $x \in X$ is called a *singular point* of f if $f(x) = 0$. Similarly, let J denotes the closed unit interval $[0, 1]$. A compact map $H : X \times J \to E$ is also called a *compact homotopy* from H_0 to H_1 where $H_t(x) = H(x, t)$ for all $(x, t) \in X \times J$. The map $h(x, t) = x - H(x, t)$ on $X \times J$ is called the *field homotopy* of H. We also write $h = I - H$. An element $x \in X$ is called a *fixed point* of H or a *singular point* of h if $H(x, t) = x$ or $h(x, t) = 0$ for some $t \in J$. For convenience, the following convention will be used whenever nothing is mentioned explicitly. If a letter in upper case denotes a compact map on X or $X \times J$, then its associated compact field or the field homotopy will be denoted by the same letter but in lower case respectively. The same convention in reserved order will be used from compact fields or field homotopy to their associated compact maps.

5-4.3. We shall study a pair of closed subsets $N \subset X$ of a normed space E. The set of all compact maps on X which have *no* fixed point in N is denoted by $C(X, N)$. Two compact maps F, G on X are said to be *homotopic* in $C(X, N)$ if there is a compact map $H : X \times J \to E$ such that $H_0 = F$, $H_1 = G$ and $H(x, t) \neq x$ for all $(x, t) \in N \times J$, i.e. H is a compact homotopy on X without any fixed point in N. In symbols, write $F \simeq G$ in $C(X, N)$. For all compact maps $F, G : X \to E$, the *affine homotopy* from F to G is defined as the map $H : X \times J \to E$ given by $H(x, t) = (1 - t)F(x) + tG(x)$. For convenience, two compact fields f, g are said to be *homotopic* in $C(X, N)$ if $F \simeq G$ in $C(X, N)$ where F, G are compact maps of f, g respectively.

5-4.4. **Theorem** If f is a compact field on X, then $f(N)$ is closed in E.

Proof. Let a be a closure point of $f(N)$. There are $x_n \in N$ such that $y_n = f(x_n) \to a$ as $n \to \infty$. Since $F(x_n)$ is a sequence in the relatively compact set $F(X)$, there are integers $n(1) < n(2) < n(3) < \cdots$ such that

$F(x_{n(j)}) \to b \in E$ as $j \to \infty$. Since $y_n = f(x_n) = x_n - F(x_n)$, we have $x_{n(j)} = y_{n(j)} + F(x_{n(j)}) \to a + b$. Because N is closed, $a + b \in N$. Now $y_{n(j)} = f(x_{n(j)}) \to f(a+b)$ by continuity. As subsequence, we also have $y_{n(j)} \to a$. Hence $a = f(a+b) \in f(N)$. Therefore $f(N)$ is closed in E. □

5-4.5. **Theorem** If $h : X \times J \to E$ be a field homotopy then the set $h(N \times J)$ is closed in E.

Proof. It is an exercise to modify the proof of last theorem. □

5-4.6. **Theorem** Let δ denote the distance from the origin to $f(N)$. If a compact map G on X satisfying $\|F(x) - G(x)\| < \delta$ for every $x \in N$, then the affine homotopy H from F to G is a compact map without fixed point in N. In particular, G has no fixed point in N.

Proof. Since $f(N)$ is a closed set which does not contain the origin, we have $\delta > 0$. Let G be a compact map satisfying the given condition. If for some $x \in N$ and $t \in J$ we have $H(x,t) = (1-t)F(x) + tG(x) = x$, then $f(x) = t\{G(x) - F(x)\}$ and a contradiction is obtained as follow:

$$\delta = d[0, f(N)] \leq \|f(x)\| = t\|G(x) - F(x)\| \leq \|G(x) - F(x)\| < \delta.$$

Therefore $H(x,t) \neq x, \forall\ (x,t) \in N \times J$. Next, to show that H is a compact map, suppose that the closures of $F(X), G(X)$ are denoted by A, B respectively. Define $\lambda(a,b,t) = (1-t)a + tb$, for all $(a,b,t) \in A \times B \times J$. Then λ is continuous on the compact set $A \times B \times J$. Now $H : X \times J \to E$ is continuous and $H(X \times J)$ is a subset of the compact set $\lambda(A \times B \times J)$. Therefore H is a compact map.□

5-4.7. **Exercise** Prove that if $F \simeq G$ and $G \simeq H$ in $C(X, N)$ then $F \simeq F$, $G \simeq F$ and $F \simeq H$ in $C(X, N)$. Therefore the relation of being homotopic is an equivalent relation.

5-4.8. **Exercise** Prove that the set of all fixed points of a compact map or a compact homotopy on a closed set is compact.

5-4.9. **Theorem** Every compact map in $C(X, N)$ can be uniformly approximated by finite dimensional compact maps in $C(X, N)$ and consequently it is also homotopic to some finite dimensional compact maps in $C(X, N)$.

Proof. Modify the formal statement and proof of the following theorem. □

5-4.10. **Theorem** Let F be a compact homotopy on X without fixed point in N. Then for every $\varepsilon > 0$ there is a finite dimensional compact homotopy G on X such that

(a) $\|F(x,t) - G(x,t)\| \leq \varepsilon$ for all $(x,t) \in X \times J$;

(b) $(1-\lambda)F(x,t) + \lambda G(x,t) \neq x, \forall\ (x,t,\lambda) \in N \times J \times J$.

Proof. Let $f = I - F$ denote the field homotopy of F. Since the origin does not belong to the closed set $f(N \times J)$, we may assume $0 < \varepsilon < d[0, f(N \times J)]$. There is a finite dimensional compact map $G : X \times J \to E$ such that

$$\|F(x,t) - G(x,t)\| \leq \varepsilon, \quad \forall\ (x,t) \in X \times J.$$

Then G is a compact homotopy on X satisfying condition (a). Next, suppose to the contrary that $(1-\lambda)F(x,t) + \lambda G(x,t) = x$ for some $(x,t,\lambda) \in N \times J \times J$. Then we have $f(x,t) = \lambda\{G(x,t) - F(x,t)\}$ and so

$$0 < \varepsilon < d[0, f(N \times J)] \leq \|f(x,t)\| = \lambda\|G(x,t) - F(x,t)\| \leq \lambda\varepsilon \leq \varepsilon$$

is a contradiction. Therefore (b) also holds. □

5-4.11. **Theorem** Let F_0, F_1 be two finite dimensional compact maps on X. If F_0, F_1 are homotopic without fixed point in N, then they are homotopic under some finite dimensional homotopy without fixed point in N.

Proof. Let F be a compact homotopy for $F_0 \simeq F_1$ in $C(X,N)$. There is a finite dimensional compact homotopy G without fixed point in N such that

$$(1-\lambda)F(x,t) + \lambda G(x,t) \neq x, \forall\ (x,t,\lambda) \in N \times J \times J.$$

For $\lambda = 1$, G is a finite dimensional homotopy, i.e. $G_0 \simeq G_1$ under a finite dimensional homotopy. Next, for $t = 0$, $F_0 \simeq G_0$ in $C(X,N)$ under the affine homotopy which is finite dimensional since both F_0, G_0 are. Finally for $t = 1$, $F_1 \simeq G_1$ under a finite dimensional affine homotopy. Therefore $F_0 \simeq F_1$ in $C(X,N)$ under the finite dimensional homotopy obtained by combining the above ones. □

5-5 Extension Property

5-5.1. To make it more acceptable to wider audiences, only elementary tools such as M-test and Tietze's Extension Theorem are used to develop a version of Homotopy Extension Theorem. It is powerful enough to define the concept of singular compact fields which will be used to extend certain finite dimensional results to infinite dimensional spaces. Intuitively, if a compact field f is singular then the equation $f(x) = 0$ has at least one solution. However if a compact field g is non-singular, the equation $g(x) = 0$ may or may not have a solution.

5-5.2. **M-Test** Let $\{f_n : n \geq 1\}$ be a sequence of continuous functions on a metric space X. Suppose $|f_n(x)| \leq M_n, \forall\ x \in X$. If the series

$\sum_{n=1}^{\infty} M_n$ of positive numbers converges then the series $\sum_{n=1}^{\infty} f_n$ of functions converges uniformly to some continuous function g on X. Furthermore we have $|g(x)| \le \sum_{n=1}^{\infty} M_n$, for all $x \in X$.

Proof. Define $g_m = \sum_{n=1}^{m} f_n$. Since $|g_m(x)| \le \sum_{n=1}^{m} |f_n(x)| \le \sum_{n=1}^{\infty} M_n < \infty$, each g_m is a bounded continuous function on X. Let $\varepsilon > 0$ be given. There is k such that for every $n \ge k$ we have $\sum_{j=1}^{p} M_{n+j} \le \varepsilon$, independent of p. Now for each x in X and each $n \ge k$ we have

$$|g_{n+p}(x) - g_n(x)| \le \sum_{j=1}^{p} |f_{n+j}(x)| \le \sum_{j=1}^{p} M_{n+j} \le \varepsilon.$$

Therefore $\{g_n\}$ is a sequence which is uniformly Cauchy in $BC_\infty(X)$. It converges uniformly to a continuous function g. Letting $p \to \infty$ in last inequality, we have $|g(x) - g_n(x)| \le \varepsilon$. Hence $g_n \to g$ uniformly, or $g = \sum_{n=1}^{\infty} f_n$ uniformly on X. The proof is completed by letting $m \to \infty$ in the following inequality: $|g_m(x)| \le \sum_{j=1}^{m} |f_j(x)| \le \sum_{j=1}^{\infty} M_j$. □

5-5.3. **Lemma** Let N be a closed subset of a metric space X. If u is a continuous function from N into $[-r, r]$ then there is a continuous function v from X into $[-\frac{1}{3}r, \frac{1}{3}r]$ such that for all x in N we have $|u(x) - v(x)| \le \frac{2}{3}r$.

Proof. Let $A = \{x \in N : u(x) \le -\frac{2}{3}r\}$ and $B = \{x \in N : u(x) \ge \frac{2}{3}r\}$. Now A, B are disjoint closed subsets of X since N is closed in X. There is a continuous function $w : X \to [0, 1]$ such that $w(A) = 0$ and $w(B) = 1$. Next define $\varphi(t) = (2t - 1)\frac{r}{3}, \forall\, t \in [-1, 1]$. Then $\varphi : [-1, 1] \to [-\frac{r}{3}, \frac{r}{3}]$ is a continuous function. Now the composite $v = \varphi w$ is a continuous function from X into $[-\frac{r}{3}, \frac{r}{3}]$ such that $v(A) = -\frac{r}{3}$ and $v(B) = \frac{r}{3}$. For $x \in A$, we have $-r \le u(x) \le -\frac{2}{3}r$ and $v(x) = -\frac{1}{3}r$. Hence $|u(x) - v(x)| \le \frac{2r}{3}$. Similarly the same inequality holds for $x \in B$. Finally, for $x \in N \setminus (A \cup B)$, we obtain $|u(x) - v(x)| \le |u(x)| + |v(x)| \le \frac{r}{3} + \frac{r}{3} \le \frac{2r}{3}$. Therefore for every x in N we have $|u(x) - v(x)| \le \frac{2r}{3}$. □

5-5.4. **Tietze's Extension Theorem** Let N be a closed subset of a metric space X. Then every continuous function $F : N \to [-1, 1]$ has a continuous extension over the whole space X into $[-1, 1]$.

Proof. There is a continuous function v_1 on X such that $|v_1(x)| \le \frac{1}{3}, \forall\, x \in X$ and $|F(a) - v_1(a)| \le \frac{2}{3}, \forall\, a \in N$. Define $G_1 = v_1$. Inductively suppose v_n, G_n are continuous functions on X satisfying

$$|v_n(x)| \le \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^{n-1}, \forall\, x \in X; \qquad |F(a) - G_n(a)| \le \left(\tfrac{2}{3}\right)^{n}, \forall\, a \in N$$

and $G_n = G_{n-1} + v_n$. For $u = F - G_n$ in last lemma, there is a continuous function v_{n+1} on X such that

$$|v_{n+1}(x)| \leq \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^n, \forall\, x \in X$$

and
$$|\{F(a) - G_n(a)\} - v_{n+1}(a)| \leq \left(\tfrac{2}{3}\right)^{n+1}, \forall\, a \in N.$$

Define $G_{n+1} = G_n + v_{n+1}$. Then G_{n+1} is a continuous function on X satisfying

$$|F(a) - G_{n+1}(a)| \leq \left(\tfrac{2}{3}\right)^{n+1}, \forall\, a \in N.$$

Since $\sum_{n=1}^{\infty}(\frac{1}{3})(\frac{2}{3})^{n-1}$ is convergent, the sequence $G_n = \sum_{j=1}^{n} v_j$ converges uniformly to some continuous function G on X by M-test. Furthermore

$$|G(x)| \leq \sum_{n=1}^{\infty}\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^{n-1} = 1, \forall\, x \in X.$$

Finally letting $n \to \infty$ in $|F(a) - G_n(a)| \leq (\frac{2}{3})^n$ we have $F(a) = G(a), \forall\, a \in N$. Therefore $G : X \to [-1, 1]$ is a continuous extension of F. □

5-5.5. **Theorem** Let N be a closed subset of a metric space X. Then every compact map F from N into a finite dimensional normed space E can be extended to a compact map on the whole space X into the closed convex hull K of $F(N)$.

Proof. Let $G : E \to \mathbb{R}^n$ be a topological isomorphism. Since F is a compact map, the set K is compact and hence $G(K)$ is bounded in $\mathbb{R}^n$. Replacing G by λG for small $\lambda > 0$, we may assume that $G(K) \subset [-1, 1]^n$. By Tietze's Extension Theorem, every coordinate function of the composite $GF : N \to \mathbb{R}^n$ can be extended to a continuous function from X into $[-1, 1]^n$, i.e. GF has an extension $H : X \to [-1, 1]^n$. Let $\varphi : E \to K$ be a retraction. Then the composite map $\varphi G^{-1} H$ is a required extension of F. □

5-5.6. **Homotopy Extension Theorem** Let $N \subset X$ be closed subsets of a normed space E. Suppose that $F, G : N \to E$ are finite dimensional compact maps which are homotopic without fixed point in N. If F can be extended to a finite dimensional compact map F^* on X without fixed point in X, then G can also be extended in the same way.

Proof. Let H be a finite dimensional homotopy for $F \simeq G$ without fixed point in N. Let E_1 be a finite dimensional vector subspace containing $H(N \times J)$. Define a map K on $T = (N \times J) \cup (X \times \{0\})$ into E by $K|(N \times J) = H$ and $K|(X \times \{0\}) = F^*$. Because $H(x, 0) = F(x) = F^*(x)$, the map $K : T \to E_1$ is well-defined. In view of $K(T) = H(N \times J) \cup F^*(X \times \{0\})$, the map K is compact. Since $N \times J$ and $X \times \{0\}$ are closed in $X \times J$, the set T is also closed in $X \times J$. So the compact map $K : T \to E_1$ can be extended to a compact map on $X \times J$ which is denoted by the same letter K for convenience. We claim that the set $M = \{x \in X : \exists\ t \in J, x = K(x,t)\}$ is closed in X. In fact, take any closure point x of M. There are $x_n \in X$ and $t_n \in J$ such that $x_n = K(x_n, t_n) \to x$ as $n \to \infty$. Since J is compact, replacing by subsequence we may assume that t_n converges to some $t \in J$. By continuity, we have $K(x_n, t_n) \to K(x,t)$. Hence $x = K(x,t)$, i.e. $x \in M$. Therefore M is closed in $X \times J$. Since H is a homotopy without fixed point in N, we have $x \neq H(x,t)$ for all $(x,t) \in N \times J$. By definition, M and N are disjoint closed subsets of X. So there is a continuous function $\lambda : X \to [0,1]$ such that $\lambda(M) = 0$ and $\lambda(N) = 1$. Define $H^*(x,t) = K(x, \lambda(x)t)$ for all $(x,t) \in X \times J$. Since K is a compact map, so is H^*. Hence H^* defines a compact homotopy on X. We assert that H^* has no fixed point. Suppose to the contrary that for some $(x,t) \in X \times J$, we have $H^*(x,t) = x$, i.e. $K(x, \lambda(x)t) = x$. From $x \in M$, we obtain $\lambda(x) = 0$. It follows that $K(x, \lambda(x)t) = K(x,0) = F^*(x)$, i.e. $F^*(x) = x$ which is a contradiction. Therefore H^* is a homotopy on X without fixed point. Define $G^*(x) = H^*(x,1)$ for all $x \in X$. Clearly, G^* is a compact map on X without fixed point. Now for all $x \in N$, since $\lambda(x) = 1$ we have $G(x) = H(x,1) = K(x, \lambda(x)1) = H^*(x,1)$, i.e. $G(x) = H^*(x,1) = G^*(x)$. Hence G^* is an extension of G. □

5-5.7. Let $N \subset X$ be closed subsets of a normed space E. A compact map F on X is said to be *non-singular* in $C(X,N)$ if $F \simeq G$ in $C(X,N)$ for some finite dimensional compact map G in $C(X,X)$, i.e. G has no fixed point in X. By definition, being non-singular is homotopy invariant. More precisely, if $F_0 \simeq F_1$ in $C(X,N)$ and if F_0 is non-singular then so is F_1. A compact map on X is said to be *singular* if it is not non-singular. For convenience, a compact field on X is *singular* in $C(X,N)$ if its compact map is singular in $C(X,N)$.

5-5.8. **Theorem** Every singular compact map F in $C(X,N)$ has a fixed point in X.

Proof. Suppose to the contrary that F belongs to $C(X,X)$. Then there is a finite dimensional compact map G such that $F \simeq G$ in $C(X,X)$. Then $F \simeq G$ in $C(X,N)$ and G has no fixed point in X. Therefore F is non-singular. □

5-5.9. **Corollary** If F is a compact map in $C(X,X)$, then it is non-singular.

5-5.10. **Theorem** If a finite dimensional compact map F in $C(X,N)$ is non-singular, then $F|N$ can be extended to a finite dimensional compact map in $C(X, X)$.

Proof. Suppose $F \simeq G$ in $C(X, N)$ for some finite dimensional compact map G in $C(X, X)$. Since both F, G are finite dimensional, there is a finite dimensional compact homotopy H in $C(X, N)$ such that $H_0 = F$ and $H_1 = G$. Let E_1 be a finite dimensional vector subspace of E containing $H(X \times J)$. Restricting our attention to E_1, we have $F|(X \cap E_1) \simeq G|(X \cap E_1)$ in $C(X \cap E_1, N \cap E_1)$ and $G|(X \cap E_1)$ is an extension of $G|(N \cap E_1)$ without any fixed point in $X \cap E_1$. Homotopy Extension Theorem ensures that $F|(N \cap E_1)$ can be extended to a compact map $A : X \cap E_1 \to E_1$. Let $T = N \cup (X \cap E_1)$. Define a map $B : T \to E_1$ by $B|N = F$ and $B|(X \cap E_1) = A$. Then B is a compact map from the closed subset T of X into the finite dimensional vector space E_1 and hence it can be extended to some compact map $F^* : X \to E_1$. If $F^*(x) = x$ for some $x \in X$, then $x \in X \cap E_1$, or $x = A(x)$ which is a contradiction. Thus F^* has no fixed point in X. Therefore, F^* is a required extension of $F|N$. □

5-6 Properties of Compact Fields in Normed Spaces

5-6.1. Since every continuous map on the closed unit ball of a finite dimensional normed space is a compact map and also a compact field, this section generalize certain corresponding results from finite to infinite dimensional spaces. Thus geometrical intuition is used as motivation. Applications to nonlinear integral equations are beyond our scope.

5-6.2. **Lemma** Let A be an open subset of a normed space E and let $\overline{A}$, ∂A denote the closure and boundary of A respectively. Let $F(x) = a$ for all $x \in \overline{A}$ where $a \notin \partial A$. If F is a non-singular compact map in $C(\overline{A}, \partial A)$, then we have $a \notin A$.

Proof. Since the finite dimensional compact map F is non-singular in $C(\overline{A}, \partial A)$, the restriction $F|\partial A$ can be extended to a compact map G in $C(\overline{A}, \overline{A})$. Define

$$G^*(x) = \begin{cases} G(x), & \text{if } x \in \overline{A}; \\ a, & \text{if } x \in E \setminus A. \end{cases}$$

Now the compact map $G^* : E \to E$ has a fixed point $b = G^*(b)$. Since $G^*|\overline{A} = G$ has no fixed point, we have $b \in E \setminus A$, i.e. $a = G^*(b) = b \notin A$. □

5-6.3. **Outward Normal Theorem** Let A be an open set containing the origin of E. If $F : \overline{A} \to E$ is a compact map such that for every $x \in \partial A$ and for every

$\lambda > 1$ we have $F(x) \neq \lambda x$, then F has a fixed point in $\overline{A}$.

Proof. Define $H(x,t) = tF(x)$ for all $(x,t) \in \overline{A} \times J$. Clearly H is a compact map on $\overline{A} \times J$. Assume $H(x,1) \neq x$ for all $x \in \partial A$ otherwise F has a fixed point at ∂A. Take any $x \in \partial A$. If $0 < t < 1$, then $F(x) \neq \lambda x$ where $\lambda = \frac{1}{t}$, i.e. $H(x,t) = tF(x) \neq x$. If $t = 0$, then $H(x,t) = 0 \neq x$ because $0 \notin \partial A$. Thus H is a compact homotopy in $C(\overline{A}, \partial A)$. Now H_0 is singular and so is H_1. Therefore, $F = H_1$ has a fixed point. □

5-6.4. **Inward Normal Theorem** Let A be an open set containing the origin of E and let f be a compact field on $\overline{A}$. If $f(x) \neq 0$ for all $x \in \overline{A}$, then there is $x \in \partial A$ and $t > 0$ such that $-tx = f(x)$.

Proof. Let $F = I - f$ be the compact map associated with f. Then F has no fixed point in $\overline{A}$. So there exist $x \in \partial A$ and $\lambda > 1$ such that $F(x) = \lambda x$, i.e. $f(x) = -tx$ where $t = \lambda - 1$. □

5-6.5. We shall study the closed unit ball B and its sphere S in a normed space E. The following exercise follows immediately from the Outward Normal Theorem.

5-6.6. **Exercise** Let F be a compact map from B into E. If for each $x \in S$, there is $\lambda > 0$ such that $(1 - \lambda)x + \lambda F(x) \in B$, then F has a fixed point.

5-6.7. **Lemma** If $F : B \to E$ is an antipodal compact map, then for every $\varepsilon > 0$ there is a finite dimensional antipodal compact map $G : B \to E$ such that $\|F(x) - G(x)\| \leq \varepsilon$ for every $x \in B$.

Proof. For every $\varepsilon > 0$ there is a finite dimensional compact map $F^* : B \to E$ such that for all $x \in B$ we have $\|F(x) - F^*(x)\| \leq \frac{1}{2}\varepsilon$. For each $x \in S$, define $H^*(x) = F^*(x) + F^*(-x)$. Then $\|H^*(x)\| \leq \|F^*(x)\| + \|F^*(-x)\| \leq \varepsilon$. Let E_1 be a finite dimensional vector subspace containing $F^*(B)$. Since H^* is a continuous map from S into the finite dimensional compact convex set $E_1 \cap \overline{\mathbb{B}}(0,\varepsilon)$, it has a continuous extension $H : B \to E_1 \cap \overline{\mathbb{B}}(0,\varepsilon)$. Now the map $G : B \to E_1$ given by $G(x) = F^*(x) - \frac{1}{2}H(x)$ is a finite dimensional compact map. For $x \in S$, we get

$$G(x) + G(-x) = F^*(x) - \tfrac{1}{2}H(x) + F^*(-x) - \tfrac{1}{2}H(-x)$$

$$= F^*(x) - \tfrac{1}{2}[F^*(x) + F^*(-x)] + F^*(-x) - \tfrac{1}{2}[F^*(-x) + F^*(x)] = 0.$$

Hence G is an antipodal compact map on B. Finally for every $x \in B$, we have

$$\|F(x) - G(x)\| = \|F(x) - F^*(x)\| + \|F^*(x) - G(x)\|$$

$= \|F(x) - F^*(x)\| + \frac{1}{2}\|H(x)\| \leq \varepsilon.$ □

5-6.8. **Theorem** Every antipodal compact map F on B is singular in $C(B, S)$ and hence has a fixed point.

Proof. Let $f = I - F$ be its associated compact field and δ the distance from the origin to $f(S)$. There is a finite dimensional antipodal compact map G such that $\|F(x) - G(x)\| < \delta$ for all $x \in B$. Then $F \simeq G$ in $C(B, S)$. Suppose to the contrary that F is non-singular. Then G is also non-singular. Hence $G|S$ can be extended to a finite dimensional compact map H in $C(B, B)$. Define $h(x) = x - H(x)$ for all $x \in B$. Then $h(x) \neq 0$ for all $x \in B$. Let E_1 be a finite dimensional vector subspace of E containing $H(B)$. Applying the finite dimensional intermediate value theorem to the antipodal map $h|(B \cap E_1)$, there is some $x \in B$ such that $h(x) = 0$. This contradiction establishes the proof. □

5-6.9. **Intermediate Value Theorem** If f is an antipodal compact field on B, then we have $f(x) = 0$ for some $x \in B$.

Proof. The compact map $F = I - f$ is antipodal and hence has a fixed point x which is also a solution to the equation $f(x) = 0$. □

5-6.10. **Parallel Vector Theorem** Let f be a compact field on B. If $f(x) \neq 0$ for all $x \in B$, then there is some $x \in S$ and $\lambda > 0$ such that $f(x) = \lambda f(-x)$.

Proof. For every $(x, t) \in B \times J$, define $H(x, t) = \frac{1}{1+t}[F(x) - tF(-x)]$ and $G(x) = H(x, 1)$. It is easy to verify that H is a compact homotopy on B. Suppose that for all $x \in S$ and $\lambda > 0$ we have $f(x) \neq \lambda f(-x)$. Then H has no fixed point in S and hence is a homotopy for $F \simeq G$ in $C(B, S)$. Because the antipodal compact map G is singular in $C(B, S)$, so is F. Therefore $f(x) = 0$ for some $x \in B$. This contradiction establishes the proof. □

5-6.11. **Exercise** Prove that the map H in last theorem is a compact map.

5-6.12. **Antipodal Theorem** Let f be a compact field on S. If $f(S)$ is contained in some vector subspace $M \neq E$ then there is some $x \in S$ such that $f(x) = f(-x)$.

Proof. Suppose to the contrary that for all $x \in S$, $f(x) - f(-x) \neq 0$. Define $g(x) = \frac{1}{2}[f(x) - f(-x)]$ for each $x \in S$. Then g is an antipodal compact field on S into M. Furthermore $g(x) \neq 0$ on S. Take any $b \in S \setminus M$ and define $L = \{\lambda b : \lambda > 0\}$. Since $g(S) \subset M$, we obtain $g(S) \cap L = \emptyset$. We claim that the distance from L to $g(S)$ is strictly positive, i.e. $d[L, g(S)] > 0$. Assuming that this is false, there are $x_n \in S$ and $\lambda_n > 0$ such that $\|g(x_n) - \lambda_n b\| \le \frac{1}{n}$. Since $G = I - g$ is a compact map, replacing by subsequence we may assume that $G(x_n) \to y \in E$. Thus

$$0 \le \lambda_n \le \|\lambda_n b\| \le \|x_n\| + \|G(x_n)\| + \tfrac{1}{n} \le \alpha < \infty$$

where α is some real constant. Hence $\{\lambda_n\}$ is a bounded sequence. Replacing by subsequence, we may assume that $\lambda_n \to \lambda$. By $\|x_n - G(x_n) - \lambda_n b\| \le \frac{1}{n}$, we obtain $x_n \to y + \lambda b \in S$. Therefore $g(y + \lambda b) = \lambda b$. Since $0 \notin g(S)$, we have $\lambda > 0$. Consequently $\lambda b \in g(S) \cap L$ offers a contradiction. Now let $0 < \varepsilon < \frac{1}{2}\min\{d[L, g(S)], d[0, g(S)]\}$. Choose a finite dimensional compact map H^* on S such that $\|G(x) - H^*(x)\| \le \varepsilon$. Then $H(x) = \frac{1}{2}\{H^*(x) - H^*(-x)\}$ defines a finite dimensional compact map on S. Since G is antipodal, we have

$$\|G(x) - H(x)\| \le \tfrac{1}{2}\|G(x) - H^*(x)\| + \tfrac{1}{2}\|G(-x) - H^*(-x)\| \le 2\varepsilon < d[0, g(S)].$$

Therefore H has no fixed point on S, i.e. $h(a) \ne 0$ for all $a \in S$ where $h = I - H$. Since $\|h(x) - \lambda b\| \ge \|g(x) - \lambda b\| - \|h(x) - g(x)\| \ge d[L, g(S)] - 2\varepsilon > 0$, we have $h(S) \cap L = \emptyset$. Let E_1 be a finite dimensional vector subspace of E containing $H(S)$ and the point b. By finite dimensional antipodal theorem, there is $a \in S$ such that $h(a) = h(-a)$. Since h is antipodal, we have $h(a) = 0$ which is a contradiction. □

5-6.13. **Theorem on Invariance of Domain** Let Y be an open subset of E. If $g : Y \to E$ is a locally injective compact field, then $g(Y)$ is open in E.

Proof. Take any $a \in Y$. There is $r > 0$ such that g is injective on $\overline{\mathbb{B}}(a, r) \subset Y$. Define $f : B \to E$ by $f(x) = \frac{1}{r}[g(a + rx) - g(a)] = x - \frac{1}{r}[G(a + rx) - G(a)]$. Then f is an injective compact field on B. In particular, $0 = f(0)$ does not belong to the closed set $f(S)$. The distance δ from 0 to $f(S)$ is strictly positive, i.e. $\delta > 0$. Take any $\|v\| < \delta$ in E. Then $f \simeq f - v$ in $C(B, S)$. Let $h : B \times J \to E$ be defined by $h(x, t) = f\left(\dfrac{x}{1+t}\right) - f\left(\dfrac{-tx}{1+t}\right)$. Clearly h is a field homotopy on B. Suppose that $h(x, t) = 0$ for some $(x, t) \in S \times J$. Then $f\left(\dfrac{x}{1+t}\right) = f\left(\dfrac{-tx}{1+t}\right)$, or $g\left(a + \dfrac{rx}{1+t}\right) = g\left(a + \dfrac{-rtx}{1+t}\right)$. By injectivity, we have $a + \dfrac{rx}{1+t} = a + \dfrac{-rtx}{1+t}$, i.e. $x = 0$ giving a contradiction. Therefore h is a homotopy in $C(B, S)$. Since h_1 is antipodal, it is singular and so is $h_0 = f$. Thus $f - v$ is also singular. There is $x \in B$ such that $f(x) - v = 0$, i.e. $g(a + rx) = g(a) + rv$. Since v is arbitrary, we have $\mathbb{B}[g(a), r\delta] \subset g(Y)$. Consequently, $g(Y)$ is open in E. □

5-6.14. **Corollary** Every locally injective compact field f on the whole space E is surjective.

Proof. Since E is open and closed in E, so is $f(E)$. By connectedness of E, we have $f(E) = E$. □

5-6.15. **Exercise** Prove that the norm of a locally injective compact field on an open subset Y of E cannot have a local maximum.

5-99. **References and Further Readings** : Borsuk, Granas-62,90, Fan-52,84, Lin, Ma-72, Steinlein, Jaworowski-89,00, Smart, Lloyd, Siegberg, Istratescu, Joshi, Borisovich-80, Gwinner, Aubin, Brown-93, Deimling-85, Mawhin, Caristi, Halpern, Browder, Kransnoselskii-84, Allgower, Alexander, Rybakowski and Bartsch.

Chapter 6

Foundation of Functional Analysis

6-1 Transfinite Induction

6-1.1. In order to show that every infinite dimensional normed space has sufficiently many continuous linear forms, elementary mathematical induction is unable to help. Zorn's Lemma seems to be quite easy to apply to situation involving transfinite induction and hence it gains the popularity. Since it touches the foundation of pure mathematics which is supposed to be built solely on set theory, its equivalence to other propositions, e.g. axiom of choice, is beyond our scope. We begin with introduction to glossary which will be needed to understand the statement of Zorn's Lemma. Then follow up with a simple application before we apply it to more involved extension problem in next section.

6-1.2. Let $\mathbb{P}$ be a set. A binary relation $\leq$ on $\mathbb{P}$ is called a *partial order* on $\mathbb{P}$ if for all $x, y, z \in \mathbb{P}$, we have
(a) $x \leq x$, reflexive
(b) $x \leq y$ and $y \leq x$ imply $x = y$, anti-symmetric
(c) $x \leq y$ and $y \leq z$ imply $x \leq z$, transitive.
A set together with a partial order is called a *partially ordered set*, or a *poset*. The symbol $x \leq y$ is also read as x dominated by y.

6-1.3. Let $\mathbb{P}$ be a partially ordered set. A non-empty subset C of $\mathbb{P}$ is called a *chain* if for all $x, y \in C$, we have either $x \leq y$ or $y \leq x$. An element $m \in \mathbb{P}$ is said to be *maximal* if $m \leq x$ in $\mathbb{P}$ implies $m = x$. Similarly, *minimal* elements are defined. The notions of upper bounds, lower bounds, suprema, infima are defined in the same way as on the real line.

6-1.4. **Zorn's Lemma** Let $\mathbb{P}$ be a non-empty partially ordered set. If every chain has an upper bound, then every element is dominated by some maximal element. In particular, $\mathbb{P}$ has at least one maximal element.

6-1.5. Let E be a vector space. A subset M of E is said to be ***linearly independent*** if every finite subset of M is linearly independent, i.e. the equation $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ where $\{x_j\}$ are distinct elements in M and $\{\alpha_j\}$ are unknown scalars, has only the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. The empty set is defined as a linearly independent set. A vector $x \in E$ is called a ***linear combination*** of vectors in M if $x = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$ for some $a_j \in M$ and some $\alpha_j \in \mathbb{K}$. The zero vector is defined as a linear combination of vectors in the empty set. A subset M is said to ***span*** E if every vector $x \in E$ is a linear combination of vectors in M. A subset M is called a ***basis*** or Hamel basis for E if it is linearly independent and spans E. These definitions agree with those in finite dimensional linear algebra. For a vector space consisting of one single zero vector, the empty set is its basis.

6-1.6. **Example** Every independent subset N of a vector space E can be extended to a basis. In particular, every vector space has a basis.

Proof. The family $\mathbb{P}$ of all independent sets of E is non-empty since the empty set belongs to $\mathbb{P}$. Suppose that $\mathbb{P}$ is ordered by inclusion, i.e. $A \leq B$ in $\mathbb{P}$ iff $A \subset B$. It is easy to see that $\mathbb{P}$ is a poset. Let $\mathcal{C}$ be a chain in $\mathbb{P}$. Let M be the union of all sets in $\mathcal{C}$. We claim that M is independent. In fact, consider the equation $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ where $x_j \in M$ and $\alpha_j \in \mathbb{K}$. Then $x_j \in A_j$ for some $A_j \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, we have either $A_i \subset A_j$ or $A_j \subset A_i$ for all $1 \leq i, j \leq n$. Hence there is some $1 \leq k \leq n$ such that $A_j \subset A_k$ for all j. Hence all $x_j \in A_k$. Since A_k is an independent set, the above equation has only the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Therefore M is linearly independent, that is $M \in \mathbb{P}$. Consequently, M is an upper bound of the chain $\mathcal{C}$. By Zorn's Lemma, the independent set N is dominated by some maximal element M of $\mathbb{P}$. Clearly $N \subset M$ and M is independent. Suppose to the contrary that there is some $y \in E$ which is not a linear combination of vectors in M. It is easy to verify that the set $V = M \cup \{y\}$ is an independent set containing M. This contradiction to the maximality of M shows that M also spans E. Therefore M is a basis for E. □

6-1.7. **Exercise** Let E, F be vector spaces. Let $\{a_i : i \in I\}$ be a basis of E and $\{b_i : i \in I\}$ an indexed subset of F. Prove that there is a unique linear map $f : E \to F$ such that $f(a_i) = b_i$ for each $i \in I$.

6-1.8. **Exercise** Prove that every infinite dimensional normed space has at least one discontinuous linear form.

6-1.9. **Exercise** Let M be a vector subspace of a vector space E and $a \in E \setminus M$. Prove that every vector in the vector subspace N generated by $M \cup \{a\}$ has a unique representation $m + \lambda a$ for some $m \in M$ and $\lambda \in \mathbb{K}$. Prove that $g(m + \lambda a) = \lambda$ is a linear form on N.

6-1.10. **Exercise** Let E, F be vector spaces and M a vector subspace of E. Prove that every linear map $g : M \to F$ has a linear extension over E.

6-1.11. **Exercise** Prove that for every $x \neq 0$ in E, there is a linear form f on E such that $f(x) \neq 0$.

6-1.12. **Exercise** Let f, g be linear forms on a vector space E. Prove that if they have the same kernel, then there is $0 \neq \lambda \in \mathbb{K}$ such that $f = \lambda g$.

6-2 Hahn-Banach Extension Theorems

6-2.1. Every linear form on a finite dimensional vector subspace of a normed space E is continuous. To extend a continuous linear form, the continuity will be isolated from linearity in terms of gauges. In this section, we start with an analytic extension theorem. It will be applied to continuous linear forms later.

6-2.2. Let p be a real-valued function on a vector space E. Then p is called a *gauge* on E if for all $x, y \in E$ we have
(a) $0 \leq p(x) < \infty$; positive
(b) $p(x + y) \leq p(x) + p(y)$; triangular inequality
(c) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$; positively homogeneous.
A gauge p on E is called a *seminorm* if $p(\alpha x) = |\alpha| p(x), \forall\, x \in E$ and $\forall\, \alpha \in \mathbb{K}$.

6-2.3. **Example** Let f be a linear form on a vector space E. Then $p(x) = |f(x)|, \forall\, x \in E$ defines a seminorm on E.

6-2.4. **Exercise** Let p, q be gauges on a vector space E and $0 \leq \alpha, \beta \in \mathbb{R}$. Show that $\max\{p, q\}$ and $\alpha p + \beta q$ are gauges on E.

6-2.5. **Exercise** Prove that if a gauge is continuous at the origin of a normed space E, then it is continuous on the whole space E.

6-2.6. **Analytic Extension Theorem** Let E be a *real* vector space and let p be a gauge on E. If g is a linear form on a vector subspace M of E satisfying $g(y) \leq p(y)$ for all $y \in M$, then g can be extended to a linear form f on E such that $f(x) \leq p(x)$ for all $x \in E$.

Proof. Let $\mathbb{P}$ be the family of all ordered pairs (H, h) where H is a vector subspace of E and h a linear form on H such that $h(y) \leq p(y)$ for all $y \in H$. Then $\mathbb{P}$ is non-empty since (M, g) is a member of $\mathbb{P}$. For all $(H, h), (K, k)$ in $\mathbb{P}$, define $(H, h) \leq (K, k)$ iff $H \subset K$ and $k|H = h$. It is easy to verify that $\mathbb{P}$ is partially ordered set. Let $\mathcal{C} = \{(H_i, h_i) : i \in I\}$ be a chain in $\mathbb{P}$. Define $K = \bigcup_{i\in I} H_i$. For every $x \in K$, there is $i \in I$ such that $x \in H_i$ and then define $k(x) = h_i(x)$. We claim that $k(x)$ is independent of the choice of i. In fact, suppose $x \in H_i$ and $x \in H_j$. Since $\mathcal{C}$ is a chain, we may assume by symmetry that $(H_i, h_i) \leq (H_j, h_j)$. Then $H_i \subset H_j$ and $h_j|H_i = h_i$. Hence $h_i(x) = h_j(x)$. Therefore $k(x)$ is independent of the choice of i for which $x \in H_i$. Next, we claim that K is a vector subspace and k is a linear form on K. Indeed, take any $x, y \in K$ and $\alpha, \beta \in \mathbb{R}$. There are $i, j \in I$ such that $x \in H_i$ and $y \in H_j$. Since $\mathcal{C}$ is a chain, we may assume by symmetry that $(H_i, h_i) \leq (H_j, h_j)$. Then $H_i \subset H_j$ and $h_j|H_i = h_i$. Hence both x, y belong to the vector subspace H_j. Therefore $\alpha x + \beta y \in H_j$, or $\alpha x + \beta y \in K$. This proves that K is a vector subspace of E. Since h_j is linear on H_j, we have

$$k(\alpha x + \beta y) = h_j(\alpha x + \beta y) = \alpha h_j(x) + \beta h_j(y) = \alpha k(x) + \beta k(y).$$

Therefore k is a linear form on K. Clearly $k(y) = h_j(y) \leq p(y), \forall\ y \in K$. Consequently (K, k) is a member of $\mathbb{P}$. Obviously $H_i \subset K$ and $k|H_i = h_i$, i.e. $(H_i, h_i) \leq (K, k)$ for all $i \in I$. Therefore every chain in $\mathcal{C}$ has an upper bound in $\mathbb{P}$. By Zorn's Lemma, there exists a maximal element $(F, f) \geq (M, g)$. It remains to prove $F = E$. Suppose to the contrary that there is $a \in E \setminus F$. Let H be the vector subspace spanned by F and a. For all $u, v \in F$, we have

$$f(u + v) \leq p(u + v),$$

or,
$$f(u) + f(v) \leq p(u - a) + p(v + a),$$

i.e.
$$f(u) - p(u - a) \leq -f(v) + (v + a).$$

Let
$$\lambda = \sup\{f(u) - p(u - a) : u \in F\}.$$

Then
$$f(u) - p(u - a) \leq \lambda \leq -f(v) + (v + a), \qquad \forall\ u, v \in F.$$

Since $a \notin F$, every vector in H can be written uniquely in the form $x = y + \alpha a$ where $y \in F$ and $\alpha \in \mathbb{R}$. Define $h(x) = f(y) + \alpha\lambda$. It is easy to verify that h is a linear form on H and $h|F = f$. If $\alpha > 0$, taking $v = y/\alpha$ we have $\lambda \leq -f\left(\frac{y}{\alpha}\right) + p\left(\frac{y}{\alpha} + a\right)$, or $\alpha\lambda \leq -f(y) + p(y + \alpha a)$, i.e. $h(x) \leq p(x)$. If $\alpha < 0$, taking $u = -y/\alpha$ we have $f\left(-\frac{y}{\alpha}\right) - p\left(-\frac{y}{\alpha} - a\right) \leq \lambda$, i.e. $-\frac{1}{\alpha}f(y) - \left(\frac{1}{-\alpha}\right)p(y+\alpha a) \leq \lambda$, or, $f(y) - p(y+\alpha a) \leq -\alpha\lambda$, that is, $h(x) \leq p(x)$ again. Clearly, $g(x) \leq p(x)$ for $\alpha = 0$ because (F, f) is in $\mathbb{P}$. Therefore $F \subset H$ and $h|F = f$. This proves that (H, h) is in $\mathbb{P}$ and $(F, f) \leq (H, h)$. Since (F, f) is a maximal element we have $(F, f) = (H, h)$. We obtain $F = H$ which is a contradiction. This completes the proof. □

6-2.7. **Exercise** Let f be a linear form on a normed space E. Prove that if there is a continuous seminorm p on E such that $|f(x)| \leq p(x)$ for all $x \in E$, then f is continuous.

6-2.8. **Exercise** Let f be a linear form on a *real* normed space E. Prove that if there is a continuous gauge p on E such that $f(x) \leq p(x)$ for all $x \in E$, then f is continuous.

6-3 Extension of Continuous Linear Forms

6-3.1. In order to combine the real and complex normed spaces into a single framework, we start to reduce the complex linear forms to the real ones and then work with seminorms instead of gauges.

6-3.2. Let E be a *complex* vector space. The vector space obtained from E by restricting the scalar multiplication to the real numbers only, is called the *real vector space associated with the complex vector space* E and it is denoted by E_r. Let f be a complex linear form on the complex vector space E. Let $f_r(x)$ be the real part of the complex number $f(x)$ for each $x \in E$. Then f_r is a real linear form on E_r and it is called the *real part of the complex linear form* f. Clearly we have $f(x) = f_r(x) - if_r(ix)$ for all $x \in E$ where $i^2 = -1$.

6-3.3. **Exercise** Let E be a complex normed space and let f be a complex linear form on E. Prove the following statements.
(a) E_r is a real normed space.
(b) f is continuous iff the real part f_r is continuous.

6-3.4. **Dominated Extension Theorem** Let p be a seminorm on a vector space E and g a linear form on a vector subspace M of E. If $|g(y)| \leq p(y), \forall\, y \in M$, then g can be extended to a linear form f on E with $|f(x)| \leq p(x)$ for all $x \in E$.

Proof. Consider a real vector space E first. Since $g(y) \leq |g(y)| \leq p(y)$ for all $y \in M$, g can be extended to a linear form f on E such that $f(x) \leq p(x)$

for all $x \in E$. If $f(x) \geq 0$, then $|f(x)| = f(x) \leq p(x)$. If $f(x) < 0$, then we obtain $|f(x)| = f(-x) \leq p(-x) = p(x)$. This completes the proof for the real case. Next, assume that E is a complex vector space. Let g_r be the real part of g. Then for every $y \in M$, we obtain $g(y) = g_r(y) - ig_r(iy)$, that is, $g_r(y) \leq |g_r(y)| \leq |g(y)| \leq p(y)$. There is a real linear form f_r on E such that $f_r(x) \leq p(x)$ for all $x \in E$ and $f_r(y) = g_r(y)$ for all $y \in M$. Define $f(x) = f_r(x) - if_r(ix)$ for each $x \in E$. Then f is a complex linear form on E and $f(y) = g(y)$ for all $y \in M$. Given $x \in E$, let $f(x) = |f(x)|e^{i\theta}$ where θ is some real number. Then the proof is complete by the following calculation:

$$|f(x)| = e^{-i\theta}f(x) = f(e^{-i\theta}x) = f_r(e^{-i\theta}x) \leq p(e^{-i\theta}x) = p(x). \qquad \square$$

6-3.5. **Exercise** Let p be a seminorm on a vector space E. Then for every $a \in E$ there is a linear form f on E such that $f(a) = p(a)$ and $|f(x)| \leq p(x)$ for all $x \in E$.

6-3.6. **Theorem** Let M be a vector subspace of a normed space E. Then every continuous linear form g on M can be extended to a continuous linear form f on E such that $\|f\| = \|g\|$.

Proof. Let $p(x) = \|g\| \, \|x\|$ for all $x \in E$. It is easy to verify that p is a seminorm on E. For each $y \in M$, we have $|g(y)| \leq \|g\| \, \|y\| = p(y)$. Hence g has a linear extension f over E such that $|f(x)| \leq p(x)$ for all $x \in E$. Now $|f(x)| \leq \|g\| \, \|x\|$ for all $x \in E$ implies the continuity of f and $\|f\| \leq \|g\|$. The proof is completed by the following calculation:

$$\|g\| = \sup\{|g(y)| : y \in M, \|y\| \leq 1\} \leq \sup\{|f(x)| : x \in M, \|x\| \leq 1\} = \|f\|. \qquad \square$$

6-3.7. **Theorem** Let M be a closed subspace of a normed space E. Then for every $b \in E \setminus M$, there is a continuous linear form f on E such that $f(M) = 0$, $\|f\| = 1$ and $f(b) = d(b, M)$, the distance from b to M.

Proof. Let N be the vector subspace spanned by M and b. Then every vector in N can be expressed uniquely in the form $y = m + \lambda b$ where $m \in M$ and $\lambda \in \mathbb{K}$. Since M is closed, we have $\delta = d(b, M) > 0$. Define $g(y) = \lambda\delta$. Clearly g is a linear form on N such that $g(M) = 0$ and $g(b) = \delta$. To compute the norm of g, let $m + \lambda b$ be in N. If $\lambda \neq 0$, then $\|\frac{m}{\lambda} + b\| \geq \delta$ and hence

$$|g(m + \lambda b)| = |\lambda\delta| = \delta|\lambda| = \frac{\delta\|m + \lambda b\|}{\|\frac{m}{\lambda} + b\|} \leq \frac{\delta\|m + \lambda b\|}{\delta} = \|m + \lambda b\|.$$

For $\lambda = 0$, we have $|g(m+\lambda b)| = |g(m)| = 0 \leq \|m+\lambda b\|$. Therefore g is continuous on N and $\|g\| \leq 1$. On the other hand, let $\varepsilon > 0$ be given. There is $m \in M$ such that $\|m-b\| \leq \delta+\varepsilon$. Then $\delta = |g(b)| = |g(m)-g(b)| \leq \|g\|\,\|m-b\| \leq \|g\|(\delta+\varepsilon)$, i.e. $\|g\| \geq \frac{\delta}{\delta+\varepsilon}$. Letting $\varepsilon \to 0$, we have $\|g\| \geq 1$. Therefore we conclude $\|g\| = 1$. Now there is a continuous linear form f on E such that $f|N = g$ and $\|f\| = \|g\| = 1$. Hence $f(M) = g(M) = 0$ and $f(b) = g(b) = \delta$. This completes the proof. □

6-3.8. **Corollary** For ever $x \neq 0$ in a normed space E, there is a continuous linear form f on E such that $\|f\| = 1$ and $f(x) = \|x\|$.

Proof. This is a special case of last theorem when $M = \{0\}$ and $b = x$. □

6-3.9. **Exercise** Find an example of a normed space E and $0 \neq f \in E'$ such that there is no $x \in E$ satisfying $\|x\| = 1$ and $f(x) = \|f\|$.

6-3.10. **Corollary** Continuous linear forms *separates points* of a normed space. More precisely, if $x \neq y$ are distinct points of E then there is a continuous linear form f on E such that $f(x) \neq f(y)$. We normally apply to the case when $y = 0$.

6-3.11. **Exercise** Let M be the closed vector subspace generated by a non-empty subset Y of a normed space E. Prove that $x \in E$ belongs to M iff for every $f \in E'$ we have $f(x) = 0$ whenever $f(y) = 0, \forall\, y \in Y$.

6-4 Closed Hyperplanes

6-4.1. Some preliminary algebraic concept will be introduced first and elementary results are stated for easy reference. Then the geometric form of Hahn-Banach Extension Theorem in terms of closed hyperplanes will follow.

6-4.2. Let E denote a vector space. A subset M of E is called a *flat* or a *linear manifold* if $M = S + a$ for some vector subspace S of E and some $a \in E$. Points, lines, planes are examples of flats.

6-4.3. **Lemma** Let M be a flat in E. Then M is a vector subspace iff the origin belongs to M.

6-4.4. A vector subspace S of E is called a *hypersubspace* if $S \neq E$ and if for any vector subspace N satisfying $S \subset N \subset E$, we have either $N = S$ or $N = E$. A translate of a hypersubspace is called a *hyperplane*. Clearly every hyperplane is a flat.

6-4.5. **Theorem** Let M be a vector subspace of E. If $M \neq E$, then the following statements are equivalent.

(a) There is a linear form f on E such that $M = \ker(f)$, the kernel of f.
(b) M is a hypersubspace.
(c) For each $a \in E \setminus M$, the whole space E is spanned by M and a.

6-4.6. **Corollary** A subset M of E is a hyperplane iff $M = f^{-1}(\lambda)$ for some non-trivial linear form f on E and some scalar $\lambda \in \mathbb{K}$.

6-4.7. **Theorem** Let M be a subset of a normed space E. Then M is a closed hyperplane iff there is a non-trivial continuous linear form f and a scalar λ such that $M = f^{-1}(\lambda)$.

Proof. Since the translate of a closed set is close, it follows immediately from the fact that a linear form is continuous iff its kernel is a closed set.

6-4.8. **Theorem** Let E be a complex normed space. A subset M of E is a closed complex hypersubspace iff there exists a closed real hypersubspace H satisfying $M = H \cap iH$, where $i^2 = -1$. Consequently every closed complex hyperplane is the intersection of two closed real hyperplanes.

Proof. ($\Leftarrow$) Let H be a closed real hypersubspace. There is a continuous real linear form g on E satisfying $H = g^{-1}(0)$. Define $f(x) = g(x) - ig(ix)$ for all $x \in E$. Then f is a continuous complex linear form on E. Suppose $f(x) = 0$. Then $g(x) = 0$ and $g(ix) = 0$. Thus $x \in H$ and $ix \in H$, i.e. $x \in -iH = iH$. Hence $x \in H \cap iH$. Clearly the argument is reversible. Therefore $H \cap iH = \ker(f)$ is a closed complex hypersubspace.
($\Rightarrow$) Suppose M is a closed complex hypersubspace. Let f be a continuous complex linear form on E such that $M = f^{-1}(0)$. Then the real part g of f is a continuous real linear form on E. Then $H = g^{-1}(0)$ is a closed real hypersubspace. As before, we have $H \cap iH = \ker(f) = M$. The last statement follows by translation.

6-4.9. A subset V of a vector space E is said to be *absorbing* if for each $x \in E$ there exists $\lambda > 0$ such that $x \in \lambda V$. Clearly the origin is in every absorbing set. The set V is said to be *balanced* if for every scalar $|\alpha| \leq 1$, we have $\alpha V \subset V$. Let V be a convex absorbing set in E. The *gauge* of V is defined as the function $p : E \to \mathbb{R}$ given by $p(x) = \inf\{\lambda > 0 : x \in \lambda V\}, \forall\, x \in E$.

6-4.10. **Exercise** Let V be a subset of a normed space E. Prove that if the origin is an interior point of V, then V is absorbing.

6-4.11. **Theorem** Let p be the gauge of a convex absorbing set V in a vector space E.

(a) p is a gauge on E.
(b) If $p(x) < 1$ then $x \in V$.
(c) If $x \in V$ then $p(x) \leq 1$.
(d) If V is balanced, then p is a seminorm.

Proof. (a) Let x, y be given points in E. From absorbing, the set $\{\lambda > 0 : x \in \lambda V\}$ is non-empty and hence $0 \leq p(x) < \infty$. Let $\varepsilon > 0$ be given. There are $\alpha, \beta > 0$ such that $\alpha \leq p(x) + \varepsilon$, $\beta \leq p(y) + \varepsilon$ and $x \in \alpha V$, $y \in \beta V$. Since V is convex, we have $x + y \in (\alpha + \beta)V$. Hence $p(x + y) \leq \alpha + \beta \leq p(x) + p(y) + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $p(x + y) \leq p(x) + p(y)$. Now if $z = 0 \in E$, then $z \in \lambda V$ for all $\lambda > 0$ and hence $p(z) = 0$. Thus $p(\alpha x) = 0 = \alpha p(x)$ holds for $\alpha = 0$. Suppose $\alpha > 0$. Then

$$p(\alpha x) = \inf\{\lambda > 0 : \alpha x \in \lambda V\} = \inf\{\alpha\mu > 0 : \alpha x \in \alpha\mu V\}, \quad \text{where } \lambda = \alpha\mu$$
$$= \alpha \inf\{\mu > 0 : x \in \mu V\} = \alpha p(x).$$

Therefore p is a gauge on E.
(b) Suppose $p(x) < 1$. There is $p(x) < \lambda < 1$ satisfying $x \in \lambda V$. Hence $x \in (1 - \lambda)0 + \lambda V \subset V$.
(c) If $x \in V$, then $x \in \lambda V$ for $\lambda = 1$ and hence $p(x) \leq \lambda = 1$.
(d) Take any $\alpha \neq 0$ in $\mathbb{K}$. Since V is balanced, $\alpha x \in \lambda V$ iff $|\alpha| x \in \lambda V$. Therefore we have

$$p(\alpha x) = \inf\{\lambda > 0 : \alpha x \in \lambda V\} = \inf\{\lambda > 0 : |\alpha| x \in \lambda V\} = p(|\alpha| x) = |\alpha| p(x). \ \square$$

6-4.12. **Geometric Extension Theorem** Let A be a non-empty open convex set in a normed space E. If M is a flat disjoint from A, then there is a closed hyperplane H containing M and disjoint from A.

Proof. Consider the real normed space E first. By translation, we may assume $0 \in A$. Then A is a convex absorbing set. Let p be the gauge of A. Since M is a flat, there is a vector subspace S and $a \in E$ such that $M = a + S$. Because $A \cap M = \emptyset$, we have $0 \notin M$, i.e. $a \notin S$. Let T be the vector subspace spanned by S and a. Then every vector in T can be written uniquely in the form $\lambda a + s$ where $s \in S$ and $\lambda \in \mathbb{R}$. Define $g(\lambda a + s) = \lambda$. Clearly g is a linear form on T. If $\lambda > 0$, we have $a + (s/\lambda) \in M$, or $a + (s/\lambda) \notin A$. It follows $p[a + (s/\lambda)] \geq 1$, i.e. $p(\lambda a + s) \geq \lambda = g(\lambda a + s)$. If $\lambda \leq 0$, then $g(\lambda a + s) = \lambda \leq 0 \leq p(\lambda a + s)$. Thus for all $y \in T$, we have $g(y) \leq p(y)$. Hence g can be extended to a linear form f on E satisfying $f(x) \leq p(x)$ for all $x \in E$. Since $0 \in A$, there is $\mathbb{B}(0, \delta) \subset A$. Take any $x \in \mathbb{B}(0, \delta)$. Then $\pm x \in A$ and hence $p(\pm x) \leq 1$. If $f(x) \geq 0$, then

$|f(x)| = f(x) \leq p(x) \leq 1$. If $f(x) < 0$, then $|f(x)| = f(-x) \leq p(-x) \leq 1$. Thus f is bounded on the ball $\mathbb{B}(0, r)$ and consequently, f is a continuous linear form on E. Let $H = \{x \in E : f(x) = 1\}$. Then H is a closed hyperplane in E. Now, suppose $x \in M$. Write $x = a + s$ where $s \in S$. Then $x \in T$. Hence $f(x) = g(x) = g(a + s) = 1$, i.e. $x \in H$. Therefore $H \subset M$. Finally, take any $x \in A$. Let $\varphi(\lambda) = \lambda x$ for all $\lambda \in \mathbb{R}$. Then φ is continuous on $\mathbb{R}$. Since $\varphi^{-1}(A)$ is an open subset of $\mathbb{R}$ and $1 \in \varphi^{-1}(A)$, there is $1 < \beta \in \varphi^{-1}(A)$, i.e. $\beta x = \varphi(\beta) \in A$. Thus $f(x) \leq p(x) \leq (1/\beta) < 1$, i.e. $x \notin H$. Therefore H is disjoint from A. This proves the case for real normed space E. Next, suppose E is a complex normed space. By translation, we may assume $0 \in M$. Let E_r be the real vector space associated with E. By the real case, there is a closed real hyperplane H_r in E_r containing M and disjoint from A. Since $0 \in H_r$, H_r is a real hypersubspace in E_r. Hence $H = H_r \cap iH$ is a closed complex hypersubspace in E. Since M is a complex vector subspace, we have $M = iM \subset iH_r$, i.e. $M \subset H$. Observe that $A \cap H = (A \cap H_r) \cap iH_r = \emptyset$. This completes the proof. □

6-4.13. **Exercise** Prove that a hyperplane in a normed space is either closed or dense.

6-4.14. **Exercise** Prove that a normed space is infinite dimensional iff there is a dense hypersubspace.

6-5 Separation by Hyperplanes

6-5.1. Let E be a vector space. Let H be a *real* hyperplane given by $H = f^{-1}(\lambda)$ where f is a *real* linear form on E and $\lambda \in \mathbb{R}$. A subset A of E is said to *lie on one side of the real hyperplane* H if we have $f(a) \leq \lambda, \forall\, a \in A$; or $f(a) \geq \lambda, \forall\, a \in A$. A subset A of E is said to lie *strictly* on one side of the real hyperplane H if we have $f(a) < \lambda, \forall\, a \in A$; or $f(a) > \lambda, \forall\, a \in A$. It can be easily proved that the definition is independent on the representation of H in terms of f, λ.

6-5.2. **Lemma** Let A be a convex set in a vector space E and H a real hyperplane in E. Then A lies strictly on one side of H iff $A \cap H = \emptyset$.

Proof. Let $H = f^{-1}(\lambda)$ where $\lambda \in \mathbb{R}$ and f a real linear form on E. Assume $A \cap H = \emptyset$. Suppose to the contrary that A does not lie strictly on one side of H. Then there are $a, b \in A$ such that $f(a) < \lambda < f(b)$. There is $0 \leq t \leq 1$ such that $\lambda = (1-t)f(a) + tf(b)$. Since A is convex, $x = (1-t)a + tb$ is in A. Also, $f(x) = \lambda$. Hence $x \in A \cap H$ which is a contradiction. Therefore A lies strictly on one side of H. The converse is obvious. □

6-5.3. **Lemma** Let H be a closed real hyperplane in a normed space E. If $A \subset E$ lies on one side of H, then its interior A^o lies strictly on one side of H.

Proof. Let $H = f^{-1}(\lambda)$ where f is a continuous real linear form on E and $\lambda \in \mathbb{R}$. Since A lies on one side of H, we may assume $f(a) \le \lambda, \forall\ a \in A$; otherwise replace f by $-f$ and λ by $-\lambda$. Since $f \ne 0$, there is $b \in E$ such that $f(b) > 0$; otherwise replace b by $-b$. Take any $a \in A^o$. There is $r > 0$ such that $\mathbb{B}(a,r) \subset A$. Let $0 < \mu < \frac{r}{\|b\|}$. Then $\|\mu b\| < r$ and hence $a + \mu b \in \mathbb{B}(a,r)$; or $a+\mu b \in A$. Therefore we have $f(a) < f(a)+\mu f(b) = f(a+\mu b) \le \lambda$. Consequently, A lies strictly on one side of H. □

6-5.4. **Exercise** Let A be a convex set that lies on one side of a hyperplane H. Prove that if A has at least one interior point, then H must be closed.

6-5.5. **Exercise** Let H be a closed hyperplane in a normed space E and let $A \subset E$ lie on one side of H. Prove that its closure also lies on one side of H.

6-5.6. **Theorem** Let A, B be two non-empty convex sets in normed space E. If the interior of A is a non-empty set disjoint from B then A, B can be separated by a closed *real* hyperplane.

Proof. Since both A, B are convex, $B - A^o$ is convex and also open because $B - A^o = \bigcup_{b \in B} b - A^o$. The condition $B \cap A^o = \emptyset$ ensures that the flat $\{0\}$ consisting of one point is disjoint from the non-empty open convex set $B - A^o$. There is a closed real hyperplane H containing $\{0\}$ and disjoint from $B - A^o$. Since H contains the origin, it is a hypersubspace. There is a continuous real linear form f one E such that $H = \ker(f)$. It follows that $B - A^o$ lies on one side of H. Without loss of generality, we may assume $f(x) \ge 0$ for all $x \in B - A^o$; otherwise replace f by $-f$. Then for all $a \in A^o$ and all $b \in B$, we have $f(a) \le f(b)$. Let $\lambda = \sup\{f(a) : a \in A^o\}$. Since f is continuous, we have $f(a) \le \lambda$ for all $a \in A^{o-} = A^-$. Hence $f(A) \le \lambda \le f(B)$. Therefore A, B are separated by the closed real hyperplane $\{x \in E : f(x) = \lambda\}$. □

6-5.7. **Theorem** Let A, B be two disjoint non-empty closed convex sets in a normed space E. If one of A, B is compact, then A, B can be strictly separated by a closed real hyperplane.

Proof. For convenience, assume that A is compact. Then the function $d(x,B)$ is continuous on A. Since B is closed, $d(x,B) > 0$ for all $x \in A$. By compactness, we have $r = \frac{1}{3}d(A,B) > 0$. Let $V = \{x \in E : d(x,A) < r\}$ and $W = \{x \in E : d(x,B) < r\}$. Since $d(x,A)$ is continuous in x, V is an open set. To show the convexity of V, take any $x, y \in V$ and $0 \leq t \leq 1$. There are $a, b \in A$ such that $\|x - a\| < r$ and $\|y - b\| < r$. Since A is convex, $(1-t)a + tb$ is in A. Hence

$$d[(1-t)x + ty, A] \leq \|[(1-t)x + ty] - [(1-t)a + tb]\|$$
$$\leq (1-t)\|x - a\| + t\|y - b\| < r,$$

i.e. $(1-t)x + ty \in V$. Therefore V is an open convex set. Similarly W is also an open convex set. By simple calculation, V, W are disjoint. They can be separated by some closed real hyperplane H. Since V is open, V lies strictly on one side of H and W strictly on the other side of H. Since $A \subset V$ and $B \subset W$, A, B are strictly separated by H. □

6-5.8. **Corollary** Let A be a non-empty closed convex set in a normed space E. Then for every $b \in E \setminus A$, there exists a continuous real linear form f on E such that $f(b) < \inf f(A)$.

Proof. The singleton $\{b\}$ is a compact convex set disjoint from the closed convex set A. There is a closed real hyperplane H which strictly separates A and b. There is a continuous real linear form f on E and $\lambda \in \mathbb{R}$ such that $f(b) < \lambda < f(a), \forall\ a \in A$. The result follows immediately by taking infimum over $a \in A$. □

6-5.9. Important topics such as closed convex hulls represented in terms of supporting hyperplanes are beyond the scope of this book. It would be better done in the framework of locally convex spaces.

6-6 Extreme Points

6-6.1. Extreme points were introduced in §4-4.10. It was proved that extreme points of a simplex are the vertices. A simplex is the closed convex hull (§4-2.7) of its extreme point. This result will be generalized to infinite dimensional spaces. It is another example of applying Zorn's Lemma. Since sophisticated applications are beyond the scope of this book, this section may be skipped without discontinuity.

6-6.2. Let A be a non-empty subset of a vector space E. A non-empty subset X of A is called an *extreme subset* of A if for all $x, y \in A$ satisfying $(x, y) \cap X \neq \emptyset$, both x, y are in X. When X is a singleton, it becomes an extreme point. The following lemma follows immediately from definition.

6-6.3. **Lemma** Let A, B, C be subsets of a vector space E.
(a) If the intersection of a family of extreme subsets of A is non-empty, then it is again an extreme subset of A.
(b) If A is an extreme subset of B and if B is an extreme subset of C, then A is an extreme subset of C.
(c) If $A \subset B \subset C$ and if A is an extreme subset of C, then A is an extreme subset of B.

6-6.4. **Lemma** Let M be a non-empty closed extreme subset of a compact convex set A in a normed space E. If X is a closed convex set in E satisfying $M \setminus X \neq \emptyset$ and $M \cap X \neq \emptyset$, then there is a closed extreme subset N of A such that $N \subset M$ and $N \subset X = \emptyset$.

Proof. Suppose $a \in M \setminus X$. There is a continuous real linear form f on E such that $f(a) < \inf f(X)$. As a closed subset of the compact set A, M is compact. There is $m \in M$ such that $f(m) = \inf f(M)$. Let $N = \{z \in M : f(z) = f(m)\}$. Since $m \in N$, N is a non-empty closed subset of M. Now for all $z \in N$, we have

$$f(z) = f(m) = \inf f(M) \leq f(a) < \inf f(X).$$

Thus $N \cap X = \emptyset$. Finally suppose x, y are in M satisfying $z = (1-t)x + ty \in N$ where $0 < t < 1$. Then $f(x) \geq \inf f(M)$, $f(y) \geq \inf f(M)$ and

$$(1-t)f(x) + tf(y) = f(z) = \inf f(M).$$

This implies $f(x) = \inf f(M)$ and $f(y) = \inf f(M)$. Hence $x, y \in N$. Therefore N is an extreme subset of M. It follows that N is also an extreme subset of A. □

6-6.5. **Krein-Milman Theorem** Every non-empty compact set A in a normed space E has an extreme point.

Proof. Let $\mathbb{P}$ be the family of all non-empty closed extreme subset of A. Then $\mathbb{P}$ is non-empty since $A \in \mathbb{P}$. It becomes a partially ordered set under inclusion. Let $\{C_i : i \in I\}$ be a chain in $\mathbb{P}$. Let $C = \bigcap_{i \in I} C_i$. As closed subsets of the compact set A, all C_i are compact. Take any finite subset J of I. By definition of a chain, $C_i \cap C_j$ is either C_i or C_j. Hence $\bigcap_{j \in J} C_j$ is actually one of C_j and thus the intersection is non-empty. By compactness, we have $C \neq \emptyset$. Consequently, C is a closed extreme subset of A. Therefore C is a lower bound of the given chain. By Zorn's Lemma, $\mathbb{P}$ has a minimal element, say M. Suppose to the contrary that M contains at least two points $a \neq b$. There is a closed extreme subset N of A such that $N \subset M$ and $a \notin N$. This contradiction to the minimality of M shows that M is a singleton. Therefore A has at least one extreme point. □

6-6.6. **Theorem** Every non-empty compact convex set A in a normed space E is the convex hull of its extreme points.

Proof. Let X be the closed convex hull of extreme points of A, i.e. the intersection of all closed convex sets containing all extreme points of A. Since A itself is a closed convex set containing all extreme points of A, we clearly have $X \subset A$. Suppose to the contrary that $A \setminus X \neq \emptyset$. Then there is a closed extreme subset B of A such that $B \cap X = \emptyset$. Let b be an extreme point of B. Then it is also an extreme point of A and hence $b \in X$ which contradicts $B \cap X = \emptyset$. Therefore $A \setminus X = \emptyset$, i.e. $A \subset X$. This completes the proof. □

6-6.7. **Exercise** Let A be the smallest convex set in $\mathbb{R}^3$ that contains the points $(1, 0, \pm 1)$ and $(\cos\theta, \sin\theta, 0)$ for $0 \leq \theta \leq 2\pi$. Show that A is compact but that the set of all extreme points of A is not compact.

6-6.8. **Exercise** Prove that for $1 < p < \infty$, every point on the unit sphere of $\mathbb{R}_p^n$ is an extreme point of the closed unit ball.

6-6.9. **Exercise** Prove that the unit sphere of c_0 has no extreme point.

6-7 Baire's Property

6-7.1. The nested property of a complete metric space will be translated into Baire's Theorem which has become an important tool to derive existence in many occasions. Originally, metric spaces were classified as first and second categories in terms of nowhere dense sets. These topics have been dropped in this book in order to shorten the path for the reader to get through the subject and then to start their own research as soon as possible.

6-7.2. **Lemma** Let M be a closed subset of X. If M has empty interior, then every open ball contains a closed ball which is disjoint from M.

Proof. Suppose to the contrary that there is an open ball B such that every closed ball $\overline{\mathbb{B}}(a, \delta) \subset B$ contains a point of M. We claim $B \subset M$. In fact, take any $a \in B$ and $r > 0$. Since B is open, there is $0 < \delta < r$ such that $\mathbb{B}(a, \delta) \subset B$. By assumption, there is some $x \in \overline{\mathbb{B}}(a, \delta) \cap M$. Hence $d(x, a) \leq \delta < r$, i.e. $x \in \mathbb{B}(a, r) \cap M$. Therefore a is a closure point of M. Since M is closed, we have $a \in M$. This proves $B \subset M$. Since B is open, $B \subset M^o$. Therefore M^o is non-empty. This contradiction establishes the proof. □

6-7.3. **Baire's Category Theorem** Let X be a complete metric space which is not empty. If it is covered by a sequence $\{A_n\}$ of closed sets, then at least one A_n has non-empty interior.

Proof. Suppose to the contrary that all A_n have empty interior. Take any $a_0 \in X$ and define $r_0 = 1$. Since A_1 has empty interior, the open ball $\mathbb{B}(a_0, r_0)$ contains a closed ball $\overline{\mathbb{B}}(a_1, \delta_1)$ which is disjoint from A_1. Let $r_1 = \min\{1/2, \delta_1\}$. Inductively, since A_n has empty interior, the open ball $\mathbb{B}(a_{n-1}, r_{n-1})$ contains a closed ball $\overline{\mathbb{B}}(a_n, \delta_n)$ which is disjoint from A_n. Let $r_n = \min\{1/2^n, \delta_n\}$. Observe that $\overline{\mathbb{B}}(a_n, r_n) \subset \overline{\mathbb{B}}(a_n, \delta_n) \subset \mathbb{B}(a_{n-1}, r_{n-1}) \subset \overline{\mathbb{B}}(a_{n-1}, r_{n-1})$. Hence $\{\overline{\mathbb{B}}(a_n, r_n)\}$ forms a decreasing sequence of closed sets. Furthermore,

$$diam\ \overline{\mathbb{B}}(a_n, r_n) \leq 2r_n \leq 1/2^{n-1} \to 0.$$

By Nested property of complete metric spaces, the set $\bigcap_{n=1}^{\infty} \overline{\mathbb{B}}(a_n, r_n)$ contains some point, say b. Since $\{A_n\}$ covers X, we have $b \in A_n$ for some n. Now the contradiction $b \in \overline{\mathbb{B}}(a_n, r_n) \cap A_n \subset \overline{\mathbb{B}}(a_n, \delta_n) \cap A_n$ establishes the proof. □

6-7.4. **Exercise** Prove that an infinite dimensional Banach space is not a countable union of compact sets.

6-7.5. **Exercise** Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions on a complete metric space X. Prove that if for each $x \in X$, $\sup_{n \geq 1} f_n(x) < \infty$, then there is a non-empty open subset V of X such that

$$\sup\{f_n(x) : n \geq 1, x \in V\} < \infty.$$

6-8 Uniform Boundedness

6-8.1. In addition to applications to some classical problems which will be formulated as exercises in this section, Uniform Boundedness Theorem asserts that pointwise bounded sets are normed bounded. This result will be used to develop complex analysis on Banach spaces and will form the foundation of weak convergence.

6-8.2. **Uniform-Boundedness Theorem** Let E, F be normed spaces and let $\{f_i : i \in I\}$ be a family of continuous linear maps from E into F. If E is a Banach space, then $\sup_{i\in I} \|f_i(x)\| < \infty, \forall\, x \in E$ implies $\sup_{i\in I} \|f_i\| < \infty$.

Proof. For each integer $n \geq 1$, define $H_n = \{x \in E : \|f_i(x)\| \leq n, \forall\, i \in I\}$. Since the function $x \to \|f_i(x)\|$ is continuous, its inverse image of the closed set $[0, n]$ is closed. Hence each H_n is closed in E. The given condition ensures $E = \bigcup_{n=1}^{\infty} H_n$. Since E is a Banach space, it follows from Category Theorem that some H_m has an interior point, say $\mathbb{B}(a, 2r) \subset H_m$ where $r > 0$. We claim $\sup_{i\in I} \|f_i\| \leq \dfrac{2m}{r}$. In fact, take any $\|x\| \leq 1$ in E and any $i \in I$. Then both a and $a + rx$ are in $\mathbb{B}(a, 2r)$ and hence in H_m. Thus,

$$\|f_i(rx)\| \leq \|f_i(a + rx)\| + \| - f_i(a)\| \leq m + m = 2m,$$

or $\|f_i(x)\| \leq \frac{2m}{r}$. Taking supremum over $\|x\| \leq 1$, we have $\|f_i\| \leq \frac{2m}{r}$ for all $i \in I$. This completes the proof. □

6-8.3. **Banach-Steinhaus Theorem** Let E, F be normed spaces and $\{f_n : n \geq 1\}$ a sequence of continuous linear maps from E into F. If E is a Banach space and if the limit $g(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$, then g is a continuous linear map from E into F. Furthermore we have

$$\|g\| \leq \liminf_{n\to\infty} \|f_n\| \leq \sup_{n\geq 1} \|f_n\| < \infty.$$

Proof. For each $x \in E$, the convergent sequence $\{f_n(x) : n \geq 1\}$ is bounded in F. Since E is a Banach space, we have $\sup_{n\geq 1} \|f_n\| < \infty$. Clearly, g is a linear map. Next, let $\varepsilon > 0$ be given. Take any $\|x\| \leq 1$ in E. Since $g(x) = \lim f_n(x)$, there is an integer k such that for all $n \geq k$, we have $\|g(x) - f_n(x)\| \leq \varepsilon$, i.e.

$$\|g(x)\| \leq \|g(x) - f_n(x)\| + \|f_n(x)\| \leq \varepsilon + \|f_n\|.$$

Hence $\|g(x)\| \leq \varepsilon + \liminf_{n\to\infty} \|f_n\|$. Taking supremum over $\|x\| \leq 1$, we get

$$\|g\| \leq \varepsilon + \liminf_{n\to\infty} \|f_n\| \leq \varepsilon + \sup_{n\geq 1} \|f_n\| < \infty.$$

Therefore g is continuous. Since $\varepsilon > 0$ is arbitrary, we have $\|g\| \leq \liminf \|f_n\|$. This completes the proof. □

6-8.4. **Corollary** Between Banach spaces, pointwise limits of continuous linear maps are continuous.

6-8.5. **Exercise** Let φ_n be the projection of a sequence to its n-th term and let $f_n = n\varphi_n$. Show that

(a) Each f_n is a continuous linear form on $\mathcal{F}_\infty$.
(b) For every $x \in \mathcal{F}_\infty$, the set $\{|f_n(x)| : n \geq 1\}$ is bounded in $\mathbb{R}$.
(c) The set $\{\|f_n\| : n \geq 1\}$ is *not* bounded.

6-8.6. **Example** Let $a = (a_1, a_2, a_3, \cdots)$ be a given sequence of numbers. If for every $x = (x_1, x_2, x_3, \cdots) \in \ell_1$ the series $\sum_{j=1}^\infty a_j x_j$ converges, then we have $a \in \ell_\infty$

Proof. Define $f_n : \ell_1 \to \mathbb{K}$ by $f_n(x) = \sum_{j=1}^n a_j x_j$. Then each f_n is a continuous linear form with $\|f_n\| = \max_{1\leq j\leq n} |a_j|$. For each $x \in \ell_1$, since the partial sums of the convergence series $\sum_{j=1}^\infty a_j x_j$ are bounded, we obtain $\sup_n |f_n(x)| = \sup_n |\sum_{j=1}^n a_j x_j| < \infty$. By Uniform Boundedness Theorem, we get $\sup_n |a_n| = \sup_n \|f_n\| < \infty$, i.e. $a \in \ell_\infty$. □

6-8.7. **Exercise** Let $a = (a_1, a_2, a_3, \cdots)$ be a given sequence of numbers and $1 < p < \infty$. Prove that if for every $x = (x_1, x_2, x_3, \cdots) \in \ell_p$ the series $\sum_{j=1}^\infty a_j x_j$ converges, then we have $a \in \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

6-8.8. **Exercise** Prove that the series $\sum_{j=1}^\infty a_j$ of complex numbers converges absolutely if for every $x = (x_1, x_2, \cdots) \in c_0$, the series $\sum_{j=1}^\infty a_j x_j$ converges.

6-8.9. **Exercise** Prove that there is no sequence $\{a_n\}$ of complex numbers with the following property: a series $\sum_{j=1}^\infty x_j$ of complex numbers converges absolutely iff the sequence $\{a_n x_n\}$ is bounded.

6-9 Open Map and Closed Graph Theorems

6-9.1. Let X, Y be metric spaces and let $f : X \to Y$ be a *surjection*. Then f is called an *open map* if for every open subset M of X, the image $f(M)$ is open in Y. Note that a bijective open map means that its inverse map is continuous.

6-9.2. **Open-Map Theorem** Let E, F be Banach spaces. Every continuous linear surjection $f : E \to F$ is an open map.

Proof. Let $V = \{x \in E : \|x\| \leq r\}$ be a ball in E. We claim there is a closed ball contained in $\overline{f(V)}$. In fact, for every $x \in E$, there is $m > \|x\|/r$, i.e. $x/m \in V$, or $x \in mV$. Hence we have $E = \bigcup_{m=1}^\infty mV$. Since f is a linear surjection, we obtain $F = \bigcup_{m=1}^\infty mf(V) \subset \bigcup_{m=1}^\infty m\overline{f(V)}$, i.e. $F = \bigcup_{m=1}^\infty m\overline{f(V)}$. Because all $m\overline{f(V)}$ are closed subsets of the Banach space F, Category theorem guarantees that $m\overline{f(V)}$ has an interior point. Since $x \to x/m$ is a homeomorphism, $\overline{f(V)}$ also has an interior point, say a. Suppose $\mathbb{B}(a, 3s) \subset \overline{f(V)}$ where

$s > 0$. Let $W = \{y \in F : \|y\| \le s\}$. To show $W \subset \overline{f(V)}$, take any $w \in W$. Then both a and $a+2w$ are in $\mathbb{B}(a,3s)$ and thus in $\overline{f(V)}$. There are x_n, y_n in V such that $f(x_n) \to a$ and $f(y_n) \to a+2w$ as $n \to \infty$. Clearly, $f(\frac{1}{2}y_n - \frac{1}{2}x_n) \to w$. On the other hand, since $\|\frac{1}{2}y_n - \frac{1}{2}x_n\| \le \frac{1}{2}\|y_n\| + \frac{1}{2}\|x_n\| \le r$, i.e. $\frac{1}{2}y_n - \frac{1}{2}x_n \in V$. Therefore $w \in \overline{f(V)}$. This proves $W \subset \overline{f(V)}$. Next, we claim $W \subset 2f(V)$. Take any $w \in W$. Then $w \in \overline{f(V)}$. There is $x_1 \in V$ such that $\|w - f(x_1)\| \le \frac{1}{2}s$. Let $y_1 = x_1$. By induction, suppose that x_n, y_n have been chosen so that

$$y_n = x_1 + \frac{1}{2}x_2 + \frac{1}{2^2}x_3 + \cdots + \frac{1}{2^{n-1}}x_n, \qquad \#1$$

and

$$\|w - f(y_n)\| \le \frac{1}{2^n}s. \qquad \#2$$

Then $2^n[w - f(y_n)]$ is in W and hence also in $\overline{f(V)}$. There is $x_{n+1} \in V$ such that $\|2^n[x - f(y_n)] - f(x_{n+1})\| \le \frac{1}{2}s$. Let $y_{n+1} = y_n + \frac{1}{2^n}x_{n+1}$. By induction, we have chosen sequences $\{x_n\}$ and $\{y_n\}$ satisfying both #1,2. Now observe that

$$\|y_n\| \le \|x_1\| + \frac{1}{2}\|x_2\| + \frac{1}{2^2}\|x_3\| + \cdots + \frac{1}{2^{n-1}}\|x_n\| \le r + \frac{r}{2} + \frac{r}{2^2} + \cdots + \frac{r}{2^{n-1}} \le 2r.$$

Hence $\{y_n\}$ converges to some $y \in E$. The continuity of f gives $f(y_n) \to f(y)$. By #2, we also have $f(y_n) \to w$. Therefore $w = f(y)$. Since $\|y\| \le 2r$, $\frac{1}{2}w = f(\frac{1}{2}y) \in f(V)$, or $w \in 2f(V)$. So we obtain $W \subset 2f(V)$ as required. Finally, let H be an open subset of E. Take any $x \in H$ and let $y = f(x)$. Since H is open, we have $\mathbb{B}(x,2r) \subset H$ for some $r > 0$. Let V be defined as above. Construct W as above. We claim $\mathbb{B}(y,\frac{1}{2}s) \subset f(H)$. In fact, take any $z \in \mathbb{B}(y,\frac{1}{2}s)$. Then $\|z - y\| \le \frac{1}{2}s$, i.e. $z - y \in \frac{1}{2}W$. There is $v \in V$ such that $z - y = f(v)$. Thus $z = y + f(v) = f(x+v)$. Now $\|(x+v) - x\| = \|v\| \le r < 2r$, i.e. $x + v \in \mathbb{B}(x,2r)$, or $x + v \in H$. Therefore $z \in f(H)$. Since z is arbitrary, we have $\mathbb{B}(y,\frac{1}{2}s) \subset f(H)$ as claimed. Because $y \in f(H)$ is arbitrary, the set $f(H)$ is open in F. This completes the proof. □

6-9.3. **Banach's Inversion Theorem** Every continuous isomorphism between two Banach spaces is a topological isomorphism.

Proof. It is left as an exercise. □

6-9.4. An exercise to show that a continuous isomorphism of which the inverse need not be continuous, was given in §3-6.3.

6-9.5. Since continuous linear maps are bounded above, it would be natural to ask questions about linear maps that are bounded below. This result will be used to deal with spectral theory in Hilbert spaces. Let E, F be Banach spaces and $f : E \to F$ a continuous linear map. Then f is said to be ***bounded below*** if there is $\delta > 0$ such that $\delta\|x\| \leq \|f(x)\|$ for all $x \in E$.

6-9.6. **Lemma** If f is bounded below, then the range $f(E)$ is closed in F.

Proof. If suffices to show that $f(E)$ is complete. Let $\{f(x_n)\}$ be a Cauchy sequence in $f(E)$. Since f is bounded below, there is $\delta > 0$ such that $\delta\|x\| \leq \|f(x)\|$ for all $x \in E$. Hence

$$\|x_m - x_n\| \leq \|f(x_m - x_n)\|/\delta = \|f(x_m) - f(x_n)\|/\delta \to 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence in the Banach space E. The limit $y = \lim x_n$ exists in E. Since f is continuous, $f(x_n) \to f(y)$ and $f(y) \in f(E)$. Therefore $f(E)$ is closed. □

6-9.7. **Theorem** The continuous linear map f is a topological isomorphism iff it is bounded below and has a dense range $f(E)$.

Proof. ($\Rightarrow$) Assume f is topological isomorphism. Then $f(E) = F$ is dense in F. Now for every $x \in E$, we obtain $\|x\| = \|f^{-1}f(x)\| \leq \|f^{-1}\| \; \|f(x)\|$. Since $1 = \|I\| = \|f^{-1}f\| \leq \|f^{-1}\| \; \|f\|$, we have $\|f^{-1}\| \neq 0$. Therefore the number $\delta = 1/\|f\| > 0$ satisfies the requirement.

($\Leftarrow$) Since f is bounded below, $f(E)$ is closed in F. Because $f(E)$ is dense in F, we have $f(E) = F$, i.e. f is surjective. Let $\delta > 0$ satisfies $\delta\|x\| \leq \|f(x)\|$ for all $x \in E$. Then $f(x) = 0$ implies $x = 0$. Hence f is injective. Now f is a continuous isomorphism. By Banach Inversion Theorem, f is a topological isomorphism. □

6-9.8. Let X, Y be metric spaces and $f : X \to Y$ a given map. Then the set $\{(x, f(x)) \in X \times Y : x \in X\}$ is called the ***graph*** of f. The map has closed graph if its graph is a closed subset of the product metric space $X \times Y$. Clearly, f has closed graph iff $x_n \to x$ in X and $f(x_n) \to y$ in Y imply $y = f(x)$. Hence whether the graph is closed in the product space, is independent of the choice of the product metric.

6-9.9. **Exercise** Prove that every continuous map from a metric space into a metric space has closed graph.

6-9.10. **Closed Graph Theorem** Let E, F be Banach spaces and $f : E \to F$ a linear map. If the graph of f is closed, then f is continuous.

Proof. Let G be the graph of f. Since f is linear, G is vector subspace of the product space $E \times F$. Since both E, F are Banach spaces, so is the

$E \times F$. Since G is closed in $E \times F$, G is also a Banach space. Let $h(x, y) = x$ and $k(x, y) = y$ be projections from $E \times F$ onto E, F respectively. Then both h, k are continuous linear maps. The restriction $h|G$ is actually a continuous isomorphism from the Banach space G onto the Banach space E and hence it is a topological isomorphism. Its inverse map $(h|G)^{-1} : E \to G$ is a continuous linear surjection. Hence the composite map $f = k(h|G)^{-1}$ is also continuous.□

6-9.11. **Exercise** Let $f : \mathfrak{F}_\infty \to \mathfrak{F}_\infty$ be defined by $f(x) = (x_1, 2x_2, 3x_3, \cdots)$, for all $x \in \mathfrak{F}_\infty$. Prove that the inverse map f^{-1} is continuous and hence f has a closed graph. Show that f is discontinuous.

6-9.12. **Exercise** Let A, B be closed vector subspaces of a Banach space E such that $A \cap B = \{0\}$. Prove that $A + B$ is closed in E iff there is $\lambda \geq 0$ such that for each $a \in A$ and $b \in B$, we have $\|a\| \leq \lambda \|a + b\|$.

6-9.13. **Exercise** Let E, F be Banach spaces and $g : E \to F$ a linear map. Show that if for every $u \in F'$, $ug : E \to \mathbb{K}$ is continuous then g is continuous.

6-99. **References and Further Readings** : Antosik, Cohen, Fan-63, Asimow, Horowitz and Holland-73.

Chapter 7

Natural Constructions

7-1 Bidual Spaces

7-1.1. Let E be a normed space. The dual space E' of E is a Banach space. Again the dual space of E' is also a Banach space which is called the *bidual space* of E and is denoted by E''. Next theorem says that E can be identified as a subspace of E''. Bidual spaces provide a very simple proof of the existence of completions. Later, it will be shown that every element $f \in E''$ can be approximated by a vector x in E on any given finite dimensional vector subspace of E' without increasing too much of its norm.

7-1.2. **Bidual Embedding Theorem** Let E be a normed space. For each $x \in E$, let $(Jx)(u) = u(x)$ for all $u \in E'$.
(a) Jx is a continuous linear form on E'.
(b) $J : E \to E''$ is a linear isometry.
(c) For every $x \in E$, we have $\|x\| = \sup\{|u(x)| : u \in E', \|u\| \leq 1\}$.

Proof. Clearly Jx is a linear form on E'. Since $|(Jx)(u)| = |u(x)| \leq \|u\| \; \|x\|$, Jx is continuous on E' and $\|Jx\| \leq \|x\|$. Therefore $Jx \in E''$. It is routine to show that J is a linear map from E into E''. Take any $x \in E$. There is $v \in E'$ such that $\|v\| = 1$ and $v(x) = \|x\|$. Hence

$$\begin{aligned}\|Jx\| &= \sup\{|(Jx)(u)| : u \in E', \|u\| \leq 1\} \\ &= \sup\{|u(x)| : u \in E', \|u\| \leq 1\} \geq v(x) = \|x\|.\end{aligned}$$

Therefore J is a linear isometry. Finally for each $x \in E$, we have

$$\|x\| = \|Jx\| = \sup\{|(Jx)(u)| : \|u\| \leq 1\} = \sup\{|u(x)| : \|u\| \leq 1\}. \qquad \square$$

7-1.3. The map J above is called the *natural embedding* from E into its bidual space E''. We shall identify E as a vector subspace of E'' via the natural embedding.

7-1.4. Let E be a normed space. Then a Banach space F is called a *completion* of E if E is isometrically isomorphic to a dense subspace of F. Clearly, every Banach space is a completion of itself.

7-1.5. **Theorem** Every normed space has a completion.

Proof. Let J be the natural embedding of a normed space E into its bidual space E''. Let F be the closure of $J(E)$ in E''. Since E'' is a Banach space, F is also a Banach space. Consequently E is isometrically isomorphic to the dense subspace $J(E)$ of F. Therefore F is a completion of E. □

7-1.6. **Generalized Orthonormalization Process** Let E^* be the algebraic dual of a vector space E. If $u_1, u_2, \cdots, u_n$ are linearly independent in E^*, then there are $x_1, x_2, \cdots, x_n$ in E such that $u_i(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. This should be compared with the corresponding result in linear algebra. Note that Chapter 13 especially §13-2.7 should be read concurrently.

Proof. We shall prove by induction on n. For $n = 1$, since u_1 is non-zero, there is $y_1 \in E$ satisfying $u_1(y_1) \neq 0$. Let $x_1 = y_1/u_1(y_1)$. Then $u_1(x_1) = 1$. Next, assume that $u_1, u_2, \cdots, u_n, u_{n+1}$ are linearly independent in E^*. By inductive assumption, there are $y_1, y_2, \cdots, y_n$ in E such that $u_i(y_j) = \delta_{ij}$ for all $1 \le i, j \le n$. Define

$$u = u_{n+1} - \sum\nolimits_{i=1}^{n} u_{n+1}(y_i)u_i.$$

Since $u_1, u_2, \cdots, u_{n+1}$ are linearly independent, we have $u \neq 0$. There is $y \in E$ such that $u(y) \neq 0$. Let

$$x = y - \sum\nolimits_{j=1}^{n} u_j(y)y_j.$$

Observe that for all $1 \le i, j \le n$, we have

$$u(y_j) = u_{n+1}(y_j) - \sum\nolimits_{i=1}^{n} u_{n+1}(y_i)u_i(y_j) = 0,$$

$$u_i(x) = u_i(y) - \sum\nolimits_{j=1}^{n} u_j(y)u_i(y_j) = 0,$$

$$u(x) = u(y) - \sum\nolimits_{j=1}^{n} u_j(y)u(y_j) = u(y) \neq 0.$$

Define $x_{n+1} = \dfrac{x}{u(x)}$ and $x_j = y_j - u_{n+1}(y_j)x_{n+1}$ for $1 \le j \le n$. Then for $1 \le i, j \le n$, we have $u_i(x_{n+1}) = u_i(x)/u(x) = 0$,

$$u_{n+1}(x_{n+1}) = \frac{u_{n+1}(x)}{u(x)} = \frac{u(x) + \sum_{i=1}^{n} u_{n+1}(y_i)u_i(x)}{u(x)} = 1,$$

$$u_{n+1}(x_j) = u_{n+1}(y_j) - u_{n+1}(y_j)u_{n+1}(x_{n+1}) = 0$$

and

$$u_i(x_j) = u_i(y_j) - u_{n+1}(y_j)u_i(x_{n+1}) = \delta_{ij}.$$

This completes the proof. □

7-1.7. **Corollary** Let E be a normed space. If $x_1, x_2, \cdots, x_n$ are linearly independent in E, then there are continuous linear forms $u_1, u_2, \cdots, u_n$ on E such that $u_i(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$.

Proof. Since $J(x_1), J(x_2), \cdots, J(x_n)$ are linearly independent in E'', there are $u_1, u_2, \cdots, u_n \in E'$ such that $u_i(x_j) = J(x_j)(u_i) = \delta_{ij}$ for all $1 \le i, j \le n$. □

7-1.8. **Bidual Approximation Theorem** Let E be a Banach space and J the natural embedding of E into its bidual space E''. Given any $f \in E''$, for any $u_1, u_2, \cdots, u_n \in E'$ and for each $\varepsilon > 0$, there is $x \in E$ such that $\|x\| \le \|f\| + \varepsilon$ and $(Jx)(u_i) = f(u_i)$, for all $1 \le i \le n$.

Proof. Since the theorem is linear in u_i, we may assume that $u_1, u_2, \cdots, u_n$ are linearly independent. For each $x \in E$, define $\varphi(x) = \big(u_1(x), u_2(x), \cdots, u_n(x)\big)$. Since all u_i are continuous on E, φ is a continuous linear map from E into $\mathbb{K}^n$. The independence of $u_1, u_2, \cdots, u_n$ offers $a_1, a_2, \cdots, a_n \in E$ such that $u_i(a_j) = \delta_{ij}$ for all $1 \le i, j \le n$. For each $r = (r_1, r_2, \cdots, r_n)$ in $\mathbb{K}^n$, let $b = \sum_{j=1}^{n} r_j a_j$. Then we have $u_i(b) = r_i$ and hence $\varphi(b) = r$. This shows that φ is a continuous surjection from the Banach space E onto the Banach space $\mathbb{K}^n$. By Open-Map Theorem, φ is an open map. Let $B = \{x \in E : \|x\| < \|f\| + \varepsilon\}$. Since B is open convex balanced, so is $\varphi(B)$ and in particular absorbing. Let p be the gauge of $\varphi(B)$. Then p is a seminorm on $\mathbb{K}^n$. Let $s = (f(u_1), f(u_2), \cdots, f(u_n))$. Suppose to the contrary that $s \notin \varphi(B)$. Then $p(s) \ge 1$. There is a linear form g on $\mathbb{K}^n$ such that $g(s) = p(s)$ and $|g(y)| \le p(y)$ for all $y \in \mathbb{K}^n$. Assume that the linear form g on $\mathbb{K}^n$ has the representation $g(y) = \sum_{i=1}^{n} \lambda_i y_i$ for all $y = (y_1, y_2, \cdots, y_n) \in \mathbb{K}^n$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are some constants in $\mathbb{K}$. Let $u = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n$. Then we get $|u(b)| = |\sum_{i=1}^{n} \lambda_i u_i(b)| = |g\varphi(b)| \le 1$ for all $b \in B$. Taking supremum over $b \in B$, we have $\|u\|(\|f\| + \varepsilon) \le 1$. Also $1 \le g(s) = \sum_{i=1}^{n} \lambda_i f(u_i) = f(u) \le \|f\| \, \|u\|$. Hence we obtain a contradiction $\|f\| + \varepsilon \le \|f\|$. Therefore $s \in \varphi(B)$. Consequently there is $x \in B$ such that $\varphi(x) = s$, i.e. $\|x\| < \|f\| + \varepsilon$ and $u_i(x) = f(u_i)$ for all $1 \le i \le n$. □

7-2 Quotient Spaces

7-2.1. Let E be a normed space and M a *closed* vector subspace of E. Let φ be the quotient map from E onto the quotient space E/M. For each $a \in E$, the *quotient norm* of $\varphi(a)$ in E/M is defined by

$$\|\varphi(a)\| = d(a, M) = \inf\{\|a - x\| : x \in M\}.$$

Clearly it is independent of the choice of a. The quotient vector space E/M together with the quotient norm is called the *quotient normed space*.

7-2.2. **Lemma** The quotient norm is a norm on the quotient vector space.

Proof. Clearly $\|\varphi(a)\| = d(a, M) \geq 0$. Observe that $\|\varphi(a)\| = d(a, M) = 0$ iff $a \in M^- = M$ iff $\varphi(a) = 0$. Next, suppose $a, b \in E$ are given. Choose any $\varepsilon > 0$. There are $x, y \in M$ such that $\|a - x\| \leq \|\varphi(a)\| + \frac{1}{2}\varepsilon$; and $\|b - y\| \leq \|\varphi(y)\| + \frac{1}{2}\varepsilon$. Hence

$$\begin{aligned}\|\varphi(a) + \varphi(b)\| &= \|\varphi(a + b)\| \leq \|(a + b) - (x + y)\|, \text{ because } x + y \in M \\ &\leq \|a - x\| + \|b - y\| \leq \|\varphi(a)\| + \|\varphi(b)\| + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\|\varphi(a) + \varphi(b)\| \leq \|\varphi(a)\| + \|\varphi(b)\|$. Next, take any scalar λ. If $\lambda = 0$, then $\|\lambda\varphi(a)\| = \|\varphi(0)\| = 0 = |\lambda|\|\varphi(a)\|$. Suppose $\lambda \neq 0$. Then

$$\begin{aligned}\|\lambda\varphi(a)\| &= \|\varphi(\lambda a)\| = \inf_{x \in M} \|\lambda a - x\| \\ &= \inf_{x \in M} |\lambda| \left\|a - \frac{x}{\lambda}\right\| = |\lambda| \inf_{y \in M} \|a - y\| = |\lambda|\|\varphi(a)\|.\end{aligned}$$

Therefore the quotient norm is a norm on the quotient space. □

7-2.3. **Theorem** The quotient map is an open continuous linear surjection.

Proof. Clearly φ is a linear surjection. Since $\|\varphi(a)\| \leq \|a\|$, φ is continuous. Let A be an open subset of E. Take any $a \in A$. Since A is open, there is $r > 0$ such that $\mathbb{B}(a, r) \subset A$. We claim $\mathbb{B}[\varphi(a), r] \subset \varphi(A)$. In fact, suppose $\|\varphi(c) - \varphi(a)\| < r$. Then $\|\varphi(c - a)\| < r$. There is $x \in M$ with $\|c - a - x\| < r$. Hence $c - x \in \mathbb{B}(a, r)$, or $c - x \in A$. Thus $\varphi(c) = \varphi(c - x) \in \varphi(A)$. This proves $\mathbb{B}[\varphi(a), r] \subset \varphi(A)$. Therefore $\varphi(A)$ is an open set in E/M. □

7-2.4. **Theorem** If E is a Banach space, then so is the quotient space E/M.

Proof. To show that E/M is complete, let $\{a_n\}$ be a sequence in E such that $\|\varphi(a_{n+1}) - \varphi(a_n)\| \leq 2^{-n}$ for all $n \geq 1$. It suffices to prove that $\{\varphi(a_n)\}$ converges in E/M. Choose $x_1 = a_1$. Inductively, assume that $x_n \in E$ has been chosen such that $\varphi(x_n) = \varphi(a_n)$. Since $\|\varphi(a_{n+1}) - \varphi(a_n)\| < 2^{-n+1}$, there is $z \in M$ such that $\|a_{n+1} - a_n - z\| < 2^{-n+1}$. Let $x_{n+1} = a_{n+1} - a_n - z + x_n$. Then

$$\begin{aligned}\varphi(x_{n+1}) &= \varphi(a_{n+1}) - \varphi(a_n) - \varphi(z) + \varphi(x_n) \\ &= \varphi(a_{n+1}) - \varphi(a_n) - 0 + \varphi(a_n) = \varphi(a_{n+1}).\end{aligned}$$

Furthermore, we have $\|x_{n+1} - x_n\| = \|a_{n+1} - a_n - z\| < 2^{-n+1}$. By completeness of E, $x_n \to a$ for some $a \in E$. Because the map φ is continuous, we obtain $\varphi(a_n) = \varphi(x_n) \to \varphi(a)$. Therefore $\{\varphi(a_n)\}$ converges in E/M. □

7-2.5. **Theorem** Let E, F be normed spaces and $f : E \to F$ a linear map from E into F. Suppose M is a closed vector subspace contained in the kernel of f; and φ the quotient map from E onto E/M.
(a) There is a unique linear map g from E/M into F such that $f = g\varphi$.
(b) The map g is continuous iff f is.

Proof. (a) It follows algebraically from $M \subset \ker(f)$.
(b) Suppose f is continuous. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that for all $\|x\| \le \delta$ in E, we have $\|f(x)\| \le \varepsilon$. Now take any $\|\varphi(a)\| \le \frac{1}{2}\delta$ in E/M. There is $y \in M$ such that $\|a - y\| < \delta$. Hence

$$\|g\varphi(a)\| = \|g\varphi(a - y)\| = \|f(a - y)\| \le \varepsilon.$$

This proves that g is continuous. The converse is obvious. □

7-2.6. **Exercise** Suppose f is continuous. Prove that the map g is an open surjection iff f is. Prove that if $M = \ker(f)$ then $\|f\| = \|g\|$.

7-2.7. **Exercise** Let E be a normed space and M a closed subspace of E. Prove that if both M and E/M are complete, then so is E.

7-2.8. **Exercise** Let M be a vector subspace of a vector space E. Prove that the vector space $(E/M) \times M$ is isomorphic to E.

7-3 Duality of Subspaces and Quotients

7-3.1. Let E be a normed space and E' its dual space. The *orthogonal complement* of a subset M of E is defined by $M^\perp = \{u \in E' : u(M) = 0\}$. On the other hand, the *orthogonal complement* of a subset H of E' is defined by $H^\perp = \{x \in E : H(x) = 0\}$. The duality theory of this section will be used to develop Riesz-Schauder theory of compact linear operators.

7-3.2. **Lemma** (a) $M^\perp$ is a closed vector subspace of E'.
(b) $H^\perp$ is a closed vector subspace of E.

Proof. (a) Let J be the natural embedding of E into E''. Because $M^\perp = \bigcap_{x \in M} \ker(Jx)$ and since all Jx are continuous on E', $M^\perp$ is therefore closed in E'.
(b) It follows from $H^\perp = \bigcap_{h \in H} \ker(h)$.

7-3.3. **Double Complement Theorem** If M is a closed vector subspace of a normed space E, then $M^{\perp\perp} = M$.

Proof. Clearly we have $M \subset M^{\perp\perp}$. Conversely, suppose $a \in E \setminus M$. Then there is $u \in E'$ such that $u(a) = d(a, M)$ and $u(M) = 0$. Hence $u \in M^{\perp}$ and $u(a) \neq 0$. This shows $a \in E \setminus M^{\perp\perp}$. Therefore $M^{\perp\perp} \subset M$. It completes the proof. □

7-3.4. **Theorem** Let B be a subset of a normed space E. If M is the closed vector subspace spanned by B, then $M^{\perp} = B^{\perp}$ and $B^{\perp\perp} = M$.

Proof. Clearly we have $B \subset M$ and hence $M^{\perp} \subset B^{\perp}$. Conversely suppose $u \in B^{\perp}$. Let x be a vector in the subspace N spanned by B. Then $x = \sum_{j=1}^{k} \lambda_j b_j$ for some scalars $\lambda_j \in \mathbb{K}$ and some $b_j \in B$. Therefore we get $u(x) = \sum_{j=1}^{k} \lambda_j u(b_j) = 0$. Next, let y be a vector in M. Then there is a sequence $\{x_n\}$ in N convergent to y. The continuity of u gives $u(y) = \lim u(x_n) = 0$. Since $y \in M$ is arbitrary, we have $u \in M^{\perp}$. Consequently $B^{\perp} \subset M^{\perp}$. The proof is completed by $B^{\perp\perp} = (B^{\perp})^{\perp} = (M^{\perp})^{\perp} = M$. □

7-3.5. **Theorem** Let M be a closed vector subspace of a normed space E. Let φ be the quotient map from E' onto the quotient space $E'/M^{\perp}$. Let f be the map from $E'/M^{\perp}$ into M' defined by $f[\varphi(u)](x) = u(x)$, for all $u \in E'$ and all $x \in M$. Then f is a linear isometry form $E'/M^{\perp}$ onto the dual space M' of a closed vector subspace M. Therefore we can identify $E'/M^{\perp}$ and M'.

Proof. Suppose $\varphi(u) = \varphi(v)$ where $u, v \in E'$. Then $u - v \in M^{\perp}$. Hence for all $x \in M$, we have $(u - v)(x) = 0$, i.e. $f[\varphi(u)](x) = u(x) = v(x) = f[\varphi(v)](x)$. Therefore $f[\varphi(u)] = f[\varphi(v)]$. Consequently, f is well-defined on $E'/M^{\perp}$. Since u is continuous linear on M, $f[\varphi(u)]$ is in M'. Take any scalars $\alpha, \beta \in \mathbb{K}$. Observe that for all $x \in M$,

$$f[\alpha\varphi(u) + \beta\varphi(v)](x) = f[\varphi(\alpha u + \beta v)](x) = (\alpha u + \beta v)(x)$$
$$= \alpha u(x) + \beta v(x) = \alpha f[\varphi(u)](x) + \beta f[\varphi(v)](x) = \{\alpha f[\varphi(u)] + \beta f[\varphi(v)]\}(x)$$

i.e. $f[\alpha\varphi(u) + \beta\varphi(v)] = \alpha f[\varphi(u)] + \beta f[\varphi(v)]$. Hence f is a linear map from $E'/M^{\perp}$ into M'. Now consider $u \in E'$, $w \in M^{\perp}$ and $x \in M$. We have

$$|f[\varphi(u)](x)| = |u(x)| = |(u - w)(x)| \leq \|u - w\| \; \|x\|.$$

Taking infimum, we obtain

$$|f[\varphi(u)](x)| \leq \inf_{w \in M^{\perp}} \|u - w\| \; \|x\| = \|\varphi(u)\| \|x\|.$$

Since $x \in M$ is arbitrary, we get $\|f[\varphi(u)]\| \leq \|\varphi(u)\|$ for all $u \in E'$. Finally, let $h \in M'$ be given. Then h has an extension $u \in E'$ such that $\|u\| = \|h\|$. Now for all $x \in M$, we obtain $f[\varphi(u)](x) = u(x) = h(x)$, i.e. $f[\varphi(u)] = h$. Hence f is surjective. Furthermore, we get $\|\varphi(u)\| \leq \|u\| = \|h\| = \|f[\varphi(u)]\|$. Therefore f is a linear isometry from $E'/M^\perp$ onto M'. □

7-3.6. **Theorem** Let M be a closed vector subspace of a normed space E. Let φ be the quotient map from E onto the quotient space E/M. Suppose that $g : M^\perp \to (E/M)'$ be given by $[g(u)][\varphi(x)] = u(x)$, for all $u \in M^\perp$ and all $x \in E$. Then g is a linear isometry form $M^\perp$ onto $(E/M)'$. Therefore we can identify $M^\perp$ as the dual space of E/M.

Proof. Suppose $\varphi(x) = \varphi(y)$ and $u \in M^\perp$. Then $x - y \in M$ and hence $u(x-y) = 0$, i.e. $[g(u)][\varphi(x)] = [g(x)][\varphi(y)]$. Hence $g(u)$ is well-defined on E/M. Next, for $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$, we have

$$[g(u)][\alpha\varphi(x) + \beta\varphi(y)] = [g(u)][\varphi(\alpha x + \beta y)] = u(\alpha x + \beta y)$$

$$= \alpha u(x) + \beta u(y) = \alpha[g(u)][\varphi(x)] + \beta[g(u)][\varphi(y)].$$

Hence $g(u)$ is linear on E/M. Furthermore, for every $z \in M$, we get

$$[g(u)][\varphi(x)] = |u(x)| = |u(x-z)| \leq \|u\| \, \|x - z\|.$$

Taking infimum over $z \in M$, we obtain $|[g(u)][\varphi(x)]| \leq \|u\| \, \|\varphi(x)\|$. Hence $g(u)$ is in $(E/M)'$. Moreover, we get $\|g(u)\| \leq \|u\|$. For $u, v \in M^\perp$ and $\alpha, \beta \in \mathbb{K}$, observe that

$$[g(\alpha u + \beta v)][\varphi(x)] = (\alpha u + \beta v)(x) = \alpha u(x) + \beta v(x)$$

$$= \alpha[g(u)][\varphi(x)] + \beta[g(u)][\varphi(x)] = [\alpha g(u) + \beta g(v)][\varphi(x)],$$

i.e.

$$g(\alpha u + \beta v) = \alpha g(u) + \beta g(v).$$

Hence $g : M^\perp \to (E/M)'$ is a linear map. Next let $h : E/M \to \mathbb{K}$ be a continuous linear form. Then $u = h\varphi$ is a continuous linear form on E. Clearly $u(M) = 0$ and hence $u \in M^\perp$. Since $[g(u)]\varphi(x) = u(x) = h[\varphi(x)]$, we have $g(u) = h$. Therefore g is surjective. Finally, for all $x \in E$ and $u \in M^\perp$, we obtain $|u(x)| = |[g(u)][\varphi(x)]| \leq \|g(u)\|\|\varphi(x)\| \leq \|g(u)\| \, \|x\|$, i.e. $\|u\| \leq \|g(u)\|$. Therefore g is a linear isometry from $M^\perp$ onto $(E/M)'$. □

7-4 Direct Sums

7-4.1. Let E be a vector space and M, N vector subspaces of E. Then E is called *an algebraic direct sum* of M, N if every vector x in E can be expressed uniquely as $x = a+b$ where $a \in M$ and $b \in N$. In this case, the map $p : E \to M$ defined by $p(x) = a$ is called a *projection* from E onto M along N. Clearly, p is a linear map and it is also an *idempotent*, i.e. $p^2 = p$. The vector subspace N is called an *algebraic complement* of M.

7-4.2. **Lemma** Let M, N be vector subspaces of a vector space E. Then the following statements are equivalent.
(a) E is the direct sum of M and N.
(b) $E = M + N$ and $M \cap N = \{0\}$
(c) The *addition map* $f(a,b) = a + b : M \times N \to E$ is an algebraic isomorphism.

7-4.3. Let M, N be vector subspaces of a normed space E. Then E is called a *topological direct sum* of M, N if the addition map $f : M \times N \to E$ is a topological isomorphism. In this case, N is called a *topological complement* of M. For convenience, write $M = E \ominus N$. See §13-10.6 for Hilbert spaces. We work with the norm $\|(a,b)\| = \|a\| + \|b\|$ for the product space $M \times N$.

7-4.4. **Lemma** Let M, N be vector subspaces of a normed space E. If $E = M \oplus N$ is a topological direct sum, then both M, N are closed.

Proof. Let $a_n \to x$ where $a_n \in M$. Write $x = a + b$ with $a \in M$ and $b \in N$. Define $b_n = 0$ and $x_n = a_n + b_n$. Then $x_n \to x$ in E. Since the addition map $f(a,b) = a + b : M \times N \to E$ is a topological isomorphism, we have $f^{-1}(x_n) \to f^{-1}(x)$, i.e. $(a_n, b_n) \to (a, b)$ in $M \times N$. Hence $a_n \to a$. Therefore $x = a \in M$. Consequently, M is closed in E. □

7-4.5. **Theorem** Let M, N be closed vector subspaces of a Banach space E. If E is the algebraic direct sum of M, N, then E is also the topological direct sum of M, N.

Proof. As closed subspaces of a Banach space E, both M, N and hence their product space are also Banach spaces. Since E is the algebraic direct sum of M, N, the addition map $f : M \times N \to E$ is a continuous algebraic isomorphism from a Banach space $M \times N$ onto a Banach space E. Therefore f is a topological isomorphism. □

7-4.6. **Theorem** Let M be a vector subspace of a normed space E. Then the following statements are equivalent.
(a) M admits a topological complement.
(b) There is a continuous linear idempotent from E onto M.
(c) There is a closed vector subspace N of E such that
  (i) E is the algebraic direct sum of M, N;
  (ii) the quotient map is a topological isomorphism from M onto E/N.

Proof. ($a \Rightarrow b$) Let N be a topological complement of M. Then the addition map $f : M \times N \to E$ is a topological isomorphism. Let g be the projection from the product space $M \times N$ onto M. Then $p = gf^{-1}$ is a continuous linear idempotent from E onto M.
($b \Rightarrow c$) Let $N = \ker(p)$. Then $E = M + N$ is an algebraic direct sum. Since p is continuous, N is closed. Let $\varphi : E \to E/N$ be the quotient map. Then $\varphi|M$ is a continuous algebraic isomorphism from M onto E/N. Since $p = (\varphi|M)^{-1}\varphi$ is continuous, $(\varphi|M)^{-1}$ is continuous. Therefore the quotient map is a topological isomorphism from M onto E/N.
($c \Rightarrow a$) Because E is the algebraic direct sum of M and N, the addition $f : M \times N \to E$ is an algebraic isomorphism. Clearly f is continuous. Let $\varphi : E \to E/N$ be the quotient map. To show that f^{-1} is continuous, let $x_n \to x$ in E. Let $x_n = a_n + b_n$ and $x = a + b$ be decompositions into $M + N$. Then $\varphi(a_n) = \varphi(x_n) \to \varphi(x) = \varphi(a)$. Since $\varphi|M$ is a topological isomorphism, we have $a_n \to a$. Hence $b_n \to b$, or $(a_n, b_n) \to (a, b)$, i.e. $f^{-1}(x_n) \to f^{-1}(x)$. Therefore f is a topological isomorphism. Consequently, M admits a topological complement. □

7-4.7. **Corollary** Let E be a normed space which is an algebraic direct sum of two vector subspaces M, N. If M is closed and N is finite dimensional, then E is the topological direct sum of M, N.

Proof. Let $\varphi : E \to E/M$ denote the quotient map. Since $E = M + N$ is an algebraic direct sum, $\varphi|N$ is an algebraic isomorphism from a finite dimensional vector space N onto E/M and hence it is a topological isomorphism. Therefore E is the topological direct sum of M, N. □

7-4.8. **Corollary** Every finite dimensional vector subspace M in a normed space E admits a topological complement.

Proof. Let $e_1, e_2, \cdots, e_n$ be a basis in M. Define the linear forms g_i on M by $g_i(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n) = \lambda_i$ for all scalars λ_i. Since M is finite dimensional, each g_i is continuous on M. Let f_i be a continuous linear extension of g_i over the whole space E. Let $p(x) = \sum_{j=1}^{n} f_j(x)e_j$ for each $x \in E$. Then p is a continuous linear idempotent onto M. Therefore M admits a topological complement. □

7-4.9. **Corollary** Let M be a closed vector subspace of a normed space E. Then for every finite dimensional vector subspace N of E, the sum $M + N$ is a closed vector subspace of E.

Proof. Let φ be the quotient map from E onto the quotient space E/M. Then $\varphi(N)$ is a finite dimensional vector subspace of the normed space E/M. Hence $\varphi(N)$ is closed in E/M. Therefore the inverse image $\varphi^{-1}[\varphi(N)] = M + N$ of the closed set $\varphi(N)$ is also closed in E.

7-4.10. **Exercise** Let E be a Banach space and P, Q two idempotents on E such that $P + Q = I$ and $PQ = QP = 0$. Let $M = \ker(P)$ and $N = \ker(Q)$. Show that $E = M \oplus N$ is a topological direct sum and that $\text{Im}(Q) = \ker(P)$.

7-4.11. Let M be a vector subspace of a vector space E. Then the dimension of the quotient space E/M is called the *codimension* of M.

7-4.12. **Theorem** Let M be a finite codimensional vector subspace of a normed space E. If M is closed and if H is a vector subspace of E satisfying $M \subset H \subset E$, then H is also closed in E.

Proof. Let $\varphi : E \to E/M$ be the quotient map. Since $\varphi(H)$ is a vector subspace of the finite dimensional vector space E/M, $\varphi(H)$ is closed in E/M. Therefore $H = H + M = \varphi^{-1}[\varphi(H)]$ is closed in E.

7-4.13. **Lemma** Let M be a vector subspace of a vector space E. Then M has finite codimension iff there is a finite dimensional vector subspace N of E such that $E = M \oplus N$ is an algebraic direct sum.

Proof. Let $\varphi : E \to E/M$ be the quotient map. Suppose $E = M \oplus N$. Then $\varphi|N : N \to E/M$ is an isomorphism, so $\dim E/M = \dim N < \infty$. Therefore M has finite codimension. Conversely, assume $\dim E/M < \infty$. There are $e_1, e_2, \cdots, e_n \in E$ such that $\varphi(e_1), \varphi(e_2), \cdots, \varphi(e_n)$ form a basis for E/M. It is easy to verify that $e_1, e_2, \cdots, e_n$ are linearly independent. Hence the vector subspace N spanned by $e_1, e_2, \cdots, e_n$ has dimension n. It is routine to show that E is the algebraic direct sum of M, N. □

7-4.14. **Theorem** Let E, F be Banach spaces and $f : E \to F$ a continuous linear map. If $f(E)$ is finite codimensional in F, then $f(E)$ is closed in F.

Proof. Let $\varphi : E \to E/\ker(f)$ be the quotient map. Since $f : E \to f(E)$ is a surjection, there is a continuous isomorphism $g : E/\ker(f) \to f(E)$ such that $f = g\varphi$. By finite codimension of $f(E)$ in F, there is a finite dimensional vector subspace N of F such that $F = f(E) \oplus N$ is an algebraic direct sum. Then the restriction of the quotient map $h : f(E) \to F/N$ is a continuous isomorphism. Now $hg : E/\ker(f) \to F/N$ is also a continuous isomorphism between two Banach spaces. By Banach-Inversion Theorem, its inverse map is continuous.

Therefore $g^{-1} = (hg)^{-1}h : f(E) \to E/\ker(f)$ is continuous. Now $f(E)$ as a topological isomorphic image of the Banach space $E/\ker(f)$ is complete and hence closed in F. □

7-5 Transposes

7-5.1. Let E', F' be the dual spaces of normed spaces E, F respectively. Let $f : E \to F$ be a continuous linear map. For every $u \in F'$, let $f^t(u) = uf$. Then $f^t(u)$ as composite of continuous linear maps is a continuous linear form on E. Clearly f^t is a linear map from F' into E'. It is called the *transpose* of f. In order to take the advantage of symmetry between E' and E, we write $< w, x > = w(x)$ for all $w \in E'$ and all $x \in E$. Similar notation will be applied to F. Then we have $< f^t u, x > = < u, fx >$ for all $u \in F'$ and all $x \in E$.

7-5.2. **Theorem** Let E, F be normed spaces and $f : E \to F$ a continuous linear map. Then the transpose f^t of f is a continuous linear map from F' into E'. Furthermore, we have $\|f^t\| = \|f\|$.

Proof. Since $\|f^t(u)\| = \|uf\| \le \|u\|\ \|f\|$, the linear map f^t is continuous and it satisfies $\|f^t\| \le \|f\|$. On the other hand, let $\varepsilon > 0$ be given. There is $x \in E$ such that $\|x\| = 1$ and $\|f\| - \varepsilon \le \|f(x)\|$. Choose $u \in F'$ such that $\|u\| = 1$ and $uf(x) = \|f(x)\|$. Now observe that

$$\|f\| - \varepsilon \le \|f(x)\| = uf(x) = |uf(x)| \le \|uf\|\ \|x\| = \|f^t(u)\| \le \|f^t\|\ \|u\| = \|f^t\|.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f\| \le \|f^t\|$. This completes the proof. □

7-5.3. **Theorem** Let E, F, G be normed spaces.
(a) $(f+g)^t = f^t + g^t$ for all $f, g \in L(E, F)$.
(b) $(\lambda f)^t = \lambda f^t$ for all $f \in L(E, F)$ and all $\lambda \in \mathbb{K}$.
(c) $(gf)^t = f^t g^t$ for all $f \in L(E, F)$ and all $g \in L(F, G)$.

7-5.4. **Theorem** Let E, F be normed spaces and let $f : E \to F$ be a continuous linear map.
(a) $(\text{Im } f)^\perp = \ker(f^t)$.
(b) $\text{Im } f \subset (\ker\ f^t)^\perp$.
(c) $\text{Im } f = (\ker\ f^t)^\perp$ iff $\text{Im } f$ is closed in F.
(d) $(\text{Im } f^t)^\perp = \ker\ f$.
(e) $\text{Im } f^t \subset (\ker\ f)^\perp$.

Proof. (a) $u \in (\text{Im } f)^\perp$ iff $< u, fx > = < f^t u, x > = 0, \forall\ x \in E$ iff $f^t u = 0$ iff $u \in \ker(f^t)$.

(b) Take any $y \in \text{Im}(f)$, i.e. $y = f(x)$ for some $x \in E$. Then for all $u \in \ker(f^t)$, we have $f^t(u) = 0$. Hence $< u, y > = < u, fx > = < f^t u, x > = 0$, or $y \in (\ker\ f^t)^{\perp}$.
(c) By (a), we get $(\ker\ f^t)^{\perp} = (\text{Im}\ f)^{\perp\perp}$ which is the closure of $\text{Im}(f)$. The result follows. Both (d) and (e) are left as exercises. □

7-5.5. **Lemma** Let E, F be Banach spaces and $f : E \to F$ a continuous linear map. If $\text{Im}(f)$ is closed in F, then there is a constant λ such that for each $y \in \text{Im}(f)$ there is $x \in E$ satisfying $y = f(x)$ and $\|x\| \leq \lambda\|y\|$.

Proof. As a closed subspace of a Banach space, $G = f(E)$ is also a Banach space and $f : E \to G$ is a continuous surjection. By Open-Map Theorem, the image $f(V)$ of the open unit ball V in E is open in G. There is $\delta > 0$ such that $G \cap \mathbb{B}(0, \delta) \subset f(V)$. Let $\lambda = \frac{1}{\delta}$. Take any $y \in F$. If $y = 0$, then choose $x = 0$ and obtain $\|x\| \leq \lambda\|y\|$. Assume $y \neq 0$ in G. Then $\frac{\delta y}{\|y\|} \in G \cap \mathbb{B}(0, \delta)$. There is $a \in V$ such that $f(a) = \frac{\delta y}{\|y\|}$. Let $x = \frac{a\|y\|}{\delta}$. Then $f(x) = y$ and $\|x\| = \frac{\|a\|\ \|y\|}{\delta} \leq \lambda\|y\|$. □

7-5.6. **Theorem** Let E, F be Banach spaces and $f : E \to F$ a continuous linear map. If $\text{Im}(f)$ is closed in F, then $\text{Im}(f^t) = (\ker\ f)^{\perp}$. Hence, $\text{Im}(f^t)$ is closed in E'.

Proof. Take any $v \in (\ker\ f)^{\perp}$. For every $f(x) \in f(E)$, define $g[f(x)] = v(x)$. Suppose $f(x) = f(a)$. Then $x - a \in \ker(f)$ and hence $v(x-a) = 0$, i.e. $v(x) = v(a)$. Thus g is well-defined on $f(E)$. Clearly g is linear on $f(E)$. There is $\lambda > 0$ such that for every $y \in f(E)$ the conditions $y = f(x)$ and $\|x\| \leq \lambda\|y\|$ for some $x \in E$. Due to $|g(y)| = |gf(x)| = |v(x)| \leq \|v\|\ \|x\| \leq \lambda\|v\|\ \|x\|$, g is a continuous linear form on $f(E)$. Let g be extended to a continuous linear form u on F. Since for every $x \in E$, $(f^t u)(x) = uf(x) = gf(x) = v(x)$, we have $f^t(u) = v$. Therefore $(\ker\ f)^{\perp} \subset \text{Im}(f^t)$. This completes the proof. □

7-5.7. **Exercise** Let E, F be normed spaces and $f : E \to F$ a continuous linear map. Identify E, F as vector subspaces of E'', F'' respectively. Prove that the second transpose $f^{tt} : E'' \to F''$ is an extension of f.

7-5.8. Let $\mathcal{E} = [e_1, \cdots, e_n]$ and $\mathcal{F} = [f_1, \cdots, f_m]$ be ordered bases for be finite dimensional vector spaces E, F in matrix form respectively. The *coordinate vector* of $x \in E$ with respect to $\mathcal{E}$ is a column vector $[x] = (x_1, x_2, \cdots, x_n)^t$

defined by $x = \mathcal{E}[x] = \sum_{j=1}^{n} e_j x_j$. The *matrix representation* of a linear map $A : E \to F$ relative to $\mathcal{E}, \mathcal{F}$ is the matrix $[A] = [\alpha_{ij}]$ defined by $A\mathcal{E} = \mathcal{F}[A]$, i.e. $[Ae_1, \cdots, Ae_n] = [f_1, \cdots, f_m]\,[A]$, or $Ae_j = \sum_{i=1}^{m} \alpha_{ij} f_i$ for each $1 \le j \le n$. Suppose that $\mathcal{G} = [g_1, \cdots, g_n]$ also an ordered basis for E. The transition matrix from $\mathcal{E}$ to $\mathcal{G}$ is the matrix representation P of the identity map $I : E_{\mathcal{E}} \to E_{\mathcal{G}}$ with the indicated ordered basis, i.e. $\mathcal{E} = \mathcal{G}P$. Clearly if $\mathcal{E}H = \mathcal{E}K$ for $n \times t$ matrices H, K, then $H = K$. In particular, coordinate vectors and matrix representations are uniquely defined.

7-5.9. **Example** The ordered basis $\{(1,3),(2,5)\}$ of $E = \mathbb{R}^2$ is represented by a matrix $\mathcal{E} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. The vector $x = (3, 8) \in E$ has the coordinate vector $[x] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because $x = \mathcal{E}[x]$. Suppose another ordered basis is given by the matrix $\mathcal{G} = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$. Then the transition matrix P from $\mathcal{E}$ to $\mathcal{G}$ is given by $I\mathcal{E} = \mathcal{G}P$, i.e. $P = \mathcal{G}^{-1}\mathcal{E} = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$.

7-5.10. **Exercise** Prove the following statements by concise matrix notation.

(a) $[Ax] = [A][x]$ is the coordinate vector of Ax with respect to the basis $\mathcal{F}$.

(b) The matrix representation of an identity map is the identity matrix and the zero map the zero matrix.

(c) The map $A \to [A]$ is an isomorphism from the vector space $L(E, F)$ of all linear maps onto the vector space $mat(m, n)$ of all $m \times n$ matrices.

(d) If B is a linear map from F into a finite dimensional vector space with a given basis, then we have $[BA] = [B][A]$.

(e) For every linear map $T : E \to E$, we have $[T]_{\mathcal{E}} = P^{-1}[T]_{\mathcal{G}}P$.

7-5.11. **Exercise** Let $x_1, x_2, \cdots, x_n$ be vectors in E and let $u_1, u_2, \cdots, u_n$ be linear forms on E. Prove that if $u_i(x_j) = \delta_{ij}$ for all $1 \le i, j \le n$ then $\{x_1, x_2, \cdots, x_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ are linearly independent. Prove that $\{x_1, x_2, \cdots, x_n\}$ is a basis of E iff $\{u_1, u_2, \cdots, u_n\}$ is a basis of the algebraic dual E^*. In this case, $\{x_1, x_2, \cdots, x_n\}$ and $\{u_1, u_2, \cdots, u_n\}$ are called *dual bases*.

7-5.12. **Exercise** Let $\mathcal{E}' = \{e_1', e_2', \cdots, e_n'\}$ and $\mathcal{F}' = \{f_1', f_2', \cdots, f_m'\}$ denote the bases for E', F' dual to $\mathcal{E}, \mathcal{F}$ respectively. Show that $[A^t] = [A]^t$ with respect to the bases $\mathcal{E}, \mathcal{F}$ and their dual bases.

7-6 Reflexive Spaces

7-6.1. Let E be a normed space and J the natural embedding from E into the bidual space E''. Then E is said to be *reflexive* if J is surjective, i.e. for every $s \in E''$, there is $x \in E$ such that $s(u) = u(x)$ for all $u \in E'$.

7-6.2. **Lemma** (a) Every finite dimensional normed space E is reflexive.
(b) Every reflexive space is a Banach space.

Proof. (a) It follows from $\dim E = \dim E' = \dim E''$.
(b) The dual space of a normed space is always complete. □

7-6.3. **Theorem** Let E be a Banach space. Then E is reflexive iff E' is.

Proof. ($\Rightarrow$) Let $J : E \to E''$ and $J' : E' \to E'''$ be the natural embeddings. Let $s \in E'''$ be given. Now the composite sJ is a continuous linear form on E. Then $u = sJ \in E'$. Take any $a \in E''$. Since E is reflexive, there is $x \in E$ such that $a = Jx$. Consider $(J'u)(a) = a(u) = u(x) = (sJ)x = s(Jx) = s(a)$. Since $a \in E''$ is arbitrary, we have $s = J'u$. Therefore J' is surjective. Consequently E' is reflexive.
($\Leftarrow$) Since E is a Banach space and J is an isometry, $J(E)$ is also a Banach space and hence it is closed in E''. Suppose to the contrary that there is $a \in E''\backslash J(E)$. Then there is $s \in E'''$ such that $s(a) \neq 0$ and $s(JE) = 0$. Since E' is reflexive, there is $u \in E'$ such that $s = J'u$. Now $a(u) = (J'u)(a) = s(a) \neq 0$. On the other hand, for all $x \in E$, we have $u(x) = (Jx)(u) = (J'u)(Jx) = s(Jx) = 0$, i.e. $u = 0$, or $a(u) = 0$. This contradiction shows $JE = E''$. Therefore E is reflexive. □

7-6.4. **Theorem** Every closed subspace M of a reflexive space E is reflexive.

Proof. Let $b \in M''$ be given. For every $u \in E'$, the restriction $u|M$ is a continuous linear form on M. Hence $b(u|M)$ is a well-defined scalar. Let $f(u) = b(u|M)$. Clearly f is a linear form in u. Furthermore,

$$|f(u)| = |b(u|M)| \le \|b\| \, \|u|M\| \le \|b\| \, \|u\|.$$

Hence f is continuous, i.e. $f \in E''$. Since E is reflexive, there is $x \in E$ such that $f = Jx$, i.e. for all $u \in E'$, we have $u(x) = (Jx)(u) = f(u) = b(u|M)$. Now suppose to the contrary that $x \notin M$. There is $u \in E'$ such that $u(M) = 0$ and $u(x) \neq 0$. Hence $u|M = 0$ and so $u(x) = b(u|M) = 0$. This contradiction establishes $x \in M$. Finally, take any $v \in M'$. Let u be an extension of v over E. Then $v(x) = u(x) = b(u|M) = b(v)$. Therefore M is reflexive. □

7-6.5. Let E be a normed space. A subset M of E is said to be *total* if the vector subspace spanned by M is dense in E.

7-6.6. **Theorem** If a normed space E has a countable total set $\{x_n\}$, then E is separable.

Proof. Let Q be a countable dense subset of the scalar field $\mathbb{K}$. Then the set of all linear combinations of $\{x_n\}$ with coefficients in Q is a countable dense subset of E. □

7-6.7. **Exercise** Let $1 \leq p < \infty$. Prove that the sequence $\{e_n : n \geq 1\}$ is total in ℓ_p and c_0. Prove that ℓ_p and c_0 are separable.

7-6.8. **Example** The space ℓ_∞ of all bounded sequences is not separable.

Proof. Suppose to the contrary that ℓ_∞ is separable. Let $\{x^n : n \geq 1\}$ be a countable dense subset of ℓ_∞. Let $x^n = (x_1^n, x_2^n, x_3^n, \cdots)$ for each n. Define $y_n = 1$ if the real part of x_n^n is negative and $y_n = -1$ otherwise. Then clearly $y = (y_1, y_2, y_3, \cdots)$ in ℓ_∞. Furthermore, $1 \leq |y_n - x_n^n| \leq \|y - x^n\|_\infty$ for each n. Therefore $\{x^n : n \geq 1\}$ is not dense in ℓ_∞ which is a contradiction. This proves that ℓ_∞ is not separable. □

7-6.9. **Theorem** Let E be a Banach space. If the dual space E' is separable, then so is E.

Proof. Let $\{u_n\}$ be a countable dense subset of the dual space E'. For each n, there is $x_n \in E$ such that $\|x_n\| = 1$ and $|u_n(x_n)| \geq \frac{1}{2}\|u_n\|$. Let M be the closed vector subspace spanned by $\{x_n\}$. Then M is separable. Suppose to the contrary that $M \neq E$. Then there is $a \in E \setminus M$. Since M is closed, we have $d(a, M) > 0$. There is $u \in E'$ such that $u(M) = 0$, $\|u\| = 1$ and $u(a) \neq 0$. Since $x_n \in M$, $u(x_n) = 0$ for all n. Hence

$$\tfrac{1}{2}\|u_n\| \leq |u_n(x_n)| = \|(u_n - u)(x_n)\| \leq \|u_n - u\|\|x_n\| = \|u_n - u\|.$$

Therefore we have

$$\begin{aligned} 1 = \|u\| &\leq \|u - u_n\| + \|u_n\| \leq \|u - u_n\| + 2\|u - u_n\| \\ &= 3\|u - u_n\|, \text{ i.e. } \|u - u_n\| \geq \tfrac{1}{3}. \end{aligned}$$

This contradiction shows $\{u_n\}$ is not dense in E', contrary to our assumption. Therefore $M = E$. Consequently E is separable. □

7-6.10. **Corollary** If E is a reflexive and separable Banach space, then so are E'' and E'.

Proof. Let $\{x_n\}$ be a countable dense subset of E. Then $\{Jx_n\}$ is a countable subset of E''. Let $a \in E''$ be given. Since E is reflexive, there is $x \in E$ such that $Jx = a$. For every $\varepsilon > 0$, there is n such that $\|x_n - x\| \leq \varepsilon$. Hence we have $\|Jx_n - a\| = \|J(x_n - x)\| = \|x_n - x\| \leq \varepsilon$. Consequently $\{Jx_n\}$ is also dense in E''. Therefore E'' is separable. As a result, E' is also reflexive and separable. □

7-6.11. **Exercise** Show that c_0 is not reflexive.

7-6.12. **Exercise** Show that ℓ_p is reflexive for $1 < p < \infty$.

7-6.13. **Exercise** Prove that if X is a compact metric space, then the Banach space $C_\infty(X)$ is separable.

7-6.14. **Example** If Q is a separable vector subspace of a Banach space E, then there is a countable set P in E' such that $\|x\| = \sup\{|v(x)| : v \in P\}$ for each $x \in Q$ and $\|v\| = 1$ for every $v \in P$. Furthermore for all $x, y \in Q$ if $v(x) = v(y)$ for all $v \in P$, then $x = y$.

Proof. Let $\{x_n\}$ be a countable dense subset of Q. For each n, let $v_n \in E'$ such that $|v_n x_n| = \|x_n\|$ and $\|v_n\| = 1$. Take any $x \in Q$. For every $\varepsilon > 0$, there is n such that $\|x_n - x\| \leq \varepsilon$. Observe that

$$\begin{aligned}\|x\| \leq \|x_n\| + \varepsilon = |v_n x_n| + \varepsilon &\leq |v_n(x_n - x)| + |v_n x| + \varepsilon \\ &\leq \|v_n\| \, \|x_n - x\| + |v_n x| + \varepsilon \leq t + 2\varepsilon\end{aligned}$$

where $t = \sup_n |v_n x|$. Letting $\varepsilon \downarrow 0$, we have $\|x\| \leq t$. It is an exercise to show $t \leq \|x\|$. Therefore $P = \{v_n\}$ is the required set. Finally, $\|x - y\| = \sup_n |v_n(x - y)| = \sup_n |v_n x - v_n y| = 0$ gives $x = y$. □

7-7 Weak Convergence

7-7.1. Let $\{x_n\}$ be a sequence in a normed space E.

(a) The sequence $\{x_n\}$ is said to be *weakly Cauchy* if for every $u \in E'$, the sequence $\{u(x_n)\}$ is a Cauchy sequence in $\mathbb{K}$. The sequence $\{x_n\}$ is said to *converge weakly* to the *weak limit* $a \in E$ if for every $u \in E'$, we have $u(x_n) \to u(a)$ in $\mathbb{K}$. Therefore the weak convergence is to measure the nearness in terms of one arbitrary direction given by u.

(b) For convenience, $\{x_n\}$ is said to be *strongly Cauchy* if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Also $\{x_n\}$ is said to *converge strongly* to $a \in E$ if $\|x_n - a\| \to 0$ as $n \to \infty$. Therefore the strong convergence is to measure the nearness in terms of norm. By $\|x_n - a\| = \sup_{\|u\| \leq 1} |u(x_n) - u(a)|$, strong convergence is also called the *uniform convergence* on the unit ball of its dual space. Uniform Boundedness Theorem will be the main tool for this and next sections.

7-7.2. **Theorem** (a) If $x_n \to a$ strongly, then $x_n \to a$ weakly.
(b) If $\{x_n\}$ is strongly Cauchy, it is weakly Cauchy.
(c) If $x_n \to a$ weakly, then $\{x_n\}$ is weakly Cauchy.
(d) If $x_n \to a$ weakly and $x_n \to b$ weakly, then $a = b$.

Proof. Both (a) and (b) follow from $|u(x)| \leq \|u\| \, \|x\|$. Part (c) follows from the completeness of $\mathbb{K}$. Part (d) follows because the dual space E' separates points of E.

7-7.3. Let M be a subset of a normed space E. Then M is *strongly bounded* or *norm bounded* if $\sup_{x \in M} \|x\| < \infty$. A set M is *weakly bounded* if for each $u \in E'$, $\sup_{x \in M} |u(x)| < \infty$. A set M is *bounded* if it is either strongly bounded or weakly bounded.

7-7.4. **Theorem** A subset M of E is strongly bounded iff it is weakly bounded.

Proof. ($\Rightarrow$) It follows from $|u(x)| \leq \|u\| \, \|x\|$.
($\Leftarrow$) Suppose M is weakly bounded. For each $a \in M$, consider the continuous linear form Ja on E' where J is the natural embedding from E into E''. Since M is weakly bounded, for each $u \in E'$,

$$\sup_{a \in M} |(Ja)u| = \sup_{a \in M} |u(a)| \leq \lambda < \infty.$$

By Uniform Boundedness Theorem, $\sup_{a \in M} \|Ja\| < \infty$, i.e. $\sup_{a \in M} \|a\| < \infty$. Therefore M is strongly bounded. □

7-7.5. **Corollary** Weakly Cauchy sequences are strongly bounded.

7-7.6. **Exercise** Prove that a subset M of a normed space is bounded if every sequence in M has a weakly Cauchy subsequence.

7-7.7. **Corollary** If $x_n \to a$ weakly, then $\{x_n\}$ is weakly bounded. Moreover, we have $\|a\| \leq \liminf \|x_n\|$.

Proof. Since $\{x_n\}$ is weakly Cauchy, it is weakly bounded. Next, for each $u \in E'$, we have $(Jx_n)u \to (Ja)u$. Hence by Banach-Steinhaus Theorem, we obtain $\|Ja\| \leq \liminf \|Jx_n\|$, i.e. $\|a\| \leq \liminf \|x_n\|$. □

7-7.8. **Theorem** Let $\{x_n\}$ be a bounded sequence in E and let $a \in E$ be given. If there is a total subset M of the dual space E' such that for every $u \in M$, $u(x_n) \to u(a)$, then we have $x_n \to a$ weakly.

Proof. Take any $v \in E'$. For the bounded sequence $\{x_n\}$, there is $\lambda > 0$ such that $\|x_n\| \leq \lambda$ for all n and also $\|a\| \leq \lambda$. Since M is total in E', the vector subspace Q spanned by M is dense in E'. For every $\varepsilon > 0$ there is $w \in Q$ such that $\|v - w\| \le \varepsilon/\lambda$. Write $w = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$ where all $\alpha_j \in \mathbb{K}$ and $u_j \in M$. Take any $\mu > \sum_{j=1}^{k} |\lambda_j|$. Since $u_j(x_n) \to u_j(a)$ as $n \to \infty$, there is an integer p such that for all $n \ge p$ we have $|u_j(x_n) - u_j(a)| \le \varepsilon/\mu$. Now for $n \ge p$, we have

$$\begin{aligned} |v(x_n) - v(a)| &\le |v(x_n) - w(x_n)| + |w(x_n) - w(a)| + |w(a) - v(a)| \\ &\le \|v - w\| \, \|x_n\| + \sum_{j=1}^{k} |\alpha_j| \, |u_j(x_n) - u_j(a)| + \|w - v\| \, \|a\| \\ &\le \frac{\varepsilon}{\lambda} \cdot \lambda + \sum_{j=1}^{k} |\alpha_j| \cdot \frac{\varepsilon}{\mu} + \frac{\varepsilon}{\lambda} \cdot \lambda \le 3\varepsilon. \end{aligned}$$

Hence $v(x_n) \to v(a)$. Since $v \in E'$ is arbitrary, we have $x_n \to a$ weakly. □

7-7.9. **Theorem** Let E, F be normed spaces and $f : E \to F$ a continuous linear map. If $x_n \to a$ weakly in E, then $f(x_n) \to f(a)$ weakly in F.

Proof. Take any $u \in F'$. Then $uf : E \to \mathbb{K}$ is a continuous linear form on E. Since $x_n \to a$ weakly, we have $uf(x_n) \to uf(a)$. Because $u \in F'$ is arbitrary, we have $f(x_n) \to f(a)$ weakly. □

7-7.10. **Exercise** Show that in ℓ_2, the sequence $\{e_n\}$ converges weakly but not in norm. Prove that in a finite dimensional normed space, every weak convergent sequence is convergent in norm.

7-7.11. **Exercise** Prove that for $1 < p < \infty$, a sequence $x_n = (x_1^n, x_2^n, x_3^n, \cdots)$ converges weakly to $a = (a_1, a_2, a_3, \cdots)$ in ℓ_p iff the following conditions hold:
(a) the sequence $\{x_n\}$ is bounded;
(b) for every j, $x_j^n \to a_j$.

7-7.12. **Exercise** Let $\{x_n\}$ and $\{y_n\}$ be given sequences in a normed space E. Prove that if $x_n \to a$ and $y_n \to b$ weakly in E, then for all $\alpha, \beta \in \mathbb{K}$, $\alpha x_n + \beta y_n \to \alpha a + \beta b$ weakly in E.

7-7.13. **Exercise** Let $\{x_n\}$ be a sequence in a normed space E. If $x_n \to a$ weakly in E, then for every $\varepsilon > 0$, there are $\lambda_i \ge 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\|a - \sum_{i=1}^{k} \lambda_i x_i\| \le \varepsilon$.

7-7.14. **Exercise** Prove that every closed convex set H of a Banach space contains the weak limit of every sequence in H.

7-8 Weak-Star Convergence

7-8.1. Let E be a normed space and $\{u_n\}$ a sequence in the dual space E'.
(a) $\{u_n\}$ is said to be *weak-star Cauchy* if for every $x \in E$, the sequence $\{u_n(x)\}$ is Cauchy in $\mathbb{K}$. The sequence $\{u_n\}$ is said to be *weak-star convergent* to the *weak-star limit* $v \in E'$ if for every $x \in E$, we have $u_n(x) \to v(x)$ in $\mathbb{K}$.
(b) The strong and weak convergences on E' were defined in last section when E' is considered as a normed space by itself. Hence the strong convergence in E' is in terms of the norm. The weak convergence in E' is in terms of the bidual E''.

7-8.2. **Theorem** Let $\{u_n\}$ be a sequence in the dual space E' of a normed space E. (a) If $u_n \to v$ weakly in E', then $u_n \to v$ in weak-star.
(b) If $u_n \to v$ in weak-star, then $\{u_n\}$ is weak-star Cauchy.
(c) Every weak-star convergence sequence has a unique weak-star limit.

Proof. We shall prove (c) only and leave (a), (b) as exercises. Let $u_n \to u$ and $u_n \to v$ in weak-star. Suppose to the contrary that $u \neq v$. There is $x \in E$ such that $u(x) \neq v(x)$. Since $u_n(x) \to u(x)$ and $u_n(x) \to v(x)$ in $\mathbb{K}$, we obtain a contradiction $u(x) = v(x)$. This completes the proof. □

7-8.3. **Theorem** Every weak-star Cauchy sequence $\{u_n\}$ in E' is weak-star convergent and norm-bounded. In other words, the dual space is always weak-star sequentially complete.

Proof. It follows immediately from Banach-Steinhaus Theorem. □

7-8.4. **Corollary** Every reflexive space E is weakly sequentially complete.

Proof. Since $E'' = J(E)$ is weak-star sequentially complete, the result follows immediately. □

7-8.5. **Theorem** Let E' be the dual space of a normed space E. If E is separable, then every bounded sequence $\{u_n : n \geq 1\}$ in E' has a weak-star convergent subsequence. In other words, bounded sets in E' are weak-star sequentially compact.

Proof. Let $\{x_n : n \geq 1\}$ be a countable dense subset of E. Since $\{u_n\}$ is bounded, there is $\lambda \in \mathbb{R}$ satisfying $\sup_{n \geq 1} \|u_n\| < \lambda < \infty$. Now observe

$$|u_n(x_1)| \leq \|u_n\| \, \|x_1\| \leq \lambda \|x_1\| < \infty, \forall \, n.$$

There is a subsequence $\{u_n^1\}$ of $\{u_n\}$ such that $\{u_n^1(x_1)\}$ converges. Again

$$|u_n^1(x_2)| \leq \|u_n^1\| \, \|x_2\| \leq \lambda \|x_2\| < \infty, \forall \, n$$

gives a subsequence $\{u_n^2\}$ of $\{u_n^1\}$ such that $\{u_n^2(x_2)\}$ converges. Similarly we can construct a subsequence $\{u_n^k\}$ of $\{u_n^{k-1}\}$ such that $\{u_n^k(x_k)\}$ converges. Let $v_n = u_n^n$ for each $n \geq 1$. Then $\{v_n\}$ is a subsequence of $\{u_n\}$ such that $\{v_n(x_k) : n \geq 1\}$ converges for each k. Now take any $x \in E$. Since $\{x_n\}$ is dense in E, fore every $\varepsilon > 0$ there is k such that $\|x - x_k\| \leq \frac{\varepsilon}{3\lambda}$. Since the convergent sequence $\{v_n(x_k)\}$ is Cauchy, there is an integer p such that for all $m, n \geq p$, we get $|v_m(x_k) - v_n(x_k)| \leq \frac{1}{3}\varepsilon$. Therefore we have

$$|v_m(x) - v_n(x)| \leq |v_m(x) - v_m(x_k)| + |v_m(x_k) - v_n(x_k)| + |v_n(x_k) - v_n(x)|$$
$$\leq \|v_m\| \; \|x - x_k\| + \tfrac{1}{3}\varepsilon + \|v_n\| \; \|x_k - x\| \leq \lambda\|x - x_k\| + \tfrac{1}{3}\varepsilon + \lambda\|x_k - x\| \leq \varepsilon.$$

Now $\{v_n\}$ is weak-star Cauchy. Hence it is weak-star convergent. This completes the proof. □

7-8.6. **Theorem** If E is a reflexive space, then every bounded sequence $\{x_n\}$ in E has a weakly convergent subsequence. In other words, the closed unit ball in E is weakly sequentially compact.

Proof. Let M be the closed vector subspace spanned by $\{x_n\}$ and $I : M \to M''$ the natural embedding. Then $\{x_n\}$ is bounded in M and hence $\{I(x_n)\}$ bounded in M''. By construction, M is separable and so is its dual M'. Therefore $\{I(x_n)\}$ has a weak-star convergent subsequence $\{I(y_n)\}$ in M''. As a closed subspace of a reflexive space E, M is also reflexive. So, there is $y \in M$ such that $I(y_n) \to I(y)$ in weak-star on M''. Now for each $v \in E'$, the restriction $v|M$ is in M'. Hence $v(y_n) = (v|M)(y_n) \to (v|M)(y) = v(y)$. Therefore the subsequence $\{y_n\}$ converges weakly to $y \in E$. □

7-8.7. **Exercise** Consider the Banach space $E = c_0$. Show that the sequence $\{e_n\}$ in $E' = \ell_1$ is weak-star convergent but not weakly convergent.

7-8.8. **Exercise** Let E, F be Banach spaces and $f_n : E \to F$ continuous linear maps. Then the following statements are equivalent:

(a) $\{\|f_n\|\}$ is bounded in $\mathbb{R}$.

(b) $\{\|f_n(x)\|\}$ is bounded for each $x \in E$.

(c) $\{\|uf_n(x)\|\}$ is bounded for each $x \in E$ and each $u \in F'$.

7-99. **References and Further Readings** : Kadison, Jarchow and Wong-92.

Chapter 8

Complex Analysis

8-1 Derivatives of Vector Maps

8-1.1. Elementary results of complex analysis will be developed in the context of vector-valued maps of a complex variable. After the introduction of spectrum, functions of an operator will be introduced. Actually, it may be more natural to do this chapter on general Banach algebra even thought we are contented with operators on Banach spaces. Meanwhile, we start with a short treatment of differential and integral calculus.

8-1.2. Let E be a Banach space. Let X be an open subset of the scalar field $\mathbb{K}$ and $f : X \to E$ a given map. Then f is said to be *differentiable* at $a \in X$ if the limit $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists, i.e. for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $0 < \|x - a\| \le \delta$, we have $x \in X$ and $\left\| \dfrac{f(x) - f(a)}{x - a} - f'(a) \right\| \le \varepsilon$. It is said to be *differentiable on* X if it is differentiable at every point of X. A map $f : X \to E$ is said to be *continuously differentiable* if its derivative $f' : X \to E$ is defined and continuous on X.

8-1.3. **Theorem** If $f : X \to E$ is differentiable, then it is continuous.

Proof. Take any $a \in X$. For $\varepsilon = 1$, there is $\delta > 0$ such that for all $|x - a| \le \delta$ we have $\|[f(x) - f(a)]/(x - a) - f'(a)\| \le 1$, i.e. $\|f(x) - f(a)\| \le (\|f'(a)\| + 1)|x - a|$. Therefore $f(x) \to f(a)$ as $x \to a$. □

8-1.4. **Exercise** Prove that if $f : X \to E$ and $g : X \to \mathbb{K}$ are differentiable at $a \in X$, then the scalar product $g.f$ is differentiable at a. Furthermore show that $(g.f)'(a) = g(a)f'(a) + g'(a)f(a)$.

8-1.5. **Exercise** Prove that linear combinations of differentiable maps are differentiable.

8-1.6. **Theorem** Let $f : X \to E$ be a map differentiable at $a \in X$. Then for every continuous linear map g from E into a Banach space F, the composite map $gf : X \to F$ is differentiable at $a \in X$. Furthermore, we have a special case of chain rule: $(gf)'(a) = g[f'(a)]$.

Proof. It follows immediately from the calculation:

$$\lim_{x\to a} \frac{gf(x) - gf(a)}{x - a} = \lim_{x\to a} g\left[\frac{f(x) - f(a)}{x - a}\right] = g \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}\right] = g[f'(a)]. \quad \square$$

8-1.7. **Exercise** Let $E = C_\infty[\pi, 2\pi]$. Define $f : \mathbb{R} \to E$ by $f(x)(t) = \sin tx$ for all $x \in \mathbb{R}$ and $t \in [\pi, 2\pi]$. Sketch the graphs of the functions $f(0), f(1)$ and also $f(x)$ for small x. Find a formula for the derivative $f'(0)$ and sketch its graph.

8-2 Integrals of Regulated Maps

8-2.1. To find the area under a curve $y = f(x)$ from $x = a$ to $x = b$, we cut the x-axis into subintervals and approximate the true area by the sum of areas of columns. This is the basic idea behind the Riemann integral. Since the treatment is slightly different from the standard undergraduate textbook, a different name will be used in order to avoid confusion. Alternatively, we cut the y-axis into small subintervals and that is the foundation of Lebesgue integral.

8-2.2. **Theorem** Let E be a Banach space. The set $B(X, E)$ of all bounded maps from a non-empty set X into E forms a Banach space under the pointwise operations and the sup-norm given by $\|f\|_\infty = \sup\{\|f(t)\| : t \in X\}$. If X is a compact metric space, then the set $C(X, E)$ of all continuous maps from X into E is a closed subspace of $B(X, E)$ and hence it is also a Banach space. In order to emphasize the sup-norm, we write $B_\infty(X, E)$ and $C_\infty(X, E)$. When E is $\mathbb{K}$, we simply write $B(X)$ and $C(X)$ instead.

8-2.3. Let E be a Banach Space and $a < b$ be given in $\mathbb{R}$. A map $f : [a, b] \to E$ is called a *step map* if there is a partition

$$P : a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

such that f is constant on every semi-interval $(t_{j-1}, t_j]$. The set $S([a, b], E)$ of all step maps forms a vector subspace of $B([a, b], E)$. For every step map f given above, its *integral* is defined by

$$\int_a^b f(t)dt = \sum_{j=1}^n f(t_j)(t_j - t_{j-1}).$$

It is easy to prove that this integral is independent of the choice of the partition P. Hence it is well-defined. In order to emphasize the sup-norm, we sometime write $S_\infty([a,b],E)$

8-2.4. **Lemma** The integral $f \to \int_a^b f(t)dt$ is a continuous linear map in f from $S_\infty([a,b],E)$ into E. Furthermore for every step map f on $[a,b]$ we have

$$\left\| \int_a^b f(t)dt \right\| \le \int_a^b \|f(t)\|dt \le (b-a)\|f\|_\infty.$$

Proof. It follows immediately from routine calculation as follow:

$$\left\| \int_a^b f(t)dt \right\| = \left\| \sum_{j=1}^n f(t_j)(t_j - t_{j-1}) \right\| \le \sum_{j=1}^n \|f(t_j)\|(t_j - t_{j-1})$$
$$= \int_a^b \|f(t)\|dt \le \sum_{j=1}^n \|f\|_\infty (t_j - t_{j-1}) = (b-a)\|f\|_\infty. \qquad \square$$

8-2.5. Let $A([a,b],E)$ denote the closure of $S([a,b],E)$ in the Banach space $B_\infty([a,b],E)$. The continuous linear map $\int_a^b f$ in f can be extended uniquely from $S([a,b],E)$ to $A([a,b],E)$. We shall use the same notation $\int_a^b f$ for the extended linear map when $f \in A([a,b],E)$. Maps in $A([a,b],E)$ can be uniformly approximated by step maps. For convenience, they are called ***regulated maps***. It is easy to show that every regulated real function is Riemann integrable. Hence in order to conform with the popular terminology, regulated maps are also said to be ***integrable*** when no ambiguity would occur.

8-2.6. **Theorem** There is a continuous linear map $f \to \int_a^b f$ from the normed space $A_\infty([a,b],E)$ into E such that the following conditions hold:

(a) $\left\| \int_a^b f(t)dt \right\| \le \int_a^b \|f(t)\|dt \le (b-a)\|f\|_\infty$;

(b) $u \int_a^b f(t)dt = \int_a^b uf(t)dt$ for all $u \in E'$.

Proof. Define $\varphi(f) = \int_a^b \|f(t)\|dt - \| \int_a^b f(t)dt\|$ for each $f \in A([a,b],E)$. Then $\varphi : A([a,b],E) \to \mathbb{R}$ is a continuous function. We have proved that $S([a,b],E)$ is contained in the closed subset $\varphi^{-1}[0,\infty)$ of $A([a,b],E)$. Since $S([a,b],E)$ is dense, we obtain $A([a,b],E) \subset \varphi^{-1}[0,\infty)$, i.e. $\int_a^b \|f(t)\|dt \ge \| \int_a^b f(t)dt\|$, $\forall\ f \in A([a,b],E)$. The proof can be completed in similar way or by passing limits to approximations by sequences of maps in $A([a,b],E)$. $\square$

8-2.7. **Exercise** Let F be a Banach space and $g : E \to F$ a continuous linear map. Prove that for every regulated map $f : [a, b] \to E$, the composite gf is a regulated map. Furthermore, we have $g \int_a^b f(t)dt = \int_a^b gf(t)dt$.

8-2.8. **Exercise** Prove that for every step map $f : [a, b] \to E$, the integral $\frac{1}{b-a} \int_a^b f(t)dt$ belongs to the convex hull of the image $f[a, b]$ of f. Also prove that for every regulated map, the integral $\dfrac{1}{b-a} \displaystyle\int_a^b f(t)dt$ belongs to the closed convex hull of $f[a, b]$.

8-2.9. **Theorem** Every continuous map $f : [a, b] \to E$ is integrable.

Proof. By uniform continuity on the compact space $[a, b]$, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in [a, b]$ satisfying $|x - y| \leq \delta$ we get $\|f(x) - f(y)\| \leq \varepsilon$. Let $P : a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition with $mesh(P) = \max_{1 \leq j \leq n}(t_j - t_{j-1}) \leq \delta$. Define $g(a) = f(a)$ and $g(t) = f(t_j)$ for all $t_{j-1} < t \leq t_j$. Then $g \in S([a, b], E)$ and $\|f(t) - g(t)\| \leq \varepsilon$ for all $t \in [a, b]$, i.e. $\|f - g\|_\infty \leq \varepsilon$. Therefore $f \in A([a, b], E)$. □

8-2.10. **Exercise** Let $f_n(x) = n^2 x(1 - x)^n$ for all $x \in [0, 1]$. Show that $f_n \to 0$ pointwise but $\int_0^1 f_n(x)dx \not\to 0$.

8-2.11. **Exercise** Let $f_n(x) = nx(1 - x)^n$ for all $x \in [0, 1]$. Show that $f_n \to 0$ pointwise but not uniformly. Also show that $\int_0^1 f_n(x)dx \to 0$.

8-2.12. Let f be a continuous map defined on a closed interval containing $a, b, c \in \mathbb{R}$. For $a < b$, the value $\int_a^b f$ has been defined. Now for $a > b$, define $\int_a^b f = - \int_b^a f$. For $a = b$, define $\int_a^b f = 0$. In this case, the value $\int_a^b f$ is called the *integral* of f from a to b. By case study, clearly $\int_a^b f = \int_a^c f + \int_c^b$ holds. Furthermore for all $a, b \in \mathbb{R}$, we have $\| \int_a^b f(t)dt\| \leq |b - a| \; \|f\|_\infty$ and also $u \int_a^b f(t)dt = \int_a^b uf(t)dt, \forall\, u \in E'$.

8-2.13. **Exercise** Let $a < b$ be given real numbers. A map $f : [a, b] \to E$ is said to be *piecewise continuous* if there is a partition

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b$$

such that f is continuous on the open interval (a_{j-1}, a_j) and both one-sided limits $\lim\limits_{x \downarrow a_{j-1}} f(x)$ and $\lim\limits_{x \uparrow a_j} f(x)$ exist for all $j = 1, 2, \cdots, n$. Define the integral of f. Show that it is independent of the choice of any particular partition. Derive some standard properties of this integral.

8-3 Fundamental Theorems of Calculus

8-3.1. Although the integrals of regulated maps were not defined in the usual way as the Riemann integrals, with the fundamental theorem of calculus we can evaluate integrals in terms of antiderivatives. Consequently, the techniques developed in elementary calculus can be used in our context.

8-3.2. **Theorem** Let X be a metric space. If $f : [a,b] \times X \to E$ is continuous, then the map given by $g(x) = \int_a^b f(t,x)dt$ is continuous on X.

Proof. Let $x_0 \in X$ and $\varepsilon > 0$ be given. By compactness of $[a,b]$, it follows from §2-7.6 that there is $\delta > 0$ such that for all $x \in \mathbb{B}(x_0, \delta)$ and $t \in [a,b]$, we have $\|f(t,x) - f(t,x_0)\| \le \varepsilon$. Therefore

$$\|g(x) - g(x_0)\| \le \int_a^b \|f(t,x) - f(t,x_0)\|dt \le \int_a^b \varepsilon dt = \varepsilon(b-a).$$

Consequently g is continuous at every point x_0 of X. □

8-3.3. **Change Order of Integration** Let $f : [a,b] \times [\alpha,\beta] \to E$ be a continuous map. Then the repeated integrals exist and they are equal.

Proof. Since $\int_\alpha^\beta f(x,y)dy$ is continuous in x, it is integrable and hence the repeated integral $\int_a^b \int_\alpha^\beta f(x,y)dydx$ exists. Similarly, the other repeated integral also exists. To show that they are equal, let $\varepsilon > 0$ be given. By uniform continuity of f on the compact set $[a,b] \times [\alpha,\beta]$, there is $\delta > 0$ such that for all $|x_1 - x_2| \le \delta$ and $|y_1 - y_2| \le \delta$ we have $|f(x_1,y_1) - f(x_2,y_2)| \le \varepsilon$. Cut the rectangle into pieces by $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ and $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta$ where $n > (b - a + \beta - \alpha)/\delta$. Define $g : [a,b] \times [\alpha,\beta] \to E$ as follow. If $x = a$ let $p = a$ and if $a_{j-1} < x \le a_j$ let $p = a_j$. Similarly if $y = \alpha$ let $q = \alpha$ and if $\alpha_{k-1} < y \le \alpha_j$ let $q = \alpha_k$. Define $g(x,y) = f(p,q)$. Then clearly $\|f(x,y) - g(x,y)\| \le \varepsilon$ for all $a \le x \le b$ and $\alpha \le y \le \beta$. Observe that

$$\int_\alpha^\beta \int_a^b g(x,y)dxdy = \sum_{j=1}^n \sum_{k=1}^n f(a_j,\alpha_k)\frac{(b-a)(\beta-\alpha)}{n^2}$$

$$= \sum_{k=1}^n \sum_{j=1}^n f(a_j,\alpha_k)\frac{(\beta-\alpha)(b-a)}{n^2} = \int_a^b \int_\alpha^\beta g(x,y)dydx.$$

On the other hand, we have

$$\left\|\int_\alpha^\beta \int_a^b f(x,y)dxdy - \int_\alpha^\beta \int_a^b g(x,y)dxdy\right\|$$

$$= \left\| \int_\alpha^\beta \int_a^b [f(x,y) - g(x,y)] dx dy \right\| \leq \int_\alpha^\beta \left\| \int_a^b [f(x,y) - g(x,y)] dx \right\| dy$$
$$\leq \int_\alpha^\beta \int_a^b \|f(x,y) - g(x,y)\| dx dy \leq \varepsilon(b-a)(\beta-\alpha).$$

Similarly, we get

$$\left\| \int_a^b \int_\alpha^\beta f(x,y) dy dx - \int_a^b \int_\alpha^\beta g(x,y) dy dx \right\| \leq \varepsilon(b-a)(\beta-\alpha).$$

Combining together, we obtain

$$\left\| \int_\alpha^\beta \int_a^b f(x,y) dx dy - \int_a^b \int_\alpha^\beta f(x,y) dy dx \right\| \leq 2\varepsilon(b-a)(\beta-\alpha).$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_\alpha^\beta \int_a^b f(x,y)dxdy = \int_a^b \int_\alpha^\beta f(x,y)dydx$ as required. Alternatively, appealing to one dimensional case, we have

$$\int_a^b \int_\alpha^\beta u f(x,y) dy dx = \int_\alpha^\beta \int_a^b u f(x,y) dx dy$$

for all $u \in E'$, i.e.

$$u \int_a^b \int_\alpha^\beta f(x,y) dy dx = u \int_\alpha^\beta \int_a^b f(x,y) dx dy$$

and the result follows because E' separates points of E. □

8-3.4. **First Fundamental Theorem of Calculus** Let $f : [a,b] \to E$ be a continuous map where $a < b$. Then the map $g : [a,b] \to E$ given by

$$g(x) = \int_a^x f(t)dt$$

is differentiable on the open interval (a,b). Furthermore we have $g'(x) = f(x)$.

Proof. Take any $x_0 \in (a,b)$. Then for any $\varepsilon > 0$ choose $\delta > 0$ such that $\|f(x) - f(x_0)\| \leq \varepsilon$ for all $|x - x_0| \leq \delta$. Suppose $0 < |x - x_0| \leq \delta$. If $x > x_0$, then we have

$$\|g(x) - g(x_0) - (x - x_0)f(x_0)\| = \left\| \int_a^x f(t)dt - \int_a^{x_0} f(t)dt - \int_{x_0}^x f(x_0)dt \right\|$$
$$= \left\| \int_{x_0}^x f(t)dt - \int_{x_0}^x f(x_0)dt \right\| \leq \int_{x_0}^x \|f(t) - f(x_0)\| dt \leq \int_{x_0}^x \varepsilon dt = \varepsilon|x - x_0|.$$

If $x < x_0$, then we get

$$\|g(x) - g(x_0) - (x - x_0)f(x_0)\| \leq \int_x^{x_0} \|f(t) - f(x_0)\| dt \leq \int_x^{x_0} \varepsilon dt = \varepsilon|x - x_0|.$$

In both cases, we obtain

$$\left\| \frac{g(x) - g(x_0)}{x - x_0} - f(x_0) \right\| \leq \varepsilon,$$

or,
$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f(x_0).$$ □

8-3.5. **Exercise** Show that the map g in last theorem is continuous on $[a, b]$.

8-3.6. **Lemma** Let $f : [a, b] \to X$ be a continuous map which is differentiable on (a, b). If $f'(x) = 0$ for each $x \in (a, b)$ then f is constant on $[a, b]$.

Proof. Take any $x \in (a, b)$ and any $u \in E'$. Then $(uf)'(x) = u[f'(x)] = 0$. By one dimensional case, the continuous function uf is constant, i.e. $uf(x) = uf(a)$ for all $x \in [a, b]$. Since E' separates points of E, we have $f(x) = f(a)$ for all $x \in [a, b]$. □

8-3.7. **Second Fundamental Theorem of Calculus** Let $f : (\alpha, \beta) \to E$ be a continuously differentiable map. Then for all $a, b \in (\alpha, \beta)$, we have

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Proof. For every $x \in (\alpha, \beta)$, define $\varphi(x) = \int_a^x f'(t)dt - f(x)$. Then φ is differentiable on (α, β) and $\varphi'(x) = 0, \forall\ x \in (\alpha, \beta)$. Hence φ is constant on (α, β). Now $\varphi(b) = \varphi(a)$ gives $\int_a^b f'(t)dt - f(b) = \int_a^a f'(t)dt - f(a)$ which is the result. It is an exercise to give an alternative proof by reducing it to one dimensional case. □

8-3.8. **Integration by Parts** Let $f : (\alpha, \beta) \to E$ and $g : (\alpha, \beta) \to \mathbb{K}$ be continuously differentiable maps. Then for all $a, b \in (\alpha, \beta)$ we have

$$\int_a^b f'(t)g(t)dt + \int_a^b f(t)g'(t)dt = f(b)g(b) - f(a)g(a).$$

Proof. It is easy to verify that $[f(t)g(t)]' = f'(t)g(t) + f(t)g'(t)$ and the result follows immediately from the Second Fundamental Theorem of Calculus. □

8-3.9. **Exercise** Let X, Y be open subsets of $\mathbb{K}$. Let $\alpha : X \to Y$ and $f : Y \to E$ be differentiable maps. Prove that the composite map $f\alpha$ is differentiable on X. Furthermore we have $(f\alpha)'(x) = f'[\alpha(x)]\alpha'(x)$.

8-3.10. **Change Variables** Let $g : (\alpha, \beta) \to (u, v)$ be a continuously differentiable function and $f : (u, v) \to E$ a continuous map. Prove that

for all $a, b \in (\alpha, \beta)$, we have $\displaystyle\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f[g(t)]g'(t)dt.$

8-3.11. **Exercise** Let $E = C_\infty[\pi, 2\pi]$. A map $g : \mathbb{R} \to E$ is defined by $g(x)(t) = t \sin tx$ for all $x \in \mathbb{R}$ and $t \in [\pi, 2\pi]$. Find a formula for the integral $\int_0^1 g(x)dx$. Note that the integral is a function from $[\pi, 2\pi]$ to $\mathbb{R}$. Sketch the graph of this function.

8-4 Holomorphic Maps of One Complex Variable

8-4.1. A *primitive contour* is a map $z : [\alpha, \beta] \to \mathbb{C}$ which has at least one partition $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_n = \beta$ such that the restriction of z to every subinterval $[\alpha_{j-1}, \alpha_j]$ has a continuously differentiable extension over an open interval containing $[\alpha_{j-1}, \alpha_j]$. In this case, z is called a parametrization. A *contour* is the formal sum of a finite number of primitive contours. Hence two disjoint circles can form a contour but both are parametrized on the same interval, say $[0, 2\pi]$. By abuse of language, we call the *image* Γ to be the contour. Equivalence of parametrizations is defined as usual in terms of changing variable. The *arc length* of Γ is well-defined and is independent of the choice of any particular parametrization. Our definition is relatively narrower but more convenient than rectifiable curves.

8-4.2. We shall work with a given *complex* Banach space E. Let Γ be a primitive contour described by a parametrization $z : [\alpha, \beta] \to \mathbb{C}$. Let $f : \Gamma \to E$ be a continuous map. Define the *contour integral* of f on Γ by $\int_\Gamma f(z)dz = \int_\alpha^\beta f[z(t)]z'(t)dt$. Clearly $\int_\Gamma f(z)dz$ is independent of the choice of equivalent parametrization. The integral of a map on a contour is defined as the sum of integrals on the corresponding primitive contour. For example, if $\Gamma = A + B$ is a formal sum to two disjoint circles A, B, then $\int_\Gamma f(z)dz = \int_A f(z)dz + \int_B f(z)dz$. We always work with contours and abandon the transitional terminology of primitive contours.

8-4.3. **Theorem** (a) $\left\| \int_\Gamma f(z)dz \right\| \le \int_\Gamma \|f(z)\| |dz| = \int_\alpha^\beta \|f[z(t)]\| |z'(t)| dt.$

(b) $u \int_\Gamma f(z)dz = \int_\Gamma uf(z)dz$ for every $u \in E'$.

8-4.4. Let X be an open subset of $\mathbb{C}$ and $f : X \to E$ a given map. Then f is said to be *holomorphic* on X if it is differentiable on X and *weakly holomorphic* if the composite uf is differentiable on X for each continuous linear form $u \in E'$. Clearly every holomorphic map is weakly holomorphic. We shall prove that the converse is also true.

8-4.5. **Lemma** If f is weakly holomorphic on X then f is continuous on X.

Proof. Let $a \in X$ be given. Since X is open, it contains some closed ball $\overline{\mathbb{B}}(a,r)$. Define a subset of E by $M = \left\{ \dfrac{f(z)-f(a)}{z-a} : 0 < |z-a| \le r \right\}$. To show that M is weakly bounded, take any $u \in E'$. Then the expression

$$g(z) = \begin{cases} \dfrac{uf(z)-uf(a)}{z-a}, & \text{if } 0 < |z-a| \le r; \\ (uf)'(a), & \text{if } z = a \end{cases}$$

defines a continuous function from the compact set $\overline{\mathbb{B}}(a,r)$ into $\mathbb{C}$. Hence the image $g[\overline{\mathbb{B}}(a,r)]$ is compact. It follows by simple calculation that $u(M) \subset g[\overline{\mathbb{B}}(a,r)]$. Since $u(M)$ is bounded for every $u \in E'$, the set M is weakly bounded and thus norm-bounded. There is $0 < \lambda < \infty$ such that for any $0 < |z-a| \le r$ we have $\left\| \dfrac{f(z)-f(a)}{z-a} \right\| \le \lambda$, i.e. $\|f(z)-f(a)\| \le \lambda |z-a|$. Consequently f is continuous at every point of X. □

8-4.6. **Theorem** If f is weakly holomorphic on X, then for every $a \in X$ we have $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(z)}{z-a}dz$ where Γ is any simple closed positively oriented contour in X enclosing a with interior contained in X.

Proof. For each $u \in E'$, the composite uf is holomorphic on X. By one dimensional case, we get

$$uf(a) = \frac{1}{2\pi i}\int_\Gamma \frac{uf(z)}{z-a}dz = u\left(\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz\right).$$

Since E' separates points of E, the result follows. □

8-4.7. **Theorem** Every weakly holomorphic map $f : X \to E$ is holomorphic. Furthermore, for each $a \in X$ we have $f'(a) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{f(z)}{(z-a)^2}dz$ where Γ is any positively oriented circle: $C(a,r) = \{z \in \mathbb{C} : |z-a| = r\}$ satisfying $\overline{\mathbb{B}}(a,r) \subset X$.

Proof. Since $C(a,r)$ is compact, write $\|f(z)\| \le \lambda < \infty$. Let $0 < \delta < \frac{1}{2}r$ and take any $0 < |b-a| < \delta$. Observe that

$$\begin{aligned} &\left\| \frac{f(b)-f(a)}{b-a} - \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^2}dz \right\| \\ &= \left\| \frac{1}{b-a}\frac{1}{2\pi i}\int_\Gamma \left[\frac{f(z)}{z-b} - \frac{f(z)}{z-a}\right]dz - \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^2}dz \right\| \\ &= \frac{1}{2\pi}\left\| \int_\Gamma \frac{f(z)}{(z-a)(z-b)}dz - \int_\Gamma \frac{f(z)}{(z-a)^2}dz \right\| \end{aligned}$$

$$= \frac{1}{2\pi}\left\|(b-a)\int_\Gamma \frac{f(z)}{(z-a)^2(z-b)}dz\right\|$$
$$\leq \frac{|b-a|}{2\pi}\int_\Gamma \frac{\|f(z)\|}{|(z-a)^2(z-b)|}|dz|$$
$$\leq \frac{\delta}{2\pi}\int_\Gamma \frac{\lambda|dz|}{r^2(\frac{1}{2}r)} = \frac{\delta}{\pi}\frac{\lambda}{r^3}(2\pi r) = \frac{2\lambda}{r^2}\delta.$$

Therefore we have

$$f'(a) = \lim_{b\to a}\frac{f(b)-f(a)}{b-a} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^2}dz. \qquad \square$$

8-4.8. **Theorem** Let X be a simply connected open subset of $\mathbb{C}$ and let $f : X \to E$ be a continuous map. Then the following statements are equivalent.
(a) The map f is holomorphic on X.
(b) For every simple closed contour Γ in X, we have $\int_\Gamma f(z)dz = 0$.
(c) For every $a \in X$, the integral $\int_a^w f(z)dz$ is independent of the contour in X from a to w.
(d) The map f is weakly holomorphic on X.

Proof. We have proved $(a \Leftrightarrow d)$. For $(d \Leftrightarrow b)$, note that $\int_\Gamma f(z)dz = 0$ iff $\int_\Gamma uf(z)dz = 0$ for every $u \in E'$. For arbitrary Γ, it is equivalent to say that uf is holomorphic on X which is (d). Similarly, we can prove $(b \Leftrightarrow c)$. $\square$

8-4.9. Note that $(a \Rightarrow b)$ of last theorem generalizes Cauchy-Goursat Theorem and $(b \Rightarrow a)$ Morera's Theorem. As a result of (b), the circle $C(a,r)$ of previous theorem can be replaced by more general contours.

8-4.10. **Exercise** Let F be a complex Banach space and $g : E \to F$ be a continuous linear map. Prove that for every holomorphic map $f : X \to E$, the composite gf is a holomorphic map. Furthermore, we have

$$g\int_\Gamma f(z)dz = \int_\Gamma gf(t)dt$$

for every contour Γ in X.

8-4.11. **Cauchy's Integral Formula** Every holomorphic map $f : X \to E$ is infinitely differentiable. Furthermore, for $n = 0, 1, 2, 3, \cdots$, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz$$

where Γ is any simple closed positively oriented contour in X enclosing a with interior contained in X.

Proof. We have proved the case when $n = 0$. Inductively, assume that theorem is true for n. Hence f is n-times differentiable on X and for each $u \in E'$, we have

$$u[f^{(n)}(a)] = \frac{n!}{2\pi i}\int_\Gamma \frac{uf(z)}{(z-a)^{n+1}}dz = (uf)^n(a), \quad \forall\, a \in X.$$

Since uf is $(n+1)$-times differentiable on X, $u[f^{(n)}] = (uf)^{(n)} : X \to \mathbb{C}$ is differentiable. Hence $f^{(n)}$ is weakly holomorphic on X, i.e. differentiable on X. Therefore f is $(n+1)$-times differentiable on X. Finally, observe that

$$\begin{aligned} &u[f^{(n+1)}(a)] = [uf^{(n)}]'(a) = [(uf)^{(n)}]'(a) = (uf)^{(n+1)}(a) \\ &= \frac{(n+1)!}{2\pi i}\int_\Gamma \frac{uf(z)}{(z-a)^{n+2}}dz = u\left[\frac{(n+1)!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-a)^{n+2}}dz\right]. \end{aligned}$$

Since $u \in E$ is arbitrary, the required formula follows. □

8-4.12. **Average Lemma** Let $f : X \to E$ be a holomorphic map. Suppose X contains the closed ball $\overline{\mathbb{B}}(a,r)$. Then for each $n = 0, 1, 2, 3, \cdots$ we have

$$f^{(n)}(a) = \frac{n!}{2\pi r^n}\int_0^{2\pi} f(a + re^{i\theta})e^{-in\theta}d\theta.$$

In particular when $n = 0$, the value $f(a)$ at the center is the average of the values on the circle.

Proof. It follows immediately from the substitution $z = a + re^{i\theta}$ into Cauchy's Integral Formula. □

8-4.13. **Cauchy's Inequality** If $\|f(z)\| \le \lambda$ for every point z on the circle $C(a,r)$, then we have $\|f^{(n)}(a)\| \le \dfrac{n!\lambda}{r^n}$, for each $n = 0, 1, 2, 3, \cdots$.

Proof. By Average Lemma, we have

$$\begin{aligned} &\|f^{(n)}(a)\| = \left\|\frac{n!}{2\pi r^n}\int_0^{2\pi} f(a+re^{i\theta})e^{-in\theta}d\theta\right\| \\ &\le \frac{n!}{2\pi r^n}\int_0^{2\pi}\|f(a+re^{i\theta})e^{-in\theta}\|d\theta \le \frac{n!}{2\pi r^n}\int_0^{2\pi}\lambda d\theta = \frac{n!\lambda}{r^n}. \end{aligned}$$ □

8-4.14. **Fundamental Theorem of Calculus** Let $f : X \to E$ be a given holomorphic map on a simply connected open subset X of $\mathbb{C}$. Then for every $a, z \in X$ we have $\dfrac{d}{dz}\displaystyle\int_a^z f(t)dt = f(z)$ and $\displaystyle\int_a^z f'(t)dt = f(z) - f(a)$.

Proof. It is left as an exercise. □

8-4.15. **Exercise** Let A, B be contours in $\mathbb{C}$ parametrized by $x(s)$ for $s \in [a,b]$ and by $y(t)$ for $t \in [\alpha,\beta]$ respectively. Let $f : A \times B \to E$ be a continuous map. Show that the repeated contour integrals $\int_A \int_B f(x,y)dydx$ and $\int_B \int_A f(x,y)dxdy$ exist. Prove that they are equal.

8-4.16. **Exercise** Let $f_n, g : X \to E$ be continuous maps on an open subset X of $\mathbb{C}$. Prove that if $f_n \to g$ uniformly on a contour Γ in X then we have

$$\int_\Gamma f_n(z)dz \to \int_\Gamma g(z)dz \text{ in } E.$$

8-4.17. **Exercise** Let X be an open subset of $\mathbb{C}$. Let $A \in E$ be given and let $g : X \to \mathbb{C}$ be a holomorphic map. Then $f : X \to E$ given by $f(z) = Ag(z)$ is holomorphic on X. Furthermore, $f'(z) = Ag'(z)$ on X.

8-5 Series Expansion

8-5.1. As in one dimensional case, holomorphic maps will be identified as analytic maps which can be expanded locally into infinite series. Taylor series expansion will be derived as from Laurent series expansion. Then polynomials will be characterized and Liouville's theorem follows as a special case. This will be used to show that the spectrum of an operator is non-empty. Finally the vector form of identity theorem will be given.

8-5.2. For convenience, let $H(X, E)$ denote the set of all holomorphic maps from an open subset X of $\mathbb{C}$ into a complex Banach space E. Write $f_n \to g$ in $H(X, E)$ if all f_n, g are maps in $H(X, E)$ and $f_n \to g$ uniformly on every compact subsets of X. We also write Dg instead of g' for the derivative of a map g in $H(X, E)$. It would be more natural if the following two theorems are stated in terms of compact open topology which we do not assume.

8-5.3. **Theorem** Let $f_n, g : X \to E$ be given maps. If all $f_n \in H(X, E)$ and if $f_n \to g$ uniformly on every compact subsets of X, then we have $g \in H(X, E)$.

Proof. Take any $a \in X$. Pick any closed ball $\overline{\mathbb{B}}(a) \subset X$. Since the continuous maps f_n converge to g uniformly on the compact set $\overline{\mathbb{B}}(a)$, g is also continuous on $\overline{\mathbb{B}}(a)$. In particular, g is continuous on the open ball $\mathbb{B}(a)$ and hence integral of g along every contour in $\mathbb{B}(a)$ is well-defined. To show that g is differentiable on the simply connected open set $\mathbb{B}(a)$, let Γ be a simple closed contour in $\mathbb{B}(a)$. Since $f_n \to g$ uniformly on Γ, we have $\int_\Gamma f_n(z)dz \to \int_\Gamma g(z)dz$. Because all f_n are holomorphic, we have $\int_\Gamma f_n(z)dz = 0$, i.e. $\int_\Gamma g(z)dz = 0$. Therefore g is differentiable on $\mathbb{B}(a)$. Since $a \in X$ is arbitrary, g is differentiable on X, i.e. holomorphic on X. □

8-5.4. **Theorem** If $f_n \to g$ in $H(X,E)$, then $Df_n \to Dg$ in $H(X,E)$. In other words, the differential operator D is continuous under the compact-open topology.

Proof. Replacing f_n by $f_n - g$, we may assume that $g = 0$. Let A be a compact subset of X. For every $x \in A$, there is a ball $\mathbb{B}(x, 3r_x) \subset X$. Now $\{\mathbb{B}(x, r_x) : x \in A\}$ is an open cover of A. By compactness, there is a finite subset J of A such that $A \subset \bigcup_{a\in J} \mathbb{B}(a, r_a)$. Now $K = \bigcup_{a\in J} \overline{\mathbb{B}}(a, 2r_a)$ is a compact subset of X. For every $\varepsilon > 0$, let $\delta = \frac{1}{2}\varepsilon \cdot \min\{r_a : a \in J\}$. Since $f_n \to 0$ uniformly on K, there is p such that for all $n \geq p$ and all $x \in K$ we have $\|f_n(x)\| \leq \delta$. Now pick any $z \in A$. Then $z \in \mathbb{B}(a, r_a)$ for some $a \in J$. Let Γ be the positively oriented circle $|w - a| = 2r_a$. It follows from Cauchy integral formula that

$$Df_n(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f_n(w)dw}{(w-z)^2}.$$

Now for all $n \geq p$, we have

$$\|Df_n(z)\| \leq \frac{1}{2\pi} \int_\Gamma \frac{\|f_n(w)\||dw|}{|w-z|^2} \leq \frac{1}{2\pi} \int_\Gamma \frac{\delta\ |dw|}{r_a^2} = \frac{1}{2\pi} \frac{(\frac{1}{2}\varepsilon r_a)(2\pi.2r_a)}{r_a^2} \leq \varepsilon.$$

Since p is independent of $z \in A$, $Df_n \to 0$ uniformly on A. □

8-5.5. **Lemma** Let Γ be a contour in $\mathbb{C}$ and K an index set. Let $h_n, h : \Gamma \times K \to C$ be maps such that for each $z \in K$, both $h_n(w,z)$ and $h(w,z)$ are continuous in $w \in \Gamma$. Suppose $h_n(w,z) \to h(w,z)$ uniformly on $\Gamma \times K$, then for every continuous map $f : \Gamma \to E$ we have

$$\int_\Gamma f(w)h_n(w,z)dw \to \int_\Gamma f(w)h(w,z)dw \text{ uniformly on } K.$$

Proof. Since $h_n(w,z) \to h(w,z)$ uniformly on $\Gamma \times K$, for each $\varepsilon > 0$ there is an integer p such that for all $(w,z) \in \Gamma \times K$ and all $n \geq p$, we obtain $|h_n(w,z) - h(w,z)| \leq \varepsilon$. The continuity of f on the compact set Γ gives $\lambda = \sup\{\|f(w)\| : w \in \Gamma\} < \infty$. Now for any $z \in K$ and any $n \geq p$ we have

$$\begin{aligned}
&\left\| \int_\Gamma f(w)h_n(w,z)dw - \int_\Gamma f(w)h(w,z)\ dw \right\| \\
&\leq \int_\Gamma \|f(w)[h_n(w,z) - h(w,z)]\|\ |dw| \\
&\leq \int_\Gamma \|f(w)\||h_n(w,z) - h(w,z)|\ |dw| \leq \lambda\varepsilon\ell
\end{aligned}$$

where ℓ denotes the arc length of the contour Γ. This completes the proof. □

8-5.6. **Laurent Series Expansion** Let $X = \{z \in \mathbb{C} : r < |z - a| < R\}$ be an annula. Let $f : X \to E$ be a holomorphic map. For any $r < t < R$, let

$A_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_t} \dfrac{f(w)dw}{(w-a)^{n+1}}$ where C_t denotes the positively oriented circle $|w-a| = t$. Then the series $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} A_n(z-a)^n$ converges uniformly on every compact subset of X.

<u>*Proof*</u>. Let K be a compact subset of X. Let $m = \min_{z\in K}|z-a|$ and $M = \max_{z\in K}|z-a|$. Choose α, β satisfying $r < \alpha < m \leq M < \beta < R$. Take any $z \in K$. By Cauchy integral formula, we obtain

$$f(z) = \frac{1}{2\pi i}\int_{C_\beta}\frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\int_{C_\alpha}\frac{f(w)dw}{w-z}$$

where C_α, C_β are positively oriented circles with center a and radii α, β respectively. Consider w on the bigger circle C_β. Since $\left|\dfrac{z-a}{w-a}\right| \leq \dfrac{M}{\beta} < 1$ is independent of (w,z) in $C_\beta \times K$, the series

$$\frac{1}{w-z} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

converges uniformly in (w,z). Hence, the following series converges uniformly on K:

$$\frac{1}{2\pi i}\int_{C_\beta}\frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_\beta}\frac{f(w)(z-a)^n dw}{(w-a)^{n+1}}$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_\beta}\frac{f(w)dw}{(w-a)^{n+1}}\right](z-a)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_t}\frac{f(w)dw}{(w-a)^{n+1}}\right](z-a)^n = \sum_{n=0}^{\infty}A_n(z-a)^n.$$

Next suppose that w lies on the smaller circle C_α. Since $\left|\dfrac{w-a}{z-a}\right| \leq \dfrac{\alpha}{m} < 1$ is independent of (w,z) in $C_\alpha \times K$, the series

$$\frac{1}{w-z} = -\sum_{n=-1}^{-\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

converges uniformly in (w,z). Hence the following series converges uniformly on K:

$$-\frac{1}{2\pi i}\int_{C_\alpha}\frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\sum_{n=-1}^{-\infty}\int_{C_\alpha}\frac{f(w)(z-a)^n dw}{(w-a)^{n+1}}$$

$$= \sum_{n=-1}^{-\infty}\left[\frac{1}{2\pi i}\int_{C_\alpha}\frac{f(w)dw}{(w-a)^{n+1}}\right](z-a)^n$$

$$= \sum_{n=-1}^{-\infty} \left[\frac{1}{2\pi i} \int_{C_t} \frac{f(w)dw}{(w-a)^{n+1}} \right] (z-a)^n = \sum_{n=-1}^{-\infty} A_n (z-a)^n.$$

Combining above two terms of $f(z)$, the proof is complete. □

8-5.7. **Theorem** Consider the series $f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$ on the open ball $\mathbb{B}(a,R)$ where $A_n \in E$. If it converges pointwise on $\mathbb{B}(a,R)$ then it converges uniformly on every smaller ball $\mathbb{B}(a,r)$ with $0 < r < R$.

Proof. For a fixed r, choose $b \in \mathbb{C}$ satisfying $r < |b-a| < R$. Since the series $f(b) = \sum_{n=0}^{\infty} A_n(b-a)^n$ converges, so is the series $uf(b) = \sum_{n=0}^{\infty} u[A_n(b-a)^n]$ for every $u \in E$. Hence the set $\{u[A_n(b-a)^n] : n \geq 0\}$ is bounded in $\mathbb{C}$. Thus $\{A_n(b-a)^n : n \geq 0\}$ is weakly bounded and therefore norm-bounded in E, i.e. $\lambda = \sup\{\|A_n(b-a)^n\| : n \geq 0\} < \infty$. For every $\varepsilon > 0$ there is an integer p such that for every $k \geq p$ we have $\dfrac{\lambda}{1-\frac{r}{|b-a|}} \left(\dfrac{r}{|b-a|} \right)^{k+1} \leq \varepsilon$. Now for every $z \in \mathbb{B}(a,r)$,

$$\left\| \sum_{m=1}^{p} A_{k+m}(z-a)^{k+m} \right\| \leq \sum_{m=1}^{p} \left\|A_{k+m}(b-a)^{k+m}\right\| \left| \frac{z-a}{b-a} \right|^{k+m}$$

$$\leq \sum_{m=1}^{\infty} \lambda \left(\frac{r}{|b-a|} \right)^{k+m} \leq \frac{\lambda}{1-\frac{r}{|b-a|}} \left(\frac{r}{|b-a|} \right)^{k+1} \leq \varepsilon.$$

Therefore we get $\left\| f(z) - \sum_{n=1}^{k} A_n(z-a)^n \right\| = \left\| \sum_{n=k+1}^{\infty} A_n(z-a)^n \right\| \leq \varepsilon$. Since it is independent of $z \in \mathbb{B}(a,r)$, the convergence $f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$ is uniform on the open ball $\mathbb{B}(a,r)$. □

8-5.8. **Corollary** Consider the series $f(z) = \sum_{n=1}^{\infty} \dfrac{A_n}{(z-a)^n}$ on an open set $X = \{z \in \mathbb{C} : |z-a| > R\}$ where $A_n \in E$. If it converges pointwise on X then it converges uniformly on every compact subset of X.

Proof. Apply last theorem to the map $g(z) = f(a + \frac{1}{z})$ on the open ball $\mathbb{B}(0, \frac{1}{R})$. □

8-5.9. **Uniqueness of Laurent Series Expansion** Consider the series $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ on an annula $X = \{z \in \mathbb{C} : r < |z-a| < R\}$ where $A_n \in E$. If it converges pointwise on X then f is holomorphic on X. Furthermore we have $A_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(w)dw}{(w-a)^{n+1}}$ where C is any positively oriented circle $|z-a| = t$ with $r < t < R$.

Proof. We have proved that $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ converges uniformly on compact subsets of X. Since $\sum_{n=-k}^{k} A_n(z-a)^n$ is holomorphic on X, so is f. To verify the formula for A_n, take any $u \in E'$. Since $uf(z) = \sum_{n=-\infty}^{\infty} u(A_n)(z-a)^n$ converges on X, the proof is completed by one dimensional case as follow:

$$u(A_n) = \frac{1}{2\pi i}\int_C \frac{(uf)(w)dw}{(w-a)^{n+1}} = u\left[\frac{1}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^{n+1}}\right]. \qquad \square$$

8-5.10. **Taylor Series Expansion** Every holomorphic map $f : \mathbb{B}(a,R) \to E$ has a series expansion $f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$ which converges uniformly on compact subsets of $\mathbb{B}(a,R)$. Furthermore, the coefficient is uniquely determined by

$$A_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(w)dw}{(w-a)^{n+1}}$$

where C is any positively oriented circle $|z-a| = r$ with $0 < r < R$.

Proof. By Laurent series expansion, we have $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ for each $0 < |z-a| < R$. Take any $n < 0$. Since $(w-a)^{-n}f(w)$ is holomorphic in $w \in \mathbb{B}(a,R)$, we obtain

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^n}dw = 0.$$

Therefore $f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$ holds for each $0 < |z-a| < R$. Obviously, the expansion also holds for $z = a$. $\square$

8-5.11. Let E be a complex Banach space. A holomorphic map defined on the entire complex plane is also called an *entire* map.

8-5.12. **Theorem** Let f be an entire map. If the set $\{f(z)/z^m : |z| \geq r\}$ is bounded in E for some integers m, r, then f is a polynomial of degree at most m.

Proof. Consider the Taylor expansion $f(z) = \sum_{n=0}^{\infty} A_n z^n$ at $0 \in \mathbb{C}$. Take any real $R > r$ and integer $n > m$. Let C be the positively oriented circle $|z| = R$. Since $\lambda = \sup\{\|f(z)/z^m\| : |z| \geq r\} < \infty$, we have

$$\|A_n\| = \left\|\frac{1}{2\pi i}\int_C \frac{f(w)dw}{w^{n+1}}\right\| \leq \frac{1}{2\pi}\int_C \frac{\|f(w)\||dw|}{|w|^{n+1}}$$

$$\leq \frac{1}{2\pi}\int_C \left(\frac{\|f(w)\|}{|w|^m}\right)\left(\frac{1}{|w|^{n-m+1}}\right)|dw| = \frac{\lambda}{R^{n-m}} \to 0$$

as $R \to \infty$. Therefore $A_n = 0$ for all $n > m$. Consequently $f(z) = \sum_{n=0}^{m} A_n z^n$ is a polynomial of degree at most m. $\square$

8-5.13. **Generalized Liouville's Theorem** Every bounded entire map is constant.

8-5.14. **Theorem on Removable Singularity** Let $f : X \to E$ be a holomorphic map on a punctured disk $X = \{z \in \mathbb{C} : 0 < |z - a| < R\}$. If f is bounded on X, then f can be extended to a holomorphic map on the open ball $\mathbb{B}(a, R)$.

Proof. Suppose $\|f(z)\| \leq \lambda < \infty$ for all $z \in X$. Consider the Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$ for each $0 < |z - a| < R$. For each $0 < r < R$, let C_r denote the positively oriented circle $|w - a| = r$. Then for every $n < 0$, we get

$$\|A_n\| = \left\| \frac{1}{2\pi i} \int_C \frac{f(w)dw}{(w-a)^{n+1}} \right\| \leq \frac{1}{2\pi} \int_C \|(w-a)^{-n-1} f(w)\| |dw|$$
$$\leq \frac{1}{2\pi} \int_C r^{-n-1} \lambda |dw| = \lambda r^{-n} \to 0$$

as $r \to 0$. Therefore $A_n = 0$ for all $n < 0$. Consequently $\sum_{n=0}^{\infty} A_n z^n$ is a holomorphic extension of f over the ball $\mathbb{B}(a, R)$. □

8-5.15. **Identity Theorem** Let f, g be holomorphic maps on an open *connected* subset of $\mathbb{C}$ into E. If $f(z) = g(z)$ for all z in a bounded infinite subset of X then $f = g$ on X.

Proof. For each $u \in E'$, both uf, ug are holomorphic on X. By one dimensional case, we have $uf(z) = ug(z)$ for each $z \in X$. Since E' separates points of E, we obtain $f(z) = g(z), \forall\, z \in X$. □

8-5.16. **Exercise** Let f be a holomorphic map on an open *connected* subset X of $\mathbb{C}$ into E. Prove that if f is not identically zero on X then the zeros of f are isolated points.

8-5.17. **Exercise** Let X be an open *connected* subset of $\mathbb{C}$. Let $f : X \to E$ and $g : X \to \mathbb{C}$ be holomorphic maps. Prove that if $f(z)g(z) = 0, \forall\, z \in X$ then either $f \equiv 0$ or $g \equiv 0$ on X. Note that the set of all holomorphic functions on X is an integral domain and all holomorphic maps a module over this integral domain.

8-6 Spectrum

8-6.1. An eigenvalue λ of a matrix A is defined by an algebraic equation $(\lambda I - A)x = 0$ for some non-zero vector x, i.e. $\lambda I - A$ is not injective. It is well known that every matrix has at least one complex eigenvalue. There is an explicit formula to express the eigenvalues of polynomials of a matrix. These results will be generalized to infinite dimensional spaces in terms of spectrum. Later in this chapter, spectrum will be used to define holomorphic functions of operators.

8-6.2. Let E, F be Banach spaces. A topological isomorphism from E onto F is also said to be *invertible*. The set of all such topological isomorphisms is denoted by $T(E, F)$ or simply by $T(E)$ if $E = F$. The identity map on E is denoted by I or I_E. A continuous linear map from E into itself is also called an *operator*.

8-6.3. **Theorem** Let A be an operator on E satisfying $\|A\| < 1$. Then $I - A$ is invertible. The series $(I-A)^{-1} = I + A + A^2 + A^3 + \cdots$ is absolutely convergent in the Banach space $L(E)$. Furthermore, we have $\|(I-A)^{-1}\| \le \frac{1}{1-\|A\|}$.

Proof. Since $\|I\| + \|A\| + \|A^2\| + \|A^3\| + \cdots \le 1 + \|A\| + \|A\|^2 + \|A\|^3 + \cdots$, the series $I + A + A^2 + A^3 + \cdots$ is absolutely convergent in $L(E)$. Letting $n \to \infty$ in

$$(I-A)(I + A + A^2 + \cdots + A^n) = I - A^{n+1},$$

we have $(I-A)\sum_{n=0}^{\infty} A^n = (\sum_{n=0}^{\infty} A^n)(I-A) = I$. Therefore $I - A$ is invertible and $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$. Furthermore,

$$\|(I-A)^{-1}\| = \left\|\sum_{n=0}^{\infty} A^n\right\| \le \sum_{n=0}^{\infty} \|A^n\| \le \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1-\|A\|}. \qquad \square$$

8-6.4. **Theorem** (a) The set $T(E, F)$ of all topological isomorphisms is open in the Banach space $L(E, F)$.

(b) The map $A \to A^{-1}$ is a continuous map from $T(E, F)$ into $L(F, E)$.

(c) Suppose $A, B \in L(E, F)$. If A is invertible and if $\|A - B\| < \frac{1}{\|A^{-1}\|}$, then B is also invertible. Furthermore we have

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\ \|A-B\|} \quad \text{and} \quad \|B^{-1} - A^{-1}\| \le \frac{\|A^{-1}\|^2\|A-B\|}{1 - \|A^{-1}\|\ \|A-B\|}.$$

Proof. Both (a) and (b) follow immediately from (c). Observe that

$$B = A - (A - B) = A[I - A^{-1}(A-B)].$$

Since $\|A^{-1}(A-B)\| \le \|A^{-1}\|\ \|A-B\| < 1$, the operator $I - A^{-1}(A-B)$ is a topological isomorphism. Therefore B itself is also a topological isomorphism from E onto F. Consequently, $T(E, F)$ is an open subset of $L(E, F)$. Next, observe

$$\|B^{-1}\| = \|[I - A^{-1}(A-B)]^{-1}A^{-1}\| \le \|[I - A^{-1}(A-B)]^{-1}\|\ \|A^{-1}\|$$

$$\leq \frac{1}{1 - \|A^{-1}(A-B)\|}\|A^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\,\|A-B\|}\ ;$$

and also

$$\|B^{-1} - A^{-1}\| = \|B^{-1}(A-B)A^{-1}\| \leq \|B^{-1}\|\,\|A-B\|\,\|A^{-1}\|$$
$$\leq \frac{\|A^{-1}\|^2\|A-B\|}{1 - \|A^{-1}\|\,\|A-B\|}.$$

This inequality proves that $A \to A^{-1}$ is continuous. □

8-6.5. **Exercise** Let A_n, A be operators on E. Suppose that all A_n are invertible. If $\sup \|A_n^{-1}\| < \infty$ and if $\lim \|A_n - A\| = 0$, then A is also invertible.

8-6.6. We shall work with a given operator A on E. A scalar λ is called a *resolvent value* of A if $\lambda I - A$ is invertible. The set of all resolvent values of A is called the *resolvent* of A and it is denoted by $\rho(A)$. The map $R(\lambda) = (\lambda I - A)^{-1}$ from $\rho(A)$ into $L(E)$ is called the *resolvent map* of A. The set $\sigma A = \mathbb{K} \setminus \rho(A)$ is called the *spectrum* of A and its elements are called *spectral values* of A. Equivalently, λ is a spectral value of A if $\lambda I - A$ is not invertible.

8-6.7. **Theorem** (a) The resolvent set $\rho(A)$ is open in $\mathbb{K}$.
(b) If $\lambda \in \mathbb{K}$ satisfies $|\lambda| > \|A\|$, then λ is a resolvent value of A.
(c) The spectrum of A is a compact subset of $\mathbb{K}$.
(d) Every eigenvalue of A belongs to the spectrum.

Proof. (a) The map $\lambda \to \lambda I - A$ is a continuous map from $\mathbb{K}$ into $L(E)$. Hence the inverse image $\rho(A)$ of the open set $T(E)$ is open.
(b) Suppose $|\lambda| > \|A\|$. Then $I - \frac{1}{\lambda}A$ is invertible. Hence $\lambda I - A$ is also invertible. Therefore λ is a resolvent value.
(c) It follows from (a) and (b) that σA is closed and bounded in $\mathbb{K}$ and hence it is compact.
(d) If λ is an eigenvalue, then $\lambda I - A$ is not injective and hence a spectral value. □

8-6.8. **Lemma** (a) The resolvent map $R(\lambda)$ is continuous.
(b) $\lim R(\lambda) = 0$ as $\lambda \to \infty$.
(c) For all resolvent values λ, μ, we have $R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$.
(d) $R(\lambda)$ is holomorphic on $\rho(A)$.

Proof. (a) As the composite of two continuous maps: $\lambda \to \lambda I - A$ on $\rho(A)$ and $Q \to Q^{-1}$ on $T(E)$, $R(\lambda)$ is continuous.
(b) Clearly, $R(\lambda)$ is well-defined for all large value of λ. Furthermore we have

$$\lim R(\lambda) = (\lim \lambda^{-1}) \lim \left[I - \frac{1}{\lambda}A\right]^{-1} = 0.I = 0.$$

(c) It follows immediately from

$$\begin{aligned} R(\lambda) - R(\mu) &= (\lambda I - A)^{-1} - (\mu I - A)^{-1} \\ &= (\lambda I - A)^{-1}[(\mu I - A) - (\lambda I - A)](\mu I - A)^{-1} \\ &= R(\lambda)(\mu I - \lambda I)R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu). \end{aligned}$$

(d) Since $\rho(A)$ is open, for each $\lambda \in \rho(A)$ and for all small h, we have $\lambda + h \in \rho(A)$. By (c), we have $R(\lambda + h) - R(\lambda) = -hR(\lambda + h)R(\lambda)$. It follows from (a) that

$$\lim_{h\to 0} \frac{R(\lambda + h) - R(\lambda)}{h} = \lim_{h\to 0} R(\lambda + h)R(\lambda) = -[R(\lambda)]^2 = -(\lambda I - A)^{-2}. \qquad \square$$

8-6.9. **Theorem** The spectrum of an operator A on a non-trivial complex Banach space E is non-empty.

Proof. Suppose to the contrary that the spectrum is empty, i.e. $\rho(A) = \mathbb{C}$. Then the resolvent map $R(\lambda)$ is bounded and entire. Hence $(\lambda I - A)^{-1}$ is a constant, or its inverse $\lambda I - A$ is a constant which is a contradiction. □

8-6.10. **Theorem** If A is invertible, then $\sigma(A^{-1}) = [\sigma(A)]^{-1}$.

Proof. Since A is invertible, $0 \in \mathbb{K}$ cannot belong to the spectrum. The right hand side is interpreted as $\{\lambda^{-1} : \lambda \in \sigma(A)\}$. Now suppose $\lambda \notin \sigma(A)^{-1}$, i.e. $\lambda^{-1} \notin \sigma(A)$. Then $\lambda^{-1}I - A$ is invertible. Hence $\lambda I - A^{-1} = -\lambda A^{-1}(\lambda^{-1}I - A)$ is also invertible. Therefore $\lambda \notin \sigma(A^{-1})$. We have proved $\sigma(A^{-1}) \subset [\sigma(A)]^{-1}$. Now replacing A by A^{-1}, we have $\sigma(A) \subset [\sigma(A^{-1})]^{-1}$, i.e. $\sigma(A)^{-1} \subset \sigma(A^{-1})$.□

8-6.11. Let A be an operator on a Banach space E. Let f be a polynomial function given by $f(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n$ where $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ are constants and λ a scalar variable. Define $f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_n A^n$. Clearly, $f(A)$ is again an operator on E. Clearly if f, g are polynomial functions which are equal in the sense $f(\lambda) = g(\lambda), \forall\ \lambda \in \mathbb{K}$, then $f(A) = g(A)$.

8-6.12. **Spectral Polynomial Theorem** Let E be a complex Banach space and A an operator on E. Then for every polynomial function f, we have

$$\sigma[f(A)] = f[\sigma(A)].$$

Proof. The right hand side is interpreted as $\{f(\lambda) : \lambda \in \sigma(A)\}$. Take any $\mu \in \sigma(A)$. Then $\mu I - A$ is not invertible. On the other hand, there is a polynomial q satisfying $f(\mu) - f(\lambda) = (\mu - \lambda)q(\lambda)$. Then

$$f(\mu)I - f(A) = (\mu I - A)q(A) = q(A)(\mu I - A)$$

is not invertible, otherwise

$$I = [f(\mu)I - f(A)]^{-1}q(A)(\mu I - A) = (\mu - AI)q(A)[f(\mu)I - f(A)]^{-1}$$

gives the left and right inverses of $\mu I - A$. Therefore, $f(\mu)$ is in the spectrum of $f(A)$. We have proved $f[\sigma(A)] \subset \sigma[f(A)]$. Conversely, take any $\mu \in \sigma[f(A)]$. Write $\mu - f(\lambda) = \alpha(\beta_1 - \lambda)(\beta_2 - \lambda)\cdots(\beta_n - \lambda)$ where $\beta_1, \beta_2, \cdots, \beta_n$ is an enumeration of all roots of $\mu - f(\lambda)$ and $\alpha \neq 0$. Hence we have

$$\mu I - f(A) = \alpha(\beta_1 I - A)(\beta_2 I - A)\cdots(\beta_n I - A).$$

Since $\mu I - f(A)$ is not invertible, there is at least one $1 \leq j \leq n$ such that $\beta_j I - A$ is not invertible, i.e. $\beta_j \in \sigma(A)$. As a result, $\mu = f(\beta_j) \in f[\sigma(A)]$. Therefore we have proved $\sigma[f(A)] \subset f[\sigma(A)]$. □

8-6.13. **Exercise** Let A_n, B be operators on a Banach space E. Suppose that all A_n are invertible satisfying $\sup \|A_n^{-1}\| < \infty$. If $\lim \|A_n - B\| = 0$, then B is also invertible.

8-7 Spectral Radius

8-7.1. Let A be an operator on a Banach space E. Then the *spectral radius* of A is defined by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. Of course, it measures the size of the spectrum. Equivalent formulas which is easier to apply will be given.

8-7.2. **Lemma** The Laurent series expansion of the resolvent map is given by $R(\lambda) = \dfrac{1}{\lambda}\displaystyle\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n$. It converges absolutely for all $|\lambda| > r(A)$.

Proof. Since $R(\lambda)$ is holomorphic on $\rho(A)$, it has a Laurent series expansion $(\lambda I - A)^{-1} = R(\lambda) = \sum_{n=-\infty}^{\infty} B_n \lambda^n$ for $|\lambda| > r(A)$. On the other hand, $(\lambda I - A)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n$ holds for all $|\lambda| > \|A\|$. By identity theorem, the corresponding coefficients must be the same. Therefore $R(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n$ also holds for $|\lambda| > r(A)$. □

8-7.3. **Theorem** $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{1/n} = \inf\limits_{n\geq 1} \|A^n\|^{1/n}$.

Proof. Let $\alpha = \inf_{n\geq 1} \|A^n\|^{1/n}$. Take any $\lambda \in \sigma(A)$. By Spectral Polynomial Theorem, $\lambda^n \in \sigma(A^n)$. Hence $|\lambda|^n \leq \|A^n\|$, or $|\lambda| \leq \|A^n\|^{1/n}$ for all n, i.e. $r(A) \leq \alpha$. Clearly, $\alpha \leq \liminf_{n\to\infty} \|A^n\|^{1/n} \leq \limsup_{n\to\infty} \|A^n\|^{1/n}$. Now take any $|\lambda| > r(A)$. Since the series $R(\lambda) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{A}{\lambda}\right)^n$ converges, the set

$\{\|A^n/\lambda^n\| : n \geq 1\}$ is bounded. There is $t \in \mathbb{R}$ such that for all $n \geq 1$, we have $\|A^n/\lambda^n\| \leq t$, i.e. $\|A^n\|^{1/n} \leq |\lambda| t^{1/n}$. Thus $\limsup \|A^n\|^{1/n} \leq |\lambda|$ for all $|\lambda| > r(A)$. Letting $|\lambda| \downarrow r(A)$, we have $\limsup \|A^n\|^{1/n} \leq r(A)$. Therefore $\lim \|A^n\|^{1/n}$ exists and the required equality is proved. □

8-7.4. **Theorem** Let A, B be operators on a Banach space E.
(a) $r(\lambda A) = |\lambda| r(A)$.
(b) $r(AB) = r(BA)$.
(c) $r(A^n) = [r(A)]^n$, for all $n \geq 1$.

Proof. (a) $r(\lambda A) = \lim \|(\lambda A)^n\|^{1/n} = \lim |\lambda| \ \|A^n\|^{1/n} = |\lambda| r(A)$.

(b) $r(AB) = \lim \|(AB)^{n+1}\|^{1/(n+1)} \leq \lim \|A\|^{1/(n+1)} \|(BA)^n\|^{1/(n+1)} \|B\|^{1/(n+1)}$

$$\leq \lim \|A\|^{1/(n+1)} \lim\{\|(BA)^n\|^{1/n}\}^{n/(n+1)} \lim \|B\|^{1/(n+1)} = r(BA).$$

Similarly, $r(BA) \leq r(AB)$. This proves (b).

(c) $r(A^n) = \lim\limits_{m\to\infty} \|A^{mn}\|^{1/m} = \left[\lim\limits_{m\to\infty} \|A^{mn}\|^{1/(mn)}\right]^n = [r(A)]^n$. □

8-7.5. **Theorem** Let A, B be operators on a Banach space E. If $AB = BA$, then $r(AB) \leq r(A)r(B)$ and $r(A+B) \leq r(A) + r(B)$.

Proof. Letting $n \to \infty$ in $\|(AB)^n\|^{1/n} = \|A^n B^n\|^{1/n} \leq \|A^n\|^{1/n} \|B^n\|^{1/n}$, we prove the first formula. For the second, take any $\alpha > r(A)$ and $\beta > r(B)$. Let $P = A/\alpha$ and $Q = B/\beta$. Then $r(P) < 1$ and $r(Q) < 1$. For each integer $n \geq 1$, there is an integer $0 \leq g(n) \leq n$ such that

$$\|P^{g(n)}\| \ \|Q^{n-g(n)}\| = \max_{0\leq k\leq n} \|P^k\| \ \|Q^{n-k}\|.$$

Let $h(n) = n - g(n)$. Then

$$r(A+B) \leq \|(A+B)^n\|^{1/n} \leq \left[\sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} \|P^k\| \ \|Q^{n-k}\|\right]^{1/n}$$

$$\leq \left[\sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k}\right]^{1/n} \|P^{g(n)}\|^{1/n} \|Q^{h(n)}\|^{1/n}$$

$$= (\alpha+\beta) \|P^{g(n)}\|^{1/n} \|Q^{h(n)}\|^{1/n}.$$

Since $0 \leq \frac{1}{n} g(n) \leq 1$, there is a sequence $n_1 < n_2 < n_3 < \cdots$ of integers such that $\lim_{j\to\infty} g(n_j)/n_j = \gamma \leq 1$. Then $\lim_{j\to\infty} h(n_j)/n_j = 1 - \gamma$. Consider the case when $g(n_j) \to \infty$ and $h(n_j) \to \infty$ as $j \to \infty$. Then

$$\lim_{j\to\infty} \left\|P^{g(n_j)}\right\|^{1/n_j} = \lim_{j\to\infty} \left\|P^{g(n_j)}\right\|^{[1/g(n_j)][g(n_j)/n_j]} = r(P)^\gamma \leq 1$$

Similarly we have

$$\lim_{j\to\infty} \left\|Q^{h(n_j)}\right\|^{1/n_j} \leq r(Q)^{1-\gamma} \leq 1$$

Hence $r(A+B) \le \alpha + \beta$. Next consider the case when $\{g(n_j) : j \ge 1\}$ is bounded. Then we have $\lim_{j\to\infty} g(n_j)/n_j = 0$, or $\lim_{j\to\infty} h(n_j)/n_j = 1$, i.e. $h(n_j) \to \infty$. Therefore

$$\lim_{j\to\infty} \|Q^{h(n_j)}\|^{1/n_j} = \lim_{j\to\infty} \|Q^{h(n_j)}\|^{[1/h(n_j)][h(n_j)/n_j]} = r(Q)^1 \le 1$$

and
$$\|P^{g(n_j)}\|^{1/n_j} \le \left[\max_{k\ge 1} \|P^{g(n_k)}\|\right]^{1/n_j} \to 1.$$

Consequently $r(A+B) \le \alpha+\beta$. Similarly when $\{h(n_j) : j \ge 1\}$ is bounded, we obtain the same result. Now letting $\alpha \downarrow r(A)$ and $\beta \downarrow r(B)$, we get $r(A+B) \le r(A) + r(B)$. □

8-7.6. **Exercise** Let A_n be an invertible operator on E. Suppose that $r(A_n^{-1}) \le \lambda < \infty$ for all n. Prove that if B is an operator on E satisfying $\lim \|A_n - B\| = 0$ and $A_n B = B A_n$ for all n, then B is invertible.

8-7.7. **Exercise** Prove that the following statements are equivalent for an operator A on E.

(a) A is *quasinilpotent*, i.e. $r(A) = 0$.

(b) $\sup_{n\ge 1} \|(\lambda A)^n\| < \infty$ for all $\lambda \in \mathbb{K}$.

(c) $\lim_{n\to\infty} \|(\lambda A)^n\| = 0$ for all $\lambda \in \mathbb{K}$.

8-7.8. **Exercise** Let A be a quasinilpotent operator on E. Prove that if $\limsup |\lambda_n|^{1/n} < \infty$, then the series $\sum_n \lambda_n A^n$ converges absolutely.

8-7.9. **Exercise** Show that $r(A) = r(B) = 0$, $r(A+B) = 1$, $r(AB) = 1$ and $AB \ne BA$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

8-8 Holomorphic Maps of an Operator

8-8.1. Let A be an operator on a complex Banach space E, V an open set containing the spectrum $\sigma(A)$ and $f : V \to \mathbb{C}$ a holomorphic function. We shall define the operator $f(A)$ by Cauchy integral formula over certain contour.

8-8.2. A bounded open subset W of $\mathbb{C}$ is said to be *Cauchy* if its boundary ∂W consists of a finite number of disjoint simple closed contours which are all oriented with respect to W according to the right handed rule. For example, the set $\{z : |z| < 1, \text{ or } 2 < |z| < 3, \text{ or } |z+9| < 1\}$ is a Cauchy open set when the circles $|z| = 1$, $|z| = 3$ and $|z+9| = 1$ are oriented counter clockwise but $|z| = 2$ clockwise.

8-8.3. **Lemma** Let K be a compact subset of an open set V in $\mathbb{C}$. Then there is a Cauchy open set W such that $K \subset W \subset \overline{W} \subset V$.

Proof. Write $z = x + iy$ and $w = u + iv$ where x, y, u, v are all real. Let $Q(z, \delta) = \{w : |u - x| < \delta, |v - y| < \delta\}$ denote the square with center z. Then for each $z \in K$ there is $\delta_z > 0$ such that $Q(z, 3\delta_z) \subset V$. By compactness of K, there is a finite subset J of K such that $K \subset \bigcup_{z \in J} Q(z, \delta_z)$. Clearly $W = \bigcup_{z \in J} Q(z, 2\delta_z)$ is a required open set. □

8-8.4. Let E be a complex Banach space. Then the space $L(E)$ of operators is also a complex Banach space. Let A be an operator on E. Suppose V is an open set in $\mathbb{C}$ containing the spectrum $\sigma(A)$ of A and $f : V \to \mathbb{C}$ a holomorphic function. There is a Cauchy open set W such that $\sigma(A) \subset W \subset \overline{W} \subset V$. Now in terms of the following line integral,

$$f(A) = \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda I - A)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\partial W} \frac{f(\lambda)}{\lambda I - A} d\lambda$$

is well-defined in the Banach space $L(E)$.

8-8.5. **Lemma** The definition of $f(A)$ is independent of the choice of W.

Proof. Let M be any Cauchy open set satisfying $\sigma(A) \subset M \subset \overline{M} \subset V$. Then $W \cap M$ is an open set containing the compact set $\sigma(A)$. There is a Cauchy open set N such that $\sigma(A) \subset N \subset \overline{N} \subset W \cap M$. Now ∂N is completely contained in the interior of ∂M and there is no singularity of $f(\lambda)(\lambda I - A)^{-1}$ between them. It follows from Cauchy-Goursat Theorem that

$$\int_{\partial N} f(\lambda)(\lambda I - A)^{-1} d\lambda = \int_{\partial M} f(\lambda)(\lambda I - A)^{-1} d\lambda.$$

Similarly, we have

$$\int_{\partial N} f(\lambda)(\lambda I - A)^{-1} d\lambda = \int_{\partial W} f(\lambda)(\lambda I - A)^{-1} d\lambda.$$

Therefore

$$\frac{1}{2\pi i} \int_{\partial W} \frac{f(\lambda)}{\lambda I - A} d\lambda = \frac{1}{2\pi i} \int_{\partial M} \frac{f(\lambda)}{\lambda I - A} d\lambda.$$ □

8-8.6. **Exercise** Prove that for every $x \in E$, we have

$$f(A)x = \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda I - A)^{-1} x d\lambda.$$

8-8.7. **Theorem** $(\alpha f + \beta g)(A) = \alpha[f(A)] + \beta[g(A)]$ for constants α, β.

Proof. Let g be a holomorphic function on an open set U containing $\sigma(A)$. There is a Cauchy open set W such that $\sigma(A) \subset W \subset \overline{W} \subset U \cap V$. Then

$$(\alpha f + \beta g)(A) = \frac{1}{2\pi i}\int_{\partial W} \frac{(\alpha f + \beta g)(\lambda)}{\lambda I - A} d\lambda$$
$$= \frac{\alpha}{2\pi i}\int_{\partial W} \frac{f(\lambda)}{\lambda I - A} d\lambda + \frac{\beta}{2\pi i}\int_{\partial W} \frac{g(\lambda)}{\lambda I - A} d\lambda = \alpha[f(A)] + \beta[g(A)]. \qquad \square$$

8-8.8. **Theorem** If f, g are holomorphic functions on an open set V containing $\sigma(A)$, then we have $(f \cdot g)(A) = [f(A)][g(A)]$.

Proof. Let M, W be Cauchy open sets satisfying

$$\sigma(A) \subset M \subset \overline{M} \subset W \subset \overline{W} \subset V.$$

Then ∂W is exterior to the contour ∂M. In particular, they are disjoint. Furthermore, for every $w \in \partial N$ we have $\int_{\partial M} \frac{f(z)}{w - z} dz = 0$. Now the result follows from the following calculation:

$$f(A)g(A) = \left\{\frac{1}{2\pi i}\int_{\partial M} \frac{f(z)}{zI - A} dz\right\}\left\{\frac{1}{2\pi i}\int_{\partial W} \frac{g(w)}{wI - A} dw\right\}$$
$$= \frac{1}{-4\pi^2}\int_{\partial W}\int_{\partial M} f(z)g(w)(zI - A)^{-1}(wI - A)^{-1} dz dw$$
$$= \frac{1}{-4\pi^2}\int_{\partial W}\int_{\partial M} f(z)g(w)(w - z)^{-1}[(zI - A)^{-1} - (wI - A)^{-1}] dz dw$$
$$= \frac{1}{2\pi i}\int_{\partial M} \frac{f(z)}{zI - A}\left\{\frac{1}{2\pi i}\int_{\partial W} \frac{g(w)}{w - z} dw\right\} dz$$
$$- \frac{1}{-4\pi^2}\int_{\partial W}\left\{\int_{\partial M} \frac{f(z)}{w - z} dz\right\} g(w)(wI - A)^{-1} dw$$
$$= \frac{1}{2\pi i}\int_{\partial M} \frac{f(z)g(z)}{zI - A} dz = (f \cdot g)(A). \qquad \square$$

8-8.9. **Theorem** Let f be a holomorphic function on an open ball $\mathbb{B}(0, r)$ containing $\sigma(A)$. If the series expansion $f(\lambda) = \sum_{n=0}^{\infty} b_n \lambda^n$ holds for every $\lambda \in \mathbb{B}(0, r)$, then we have $f(A) = \sum_{n=0}^{\infty} b_n A^n$ in norm.

Proof. By compactness of $\sigma(A)$, there is $t \in \mathbb{R}$ such that $\sigma(A) \subset \mathbb{B}(0, t)$ and $t < r$. Then the series $f(\lambda)$ converges uniformly on the positively oriented circle Γ given by $\{\lambda : |\lambda| = t\}$. Therefore,

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\lambda)}{\lambda I - A} d\lambda = \frac{1}{2\pi i}\int_{\Gamma} \sum_{n=0}^{\infty} b_n \lambda^n (\lambda I - A)^{-1} d\lambda$$
$$= \sum_{n=0}^{\infty} \frac{b_n}{2\pi i}\int_{\Gamma} \lambda^n (\lambda I - A)^{-1} d\lambda = \sum_{n=0}^{\infty} \frac{b_n}{2\pi i}\int_{\Gamma} \lambda^n \left(\sum_{k=0}^{\infty} A^k/\lambda^{k+1}\right) d\lambda$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_n A^k \frac{1}{2\pi i}\int_{\Gamma} \lambda^{n-k-1} d\lambda = \sum_{n=0}^{\infty} b_n A^n. \qquad \square$$

8-8.10. When f is a polynomial, we have proved that the functional definition of $f(A)$ agrees with the algebra of operators. The exercise below follows by argument similar to last theorem. After that, the spectral polynomial theorem will be generalized.

8-8.11. **Exercise** Let f_n be a sequence of holomorphic functions on an open set V containing $\sigma(A)$. Prove that if $f_n \to f$ uniformly on compact subsets of V, then $f_n(A) \to f(A)$ in norm.

8-8.12. **Spectral Function Theorem** For every holomorphic function f on an open set containing $\sigma(A)$, we have $\sigma[f(A)] = f[\sigma(A)]$.

Proof. Suppose $\mu \notin f[\sigma(A)]$. Then for every $\lambda \in \sigma(A)$, we have $\mu \neq f(\lambda)$. Since $g(\lambda) = [\mu - f(\lambda)]^{-1}$ is holomorphic on $\sigma(A)$, it is holomorphic on some open set W satisfying $\sigma(A) \subset W \subset V$. Now $g(\lambda)[\mu - f(\lambda)] = 1$ on W gives

$$g(A)[\mu I - f(A)] = I = [\mu I - f(A)]g(A).$$

Hence $\mu I - f(A)$ is invertible, i.e. $\mu \notin \sigma[f(A)]$. Therefore, we have $\sigma[f(A)] \subset f[\sigma(A)]$. Conversely, assume $\mu \in \sigma(A)$. Then the function

$$g(\lambda) = \frac{f(\mu) - f(\lambda)}{\mu - \lambda}$$

is holomorphic on V. Now $f(\mu) - f(\lambda) = (\mu - \lambda)g(\lambda)$ gives

$$f(\mu)I - f(A) = (\mu I - A)g(A).$$

Thus $f(\mu)I - f(A)$ is not invertible, otherwise

$$(\mu I - A)g(A)[f(\mu)I - f(A)]^{-1} = I = [f(\mu)I - f(A)]^{-1}g(A)(\mu I - A)$$

provides an inverse for $\mu I - A$. Therefore $f(\mu) \in \sigma[f(A)]$. Consequently, we obtain $f[\sigma(A)] \subset \sigma[f(A)]$. □

8-8.13. **Theorem** Let f be a holomorphic function on an open set V containing $\sigma(A)$ and g a holomorphic function on an open set U containing $\sigma[f(A)]$. Then we have $(gf)(A) = g[f(A)]$.

Proof. Let M be a Cauchy open set satisfying $\sigma[f(A)] \subset M \subset \overline{M} \subset U$. Then $\sigma(A) \subset f^{-1}(M) \subset V$. Since $f^{-1}(M)$ is open, there is a Cauchy open set W such that $\sigma(A) \subset W \subset \overline{W} \subset f^{-1}(M)$. Now take any $\lambda \in \overline{W}$. Then $f(\lambda) \in M$. Hence for every $z \in \partial M$, we have $z - f(\lambda) \neq 0$. Thus $\dfrac{1}{z - f(\lambda)}$ is holomorphic in λ on $\overline{W}$. Therefore,

$$[zI - f(A)]^{-1} = \frac{1}{2\pi i} \int_{\partial W} \frac{d\lambda}{[z - f(\lambda)](\lambda I - A)}.$$

Now the result follows from the calculation:

$$\begin{aligned} g[f(A)] &= \frac{1}{2\pi i}\int_{\partial M} g(z)[zI - f(A)]^{-1}dz \\ &= \frac{1}{2\pi i}\int_{\partial M} g(z)\left\{\frac{1}{2\pi i}\int_{\partial W}\frac{d\lambda}{[z-f(\lambda)](\lambda I - A)}\right\}dz \\ &= \frac{1}{2\pi i}\int_{\partial W}\left\{\frac{1}{2\pi i}\int_{\partial M}\frac{g(z)dz}{z-f(\lambda)}\right\}(\lambda I - A)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial W}(gf)(\lambda)(\lambda I - A)^{-1}d\lambda = (gf)(A). \end{aligned}$$

□

8-8.14. **Exercise** Let A, B be operators on E. Let f be a holomorphic function on an open set V containing $\sigma(A)$ and g a holomorphic function on an open set U containing $\sigma(B)$. Prove that if $AB = BA$, then we have $f(A)g(B) = g(B)f(A)$.

8-8.15. **Exponential Function of Operators** Since the exponential function is holomorphic on the spectrum of A, the operator e^A is well-defined.

(a) The series $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$ converges in norm.
(b) $e^0 = I$ and $\frac{d}{d\lambda}e^{A\lambda} = Ae^{A\lambda} = e^{A\lambda}A$ for all $\lambda \in \mathbb{C}$.
(c) If $AB = BA$ then $e^AB = Be^A$ and $e^{A+B} = e^Ae^B$.
(d) e^A is invertible and its inverse is given by $(e^A)^{-1} = e^{-A}$.

Proof. (b) follows immediately from differentiating the following series term by term: $e^{A\lambda} = I + A\lambda + \dfrac{1}{2!}A^2\lambda^2 + \dfrac{1}{3!}A^3\lambda^3 + \dfrac{1}{4!}A^4\lambda^4 + \cdots$.

(c,d) Suppose $AB = BA$. Then $e^AB = Be^A$ follows from last exercise. Next, let $F(\lambda) = e^{(A+B)\lambda}e^{-B\lambda}e^{-A\lambda}$ for all $\lambda \in \mathbb{C}$. Then $F'(\lambda)$ is given by

$$(A+B)e^{(A+B)\lambda}e^{-B\lambda}e^{-A\lambda} + e^{(A+B)\lambda}(-B)e^{-B\lambda}e^{-A\lambda} + e^{(A+B)\lambda}e^{-B\lambda}(-A)e^{-A\lambda}.$$

It follows from (b) that $F'(\lambda) = 0$. Thus $F(1) = F(0)$, i.e. $e^{A+B}e^{-B}e^{-A} = I$. Taking $B = 0$, we have $e^Ae^{-A} = I$. Replacing A by $-A$, $e^{-A}e^A = I$. Therefore e^A is invertible and we have $(e^A)^{-1} = e^{-A}$. Since $e^Ae^Be^{-B}e^{-A} = I$, we obtain $e^{A+B} = e^Ae^B$ by uniqueness of left inverses. □

8-8.16. Finally, we shall present a practical way to evaluate a holomorphic function of a square matrix. To do so, we need the following standard result from linear algebra. It will be used again by Fulmer's Method in §11-4.2.

8-8.17. **Cayley-Hamilton Theorem** Let A be an $n \times n$ matrix with characteristic polynomial given by

$$p(\lambda) = det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0.$$

Then we have $\qquad p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I = 0.$

8-8.18. Let A be a square matrix of order n and let f be a function analytic on the spectrum of A. Suppose that the characteristic function of A is factorized into the form $p(\lambda) = det(\lambda I - A) = \prod_{j=1}^{s}(\lambda - \lambda_j)^{m(j)}$ where $\lambda_1, \lambda_2, \cdots, \lambda_s$ are eigenvalues of A and each $m(j)$ is an integer ≥ 1. By partial fraction, there are polynomials $u_j(\lambda)$ of degrees $< m(j)$ respectively such that

$$\frac{1}{p(\lambda)} = \sum_{j=1}^{s} \frac{u_j(\lambda)}{(\lambda - \lambda_j)^{m(j)}}.$$

Define
$$v_k(\lambda) = \prod_{j=1, j\neq k}^{n} (\lambda - \lambda_j)^{m(j)}.$$

Let
$$T_j(\lambda) = \sum_{k=0}^{m(j)-1} \frac{f^{(k)}(\lambda_j)}{k!} (\lambda - \lambda_j)^k$$
denote the Taylor's expansion of f at λ_j truncated to the degree $m(j)-1$. Then we have the following practical expression for function of a square matrix:

$$f(A) = \sum_{j=1}^{s} u_j(A)v_j(A)T_j(A). \qquad \#1$$

Indeed, from
$$\sum_{j=1}^{s} u_j(\lambda)v_j(\lambda) = 1$$
we get

$$f(\lambda) = \sum_{j=1}^{s} u_j(\lambda)v_j(\lambda)f(\lambda) = \sum_{j=1}^{s} u_j(\lambda)v_j(\lambda) \sum_{j=0}^{\infty} \frac{f^{(k)}(\lambda_j)}{k!} (\lambda - \lambda_j)^k$$
$$= \sum_{j=1}^{s} \sum_{j=0}^{m(j)-1} \frac{f^{(k)}(\lambda_j)}{k!} u_j(\lambda)v_j(\lambda)(\lambda - \lambda_j)^k$$
$$+ \sum_{j=1}^{s} \sum_{j=m(j)}^{\infty} \frac{f^{(k)}(\lambda_j)}{k!} u_j(\lambda)p(\lambda)(\lambda - \lambda_j)^{j-m(j)}$$
$$= \sum_{j=1}^{s} u_j(\lambda)v_j(\lambda)T_j(\lambda) + p(\lambda) \sum_{j=1}^{s} \sum_{j=m(j)}^{\infty} \frac{f^{(k)}(\lambda_j)}{k!} u_j(\lambda)(\lambda - \lambda_j)^{j-m(j)}.$$

Now replacing λ by the matrix A, since $p(A) = 0$, we prove the formula #1.

8-8.19. **Exercise** Find e^A where $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 2 & 1 & 0 & -2 \\ 2 & 2 & -1 & -2 \\ 3 & 1 & 0 & -2 \end{bmatrix}$. Compare the result in §11-4.3.

8-99. **References and Further Readings** : Taylor-71, Zhu, Murphy, Stout, Gamelin-69, Garnett, Gohberg-82 and Aron-91.

Chapter 9

Differentiation in Banach Spaces

9-1 Differentiable Maps

9-1.1. Let f be a map from a Banach space E into a Banach space. We cannot define the derivative at a point in terms of difference quotient because it does not make sense to divide a vector $f(x) - f(a)$ by another vector $x - a$. However the following lemma gives an equivalent formulation which motivates the definition of derivatives as linear maps. Elementary properties such as continuity of differentiable maps, chain rule will be given in this section.

9-1.2. **Lemma** Let f be a map from an open subset X of the scalar field $\mathbb{K}$ into a Banach space F. Then the limit $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists for $a \in X$ iff there is a continuous linear map $Df(a) : \mathbb{K} \to F$ such that for every $\varepsilon > 0$ there is $\delta > 0$ satisfying $\|f(x) - f(a) - Df(a)(x-a)\| \le \varepsilon|x-a|$ whenever $|x - a| \le \delta$. In this case, we have $Df(a)t = tf'(a)$ for all $t \in \mathbb{K}$.

Proof. ($\Leftarrow$) Suppose that $Df(a), \varepsilon, \delta$ satisfy the given conditions. Let $f'(a) = Df(a)1$ where $1 \in \mathbb{K}$. Then for all $|x - a| \le \delta$, we have

$$\begin{aligned}
&\left\|\frac{f(x) - f(a)}{x - a} - f'(a)\right\| = \frac{\|f(x) - f(a) - (x-a)f'(a)\|}{|x-a|} \\
&= \frac{\|f(x) - f(a) - (x-a)Df(a)1\|}{|x-a|} = \frac{\|f(x) - f(a) - Df(a)[(x-a)1]\|}{|x-a|} \\
&= \frac{\|f(x) - f(a) - Df(a)(x-a)\|}{|x-a|} \le \varepsilon.
\end{aligned}$$

($\Rightarrow$) Define $Df(a)t = tf'(a)$ for all $t \in \mathbb{K}$. Then $Df(a) : \mathbb{K} \to F$ is a continuous linear map. Since $f'(a) = \lim[f(x) - f(a)]/(x-a)$, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $0 < |x-a| \le \delta$ we have $\|[f(x) - f(a)]/(x-a) - f'(a)\| \le \varepsilon$, i.e. $\|f(x) - f(a) - (x-a)f'(a)\| \le \varepsilon|x-a|$, or $\|f(x) - f(a) - Df(a)(x-a)\| \le \varepsilon|x-a|$. Note that last inequality is also true for $|x - a| = 0$. The proof is complete. □

9-1.3. Let E, F be Banach spaces and $L(E, F)$ the vector space of all continuous linear maps from E into F. Let X be an open subset of E and

$f : X \to F$ a given map. Then f is said to be *differentiable at* $a \in X$ if there is $Df(a)$ in $L(E, F)$ such that for every $\varepsilon > 0$ there is $\delta > 0$ satisfying

$$\|f(x) - f(a) - Df(a)(x-a)\| \le \varepsilon \|x - a\|$$

whenever $\|x - a\| \le \delta$, i.e.

$$\lim_{x \to a} \frac{f(x) - f(a) - Df(a)(x-a)}{\|x-a\|} = 0.$$

In this case, $Df(a)$ is called the *derivative* of f at a or *total derivative* in order to distinguish from the partial derivatives introduced later. The map f is said to be *differentiable on* X if it is differentiable at every point of X. A *holomorphic* map refers specifically to differentiable maps among *complex* Banach spaces.

9-1.4. **Theorem** If $f : X \to F$ is differentiable at $a \in X$, then its derivative is unique.

Proof. Let A, B be derivatives of f at $a \in X$. For every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in \mathbb{B}(a, 2\delta)$ we have $\|f(x) - f(a) - A(x-a)\| \le \varepsilon\|x - a\|$ and $\|f(x) - f(a) - B(x-a)\| \le \varepsilon\|x-a\|$. Hence $\|(A-B)(x-a)\| \le 2\varepsilon\|x-a\|$. Now take any $\|v\| \le 1$ in E. Let $x = a + \delta v$. Since $x \in \mathbb{B}(a, 2\delta)$, we have $\|(A-B)(\delta v)\| \le 2\varepsilon\delta$, i.e. $\|A - B\| = \sup_{\|v\| \le 1} \|(A-B)v\| \le 2\varepsilon$. Because ε is arbitrary, $\|A - B\| = 0$, i.e. $A = B$. □

9-1.5. **Exercise** Let $f : X \to F$ be given by $f(x) = Ax + b$ where $A : E \to F$ is a continuous linear map and $b \in F$ a constant. Prove that f is differentiable on X and $Df(a) = A$ for each $a \in X$. In particular when $A = 0$, the derivatives of constant maps are zero.

9-1.6. **Theorem** If f is differentiable at $a \in X$, then f is continuous at a. More precisely, for every $\lambda > \|Df(a)\|$, there is $\delta > 0$ such that

$$\|f(x) - f(a)\| \le \lambda\|x - a\| \text{ for all } \|x - a\| \le \delta.$$

Proof. For every $0 < \varepsilon < \lambda - \|Df(a)\|$, there is $0 < \delta < \varepsilon/\lambda$ such that for all $\|x - a\| \le \delta$ we have $\|f(x) - f(a) - Df(a)(x-a)\| \le \varepsilon\|x - a\|$, or $\|f(x) - f(a)\| - \|Df(a)(x-a)\| \le \varepsilon\|x-a\|$, that is,

$$\begin{aligned} &\|f(x) - f(a)\| \le \|Df(a)(x-a)\| + \varepsilon\|x-a\| \\ &\le \|Df(a)\|\|(x-a)\| + \varepsilon\|x-a\| \le \lambda\|x-a\| \le \varepsilon. \end{aligned}$$ □

9-1.7. **Exercise** Show that the function $h(z) = \overline{z}$ is continuous but not holomorphic from the complex plane into itself.

9-1.8. **First Order Chain Rule** Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. Let $f : X \to Y$ and $g : Y \to G$ be two given maps. If f is differentiable at $a \in X$ and g differentiable at $b = f(a) \in Y$, then the composite map $gf : X \to G$ is differentiable at $a \in X$. Furthermore, we have $D(gf)(a) = Dg(b)Df(a)$.

Proof. Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that for all $\|x - a\| \le \delta$ we have

$$\|f(x) - f(a) - Df(a)(x - a)\| \le \varepsilon\|x - a\| \ ,$$

$$\|f(x) - f(a)\| \le \{\|Df(a)\| + 1\}\|x - a\| \ ,$$

and also $$\|g(y) - g(b) - Dg(b)(y - b)\| \le \varepsilon\|y - b\|$$

for all $\|y - b\| \le (\|Df(a)\| + 1)\delta$. Writing $y = f(x)$, it follows that

$$\begin{aligned} &\|gf(x) - gf(a) - Dg(b)Df(a)(x - a)\| \\ &\le \|g(y) - g(b) - Dg(b)(y - b)\| + \|Dg(b)\|\|f(x) - f(a) - Df(a)(x - a)\| \\ &\le \varepsilon\|y - b\| + \varepsilon\|Dg(b)\| \ \|x - a\| \\ &\le \varepsilon\{\|Df(a)\| + 1\}\|x - a\| + \varepsilon\|Dg(b)\| \ \|x - a\| \\ &\le \varepsilon\{\|Df(a)\| + 1 + \|Dg(b)\|\}\|x - a\|. \end{aligned}$$

Therefore the composite of operators $Dg(b)Df(a)$ is the derivative of gf at $a \in X$, i.e. $D(gf)(a) = Dg(b)Df(a)$. □

9-1.9. **Exercise** Let $\varphi : E_1 \to E$ and $\psi : F \to F_1$ be topological isomorphisms. Prove that $f : X \to F$ is differentiable iff the composite $\psi f \varphi : \varphi^{-1}(X) \to F_1$ is differentiable. In particular, if a map from $\mathbb{K}^n$ into $\mathbb{K}^m$ is differentiable with respect to some norms on $\mathbb{K}^n, \mathbb{K}^m$, then it is differentiable with respect to every norms.

9-1.10. Let $L(E, F)$ denote the Banach space of all continuous linear maps from E into F. A map $f : X \to F$ is said to be *continuously differentiable* if f is differentiable on X and its derivative $Df : X \to L(E, F)$ is a continuous map. Continuously differentiable maps are also called C^1*-maps*.

9-1.11. **Theorem** Composites of continuously differentiable maps are continuously differentiable.

Proof. Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. Suppose $f : X \to Y$ and $g : Y \to G$ are C^1-maps. Then $y = f(x)$ and $Dg(y)$ are continuous in x, y respectively. Hence the composite map

$(Dg)[f(x)]$ is continuous in x. Since the product $L(E,F) \times L(F,G) \to L(E,G)$ is jointly continuous, $D(gf)(x) = (Dg)[f(x)]Df(x)$ is continuous in x. □

9-1.12. **Exercise** Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(0) = g(0) = 0$ and for all $x \neq 0$, $f(x) = x \sin \frac{1}{x}$, $g(x) = x^2 \sin \frac{1}{x}$. Show that f is continuous but not differentiable. Show that g is differentiable but not continuously differentiable.

9-1.13. **Theorem** Let E, F, G be Banach spaces and X an open subset of E. The product space $F \times G$ consists of all column vectors $\begin{bmatrix} u \\ v \end{bmatrix}$ where $u \in F$ and $v \in G$. Suppose $f : X \to F$ and $g : X \to G$ are given maps and $p : X \to F \times G$ is given by $p(x) = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$ for each $x \in X$. Then p is continuously differentiable on X iff both f, g are. In this case, the derivative of p is given in matrix form:

$$Dp(x) = \begin{bmatrix} Df(x) \\ Dg(x) \end{bmatrix}.$$

Proof. Let $\pi : F \times G \to F$ be the projection onto the first coordinate. Let $\varphi : F \to F \times G$ be the natural injection given by $\varphi(u) = \begin{bmatrix} u \\ 0 \end{bmatrix}$ and $\psi : G \to F \times G$ by $\psi(v) = \begin{bmatrix} 0 \\ v \end{bmatrix}$. Then $f = \pi p$ and $p = \varphi f + \psi g$ with chain rule give the result. □

9-2 Mean-Value Theorem

9-2.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = \begin{bmatrix} x^2 \\ x^3 \end{bmatrix}$. It is easy to show that there is no $t \in [0, 1]$ satisfying $f(1) - f(0) = Df(t)(1 - 0)$. Hence the classical form of Mean-Value Theorem in one dimensional case fails in $\mathbb{R}^2$. There will be several versions of generalization of this important theorem. We start with the following preparation. Let E, F be Banach spaces and X an open subset of E. We shall continue to work on a map $f : X \to F$.

9-2.2. **Lemma** Let M be an open subset of $\mathbb{K}$. Suppose $g : M \to X$ and $f : X \to F$ are differentiable maps. Then for each $t \in M$ we have $(fg)'(t) = Df(x)g'(t)$ where $x = g(t)$ and both $(fg)'(t), g'(t)$ are defined as the limits of difference quotients.

Proof. By Chain Rule, $D(fg)(t) = Df(x)Dg(t)$, i.e. for each vector $h \in \mathbb{K}$, we have $D(fg)(t)h = Df(x)Dg(t)h$. Regarding h as a scalar and $(fg)'(t), g'(t)$ as vectors, we obtain $h(fg)'(t) = Df(x)[hg'(t)]$, i.e. $h(fg)'(t) = hDf(x)g'(t)$. The result follows by letting $h = 1$. □

9-2.3. **Inequality Mean-Value Theorem** Let f be differentiable on X. Suppose the line segment $[a, b] = \{(1-t)a + tb : 0 \leq t \leq 1\}$ is contained in X. Then we have $\|f(b) - f(a)\| \leq \|b - a\| \sup\{\|Df(z)\| : z \in [a, b]\}$.

Proof. It suffices to prove the real case because every complex Banach space is also a real one. Define $g(t) = (1-t)a + tb$ for all $t \in R$. Then $g : \mathbb{R} \to E$ is differentiable. Applying the Chain Rule to the composite map fg, we have $(fg)'(t) = Df(x)g'(t)$. Since $[0, 1]$ is a subset of the open set $g^{-1}(X)$, it follows from the mean-value theorem for $\mathbb{R}$ that for every $u \in F'$ there is $t \in (0, 1)$ such that $(ufg)(1) - (ufg)(0) = (ufg)'(t)$, i.e. $uf[g(1)] - uf[g(0)] = uDf(x)g'(t)$ where $x = g(t) \in [a, b]$. Hence $u[f(b) - f(a)] = u[Df(x)(b-a)]$. For any $\|u\| \leq 1$, we have $|u[f(b) - f(a)]| \leq \|u\| \; \|Df(x)\| \; \|b - a\| \leq \|b - a\| \sup_{z \in [a,b]} \|Df(z)\|$. The result follows by taking supremum over $\|u\| \leq 1$. □

9-2.4. **Exercise** Let $g : X \to \mathbb{R}$ a differentiable map. Prove that if the line segment $[a, b] = \{(1-t)a + tb : 0 \leq t \leq 1\}$ is contained in X show that there is $x \in [a, b]$ such that $g(b) - g(a) = Dg(x)(b - a)$.

9-2.5. **Exercise** Prove that if M, N are disjoint non-empty open subsets of a normed space E, then $M \cup N$ is disconnected. Also prove that every pair of points in an open connected subset Y of E can be joined by a broken line contained in Y.

9-2.6. **Exercise** Let $f : X \to F$ be a differentiable map. If the derivative $Df(x) = A$ is a constant on X and if X is connected, then there is some $b \in E$ such that for every $x \in X$ we have $f(x) = Ax + b$.

9-2.7. **Integral Mean-Value Theorem** Let $f : X \to F$ be continuously differentiable. If the line segment $[a, a + h] = \{a + th : 0 \leq t \leq 1\}$ is contained in X, then we have $f(a + h) = f(a) + \int_0^1 Df(a + th)h\,dt$.

Proof. The map $g : \mathbb{R} \to E$ defined by $x = g(t) = a + th$ for all $t \in R$ is continuously differentiable. The composite map fg has continuous derivative: $(fg)'(t) = Df(x)g'(t) = Df(a+th)h$. Now the Second Fundamental Theorem of Calculus gives $f(b) - f(a) = fg(1) - fg(0) = \int_0^1 Df(a + th)h\,dt$. □

9-2.8. **Uniform Mean-Value Theorem** Let f be a continuously differentiable map from an open subset X of E into F. Then for every $a \in X$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in \mathbb{B}(a, \delta)$ we have

$$\|f(x) - f(y) - Df(a)(x - y)\| \le \varepsilon\|x - y\|.$$

Proof. Define $g(x) = f(x) - Df(a)x$ for all $x \in X$. Given that f is continuously differentiable, so is g. Since its derivative $Dg(x) = Df(x) - Df(a)$ vanishes at a, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $\|z - a\| \le \delta$ in X, we obtain $\|Dg(z)\| = \|Df(z) - Df(a)\| \le \varepsilon$. Now for any $x, y \in \mathbb{B}(a, \delta)$, the following estimation completes the proof:

$$\|f(x) - f(y) - Df(a)(x - y)\| = \|g(x) - g(y)\|$$
$$\le \|x - y\| \sup\{\|Dg(z)\| : z \in [x, y]\} \le \|x - y\|\varepsilon. \qquad \square$$

9-2.9. **Exercise** Let $f : X \to F$ be a continuously differentiable map and Q a compact subset of X. Prove that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in Q$ with $\|x - y\| \le \delta$ we have $\|f(y) - f(x) - Df(x)(y - x)\| \le \varepsilon\|y - x\|$.

9-2.10. **Lemma** Let X be a *convex* open subset of E and $f_n : X \to F$ a sequence of differentiable maps. Suppose for some $a \in X$, the sequence $f_n(a)$ converges. If $Df_n \to g$ converges uniformly on X for some $g : X \to L(E, F)$, then $f_n \to f$ converges uniformly on bounded subsets of X for some $f : X \to F$. Furthermore we have $Df = g$ on X.

Proof. By uniform convergence of $Df_n \to g$ on X, $\{Df_n\}$ is uniformly Cauchy on X, i.e. for every $\varepsilon > 0$ there is an integer p such that for all $m, n \ge p$ and for all $x \in X$ we have $\|Df_m(x) - Df_n(x)\| \le \varepsilon$. For each $x \in X$, since X is convex we have

$$\|[f_m(x) - f_m(a)] - [f_n(x) - f_n(a)]\| = \|[f_m(x) - f_n(x)] - [f_m(a) - f_n(a)]\|$$
$$\le \|x - a\| \sup\{\|Df_m(z) - Df_n(z)\| : z \in [x, a]\} \le \|x - a\|\varepsilon.$$

Thus $\{f_n(x) - f_n(a)\}$ is Cauchy in the Banach space F and hence it converges. Since $\{f_n(a)\}$ converges, we may write $f_n(x) \to f(x) \in F$. Now a map $f : X \to F$ has been defined. The above inequality shows that $\{f_n(x) - f_n(a)\}$ converges uniformly on every bounded subset of X and so does $\{f_n(x)\}$. Take any $x_0 \in X$. We want to show that $g(x_0)$ is the derivative of f at x_0. Repeating what we have proved by replacing a with x_0, for every $\varepsilon > 0$ there is an integer p such that for all $m, n \ge p$ we have

$$\|[f_m(x) - f_m(x_0)] - [f_n(x) - f_n(x_0)]\| \le \|x - x_0\|\varepsilon.$$

Letting $m \to \infty$, we obtain $\|[f(x) - f(x_0)] - [f_n(x) - f_n(x_0)]\| \le \|x - x_0\|\varepsilon$. Since $Df_n(x_0) \to g(x_0)$, there is an integer q such that for all $n \ge q$, we get $\|Df_n(x_0) - g(x_0)\| \le \varepsilon$. Now choose $n = p + q$. By differentiability of f_n at x_0, there is $\delta > 0$ such that for all $\|x - x_0\| \le \delta$ in X we get

$$\|f_n(x) - f_n(x_0) - Df(x_0)(x - x_0)\| \le \varepsilon\|x - x_0\|.$$

The following estimation

$$\begin{aligned}
&\|f(x) - f(x_0) - g(x_0)(x - x_0)\| \\
&\le \|[f(x) - f(x_0)] - [f_n(x) - f_n(x_0)]\| \\
&\qquad + \|f_n(x) - f_n(x_0) - Df_n(x_0)(x - x_0)\| \\
&\qquad + \|Df_n(x_0)(x - x_0) - g(x_0)(x - x_0)\| \\
&\le 3\varepsilon\|x - x_0\|,
\end{aligned}$$

shows that f is differentiable at x_0 and $Df(x_0) = g(x_0)$. □

9-2.11. **Closed Graph of Differential Operators** Let X be an open subset of E and let $f_n, f : X \to F$ be given maps. Suppose each f_n is continuously differentiable and $f_n \to f$ pointwise on X. If $\{Df_n\}$ converges locally uniformly on X, then so is $\{f_n\}$. Furthermore, f is continuously differentiable on X and $Df = \lim Df_n$ on X.

Proof. Let $a \in X$ be given. Then there is $\delta > 0$ such that $Df_n \to g$ uniformly on $\mathbb{B}(a, \delta) \subset X$. It follows from last lemma that $f_n \to f$ uniformly on the bounded convex set $\mathbb{B}(a, \delta)$. Since $a \in X$ is arbitrary, $f_n \to f$ locally uniformly on X. Also f is differentiable and $Df = g = \lim Df_n$. As uniform limit of continuous maps Df_n on $\mathbb{B}(a, \delta)$, the derivative Df is also continuous on $\mathbb{B}(a, \delta)$. Since $a \in X$ is arbitrary, Df is continuous on X. □

9-2.12. **Exercise** Let $f_n(x) = xe^{-nx^2}$ and $g(x) = 0$ for all $x \in [-1, 1]$. Show that $f_n \to g$ uniformly on $[-1, 1]$ but $\lim_{n\to\infty} f_n'(0) \ne g'(0)$.

9-2.13. **Exercise** Let $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ for all $x \in \mathbb{R}$. Show that $f_n \to 0$ uniformly on $\mathbb{R}$ but $\lim_{n\to\infty} f_n'(x)$ does not exist at any point.

9-3 Partial Derivatives

9-3.1. In this section, continuously differentiable maps will be characterized in terms of partial derivatives. As a result of §§9-1.2, 3.7, finite dimensional derivatives are identified as matrices. Differentiation under integral sign will be done.

9-3.2. Let E, G be Banach spaces and Y a given set. Let X be an open subset of E. The product space $E \times Y$ consists of all columns $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in E, y \in Y \right\}$

which will be written as rows (x, y) for the sake of convenience unless matrix operations are involved. Let $f : X \times Y \to G$ be a given map and (a, b) a given point of $X \times Y$. Then the *partial map* $f^b : X \to G$ is defined by $f^b(x) = f(x, b)$. If f^b is differentiable at $a \in X$, then its derivative is called the *partial derivative* of f at (a, b) which will be denoted by $\partial_x f(a, b) = Df^b(a)$. Similarly, f^a and $\partial_y f(a, b)$ are defined when Y is an open subset of a Banach space.

9-3.3. **Theorem** Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. If $f : X \times Y \to G$ is differentiable, then both partial derivatives exist. Furthermore we have in matrix form: $Df(a, b) = [\partial_x f(a, b), \partial_y f(a, b)]$. Equivalently, if π, ψ are projections of $E \times F$ onto E, F respectively, then we have $Df(a, b) = \partial_x f(a, b)\pi + \partial_y f(a, b)\psi$.

Proof. Let $\sigma : E \to E \times F$ and $\tau : F \to E \times F$ be the natural injections. Consider the composite map $f^b = fg$ where $g(x) = \sigma + \begin{pmatrix} 0 \\ b \end{pmatrix}$. Since f, g are differentiable, so is f^b. Furthermore $\partial_x f(a, b) = Df^b(a) = Df(a, b)\sigma$. Similarly f^a is differentiable at $b \in Y$ and $\partial_y f(a, b) = Df(a, b)\tau$. Consequently for every $h \in E$ and $k \in F$ we have

$$Df(a, b)\begin{bmatrix} h \\ k \end{bmatrix} = Df(a, b)\begin{bmatrix} h \\ 0 \end{bmatrix} + Df(a, b)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Df(a, b)\sigma(h) + Df(a, b)\tau(k)$$

$$= \partial_x f(a, b)h + \partial_y f(a, b)k = [\partial_x f(a, b), \partial_y f(a, b)\ ]\begin{bmatrix} h \\ k \end{bmatrix}. \qquad \square$$

9-3.4. **Theorem** Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. Then $f : X \times Y \to G$ is continuously differentiable on $X \times Y$ iff both partial derivatives $\partial_x f$ and $\partial_y f$ exist and are continuous on $X \times Y$.

Proof. Let $\sigma : E \to E \times F$ and $\tau : F \to E \times F$ be the natural injections. If f is continuously differentiable on $X \times Y$, then the partial derivative $\partial_x f(a, b) = Df(a, b)\sigma$ is a composite of continuous maps and hence it is a continuous map of (a, b). Conversely assume that both $\partial_x f(a, b)$ and $\partial_y f(a, b)$ are continuous maps of (a, b). For each $(a, b) \in X \times Y$, define a linear map $A(a, b) : E \times F \to G$ by $A(a, b) = \partial_x f(a, b)\sigma + \partial_y f(a, b)\tau$. Fix $(a, b) \in X \times Y$. By continuity of partial derivatives, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $\|h\| \le \delta$ in E and all $\|k\| \le \delta$ in F we have $(a + h, b + k) \in X \times Y$; $\|\partial_x f(a + h, b + k) - \partial_x f(a, b)\| \le \varepsilon$ and $\|\partial_y f(a + h, b + k) - \partial_y f(a, b)\| \le \varepsilon$. Observe that

$$\|f(a + h, b + k) - f(a, b) - A(a, b)(h, k)\|$$
$$\le \|f(a+h, b+k) - f(a, b+k) - \partial_x f(a, b)h\| + \|f(a, b+k) - f(a, b) - \partial_y f(a, b)k\|$$

$$= \left\| \int_0^1 [\partial_x f(a+th, b+k) - \partial_x f(a,b)] h dt \right\| \\ + \left\| \int_0^1 [\partial_y f(a, b+tk) - \partial_y f(a,b)] k dt \right\|$$

$$\leq \int_0^1 \|[\partial_x f(a+th, b+k) - \partial_x f(a,b)]\| \, \|h\| dt \\ + \int_0^1 \|[\partial_y f(a, b+tk) - \partial_y f(a,b)]\| \, \|k\| dt$$

$$\leq \int_0^1 \varepsilon \|h\| dt + \int_0^1 \varepsilon \|k\| dt = \varepsilon(\|h\| + \|k\|).$$

Therefore f is differentiable at (a,b) and its derivative is given by

$$Df(a,b) = A(a,b) = \partial_x f(a,b)\sigma + \partial_y f(a,b)\tau.$$

As a composite of continuous maps of the point (a,b), the derivative $Df(a,b)$ is also continuous in (a,b). □

9-3.5. **Exercise** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that at $(0,0)$, the partial derivatives exist but the total derivative does not.

9-3.6. **Exercise** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2) \sin \dfrac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that at $(0,0)$, the total derivative exists. Show that the partial derivatives exist but not continuous at $(0,0)$. Derive that f is not continuously differentiable.

9-3.7. **Exercise** Let $E = \prod_{i=1}^n E_i$, $F = \prod_{j=1}^m F_j$ denote the product spaces of Banach spaces $E_1, E_2, \cdots, E_n$ and $F_1, F_2, \cdots, F_m$ respectively. Let X be an open subset of E and $f : X \to F$ a differentiable map. Write

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} \in X, x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \in E, h = \begin{bmatrix} h_1 \\ h_2 \\ \cdots \\ h_n \end{bmatrix} \in E, f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \cdots \\ f_m(x) \end{bmatrix} \in F.$$

Show that the partial derivatives $\partial_j f_i(a)$ of the map $f_i : X \to F_i$ exist. Furthermore verify that

$$Df(a)h = \begin{bmatrix} \partial_1 f_1(a)h_1 + \partial_2 f_1(a)h_2 + \cdots + \partial_n f_1(a)h_n \\ \partial_1 f_2(a)h_1 + \partial_2 f_2(a)h_2 + \cdots + \partial_n f_2(a)h_n \\ \cdots\cdots\cdots\cdots \\ \partial_1 f_m(a)h_1 + \partial_2 f_m(a)h_2 + \cdots + \partial_n f_m(a)h_n \end{bmatrix}$$

$$= \begin{bmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \cdots & \partial_n f_1(a) \\ \partial_1 f_2(a) & \partial_2 f_2(a) & & \partial_n f_2(a) \\ \cdots & \cdots & & \cdots \\ \partial_1 f_m(a) & \partial_2 f_m(a) & \cdots & \partial_n f_m(a) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \cdots \\ h_n \end{bmatrix}.$$

Therefore in terms of matrix of linear maps we may write

$$Df(a) = \begin{bmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \cdots & \partial_n f_1(a) \\ \partial_1 f_2(a) & \partial_2 f_2(a) & & \partial_n f_2(a) \\ \cdots & \cdots & & \cdots \\ \partial_1 f_m(a) & \partial_2 f_m(a) & & \partial_n f_m(a) \end{bmatrix}.$$

9-3.8. **Example** We always identify an $m \times n$ matrix as a linear map from $\mathbb{K}^n$ to $\mathbb{K}^m$. The total derivative of the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} v \sin u \\ e^{uv} \\ u^3 + v^4 \end{bmatrix}$ is given by the expression

$$Df\begin{pmatrix} u \\ v \end{pmatrix} = [\partial_u f, \partial_v f] = \begin{bmatrix} v \cos u & \sin u \\ ve^{uv} & ue^{uv} \\ 3u^2 & 4v^3 \end{bmatrix}.$$

Therefore the derivative of f at $a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the matrix $Df(a) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 4 \end{bmatrix}$.

9-3.9. **Differentiation under Integral Sign** Let X be an open subset of E and let $\alpha < \beta$ be real numbers. Suppose $f : [\alpha, \beta] \times X \to F$ is a continuous map. If the partial derivative $\partial_x f : [\alpha, \beta] \times X \to L(E, F)$ exists and is continuous, then the map $g : X \to F$ given by $g(x) = \int_\alpha^\beta f(t, x)dt$ is continuously differentiable on X. Furthermore we have $\quad Dg(x) = \int_\alpha^\beta \partial_x f(t, x)dt.$

Proof. For fixed $a \in X$, $\partial_x f(t, a)$ is continuous in $t \in [\alpha, \beta]$ and hence $A = \int_\alpha^\beta \partial_x f(t, a)dt$ is well-defined in $L(E, F)$. To show that A is the derivative of g at a, observe that

$$\begin{aligned} &g(a + h) - g(a) - Ah \\ &= \int_\alpha^\beta f(t, a + th)dt - \int_\alpha^\beta f(t, a)dt - \left[\int_\alpha^\beta \partial_x f(t, a)dt\right] h \\ &= \int_\alpha^\beta [f(t, a + h) - f(t, a)]dt - \left[\int_\alpha^\beta \partial_x f(t, a)dt\right] h \end{aligned}$$

$$= \int_\alpha^\beta \left[\int_0^1 \partial_x f(t, a + \theta h)hd\theta\right] dt - \left[\int_\alpha^\beta \int_0^1 \partial_x f(t,a)d\theta dt\right] h$$
$$= \left[\int_\alpha^\beta \int_0^1 Dxf(t, a + \theta h)hd\theta\right] h - \left[\int_\alpha^\beta \int_0^1 \partial_x f(t,a)d\theta dt\right] h$$
$$= \left[\int_\alpha^\beta \int_0^1 \{\partial_x f(t, a + \theta h) - \partial_x f(t,a)\}d\theta dt\right] h.$$

Since $\partial_x f : [\alpha, \beta] \times X \to L(E, F)$ is continuous and $[\alpha, \beta]$ is compact, it follows from §2-7.6 that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $t \in [\alpha, \beta]$, we have $\|\partial_x f(t, a + h) - \partial_x f(t, a)\| \leq \varepsilon$ whenever $\|h\| \leq \delta$. Therefore we obtain

$$\|g(a + h) - g(a) - Ah\|$$
$$\leq \left[\int_\alpha^\beta \int_0^1 \|\partial_x f(t, a + \theta h) - \partial_x f(t,a)\| d\theta dt\right] \|h\| \leq (\beta - \alpha)\varepsilon\|h\|,$$

i.e.
$$Dg(a) = A = \int_\alpha^\beta \partial_x f(t, a)dt.$$

Since $\partial_x f(t, a)$ is continuous in (t, a), $\int_\alpha^\beta \partial_x f(t, a)dt$ is continuous in a. Therefore g is continuously differentiable on X. □

9-3.10. **Exercise** Let $f(x) = \left[\int_0^x e^{-t^2}dt\right]^2$ and $g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2 + 1}dt.$ Show that both $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable. Evaluate $f'(x)+g'(x)$. Deduce $f(x) + g(x) = \frac{1}{4}\pi$. Hence prove that $\lim_{x\to\infty} \int_0^x e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}$.

9-4 Fixed Points of Contractions

9-4.1. Let X be a metric space and $A : X \to X$ a given map. Then A is called a *contraction* on X if there is $0 \leq \lambda < 1$, called a *contraction constant* of A, such that $d(Ax, Ay) \leq \lambda d(x, y)$ for all $x, y \in X$. Clearly every contraction is uniformly continuous. Contraction is one of the most popular tool to provide existence of solutions in many problems.

9-4.2. **Contraction Fixed-Point Theorem** Let A be a contraction on a nonempty metric space X. If X is complete, then A has a unique fixed-point.

Proof. Choose any $x_0 \in X$. Define $x_n = Ax_{n-1}$ for all $n \geq 1$. Let λ be a contraction constant of A. Then for each $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(Ax_n, Ax_{n-1}) \leq \lambda d(x_n, x_{n-1})$$

$\leq \lambda^2 d(x_{n-1}, x^{n-2}) \leq \cdots \leq \lambda^n d(x_1, x_0)$. Next, for each $k \geq 1$, we have

$$d(x_{n+k}, x_n) \leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \cdots + d(x_{n+1}, x_n)$$
$$\leq (\lambda^{n+k-1} + \lambda^{n+k-2} + \cdots + \lambda^n) d(x_1, x_0) \leq \frac{\lambda^n d(x_1, x_0)}{1 - \lambda}.$$

Since $0 \leq \lambda < 1$, the last term tends to zero as $n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in the complete metric space X. Let $x_n \to y$ in X. Since A is continuous, we have $Ax_n \to Ay$, i.e. $x_{n+1} \to Ay$, or $x_n \to Ay$. By uniqueness of limit, we have $Ay = y$. Therefore A has at least one fixed point. Finally, suppose x, y are fixed points of A. Then $d(x, y) = d(Ax, Ay) \leq \lambda d(x, y)$, i.e. $0 \geq (1 - \lambda) d(x, y)$. Since $0 \leq \lambda < 1$, we have $d(x, y) = 0$, i.e. $x = y$. □

9-4.3. **Example** Let $X = [1, \infty)$ and $Ax = x + \frac{1}{x} : X \to X$. As a closed subset of the complete space $\mathbb{R}$, X is a complete metric space. Simple calculation shows that $|Ax - Ay| < |x - y|$ on X but A has no fixed point.

9-4.4. **Theorem** Let X be a complete metric space and $A : X \to X$ a given map. If A^p is a contraction for some integer p, then A has a unique fixed-point.

Proof. Because A^p is a contraction, there is $0 < \lambda < 1$ such that $d(A^p x, A^p y) \leq \lambda d(x, y)$ for all $x, y \in X$. Let x be a fixed point of A^p. Then

$$d(Ax, x) = d(AA^p x, A^p x) = d(A^p Ax, A^p x) \leq \lambda d(Ax, x)$$

gives $d(Ax, x) = 0$. Therefore x is also a fixed point of A. Since every fixed point of A is also a fixed point of A^p, the uniqueness follows immediately. □

9-4.5. **Exercise** Let A be an operator on a Banach space E and $b \in E$ a given vector. Prove that if $r(A) < 1$ then the equation $x = Ax + b$ has a unique solution $x \in E$.

9-5 Inverse and Implicit Mapping Theorems

9-5.1. In this section, we shall apply the Contraction Fixed-Point Theorem to establish Inverse and Implicit Mapping Theorems. An advantage of contraction allows us to estimate the error but numerical flavor is beyond our scope even we make provision in the following contraction lemma.

9-5.2. **Contraction Lemma** Let E, F be Banach spaces, X an open subset of E and $f : X \to F$ a continuously differentiable map. If the derivative $Df(a)$ at some point $a \in X$ is invertible, then there is $\delta > 0$ such that for each $\|v\| \leq \frac{1}{2}\delta$, the map $g(x) = x + v - [Df(a)]^{-1}[f(x) - f(a)]$ is a contraction on the closed ball $\overline{\mathbb{B}}(a, \delta)$.

Proof. Let $A = Df(a)$ for simplicity. Define $h(x) = x - A^{-1}f(x)$ for all $x \in X$. Then h is differentiable on X. Moreover, $Dh(x) = I - A^{-1}Df(x)$ and $Dh(a) = 0$. Since f is continuously differentiable, so is h. Therefore for $\varepsilon = \frac{1}{2}$ there is $\delta > 0$ such that for all $\|x - a\| \le \delta$ we have $\|Dh(x)\| \le \frac{1}{2}$. Take any $x \in \overline{\mathbb{B}}(a, \delta)$ and any $\|v\| \le \frac{1}{2}\delta$. Observe that

$$\begin{aligned}&\|g(x) - a\| = \|x + v - A^{-1}[f(x) - f(a)] - a\|\\&\le \|v\| + \|[x - A^{-1}f(x)] - [a - A^{-1}f(a)]\| \le \tfrac{1}{2}\delta + \|h(x) - h(a)\|\\&\le \tfrac{1}{2}\delta + \|x - a\| \sup\{\|Dh(z)\| : z \in [a, x]\} \le \tfrac{1}{2}\delta + \delta \cdot \tfrac{1}{2} = \delta.\end{aligned}$$

Therefore g carries $\overline{\mathbb{B}}(a, \delta)$ into itself. Finally to show that g is a contraction, take any $x_1, x_2 \in \overline{\mathbb{B}}(a, \delta)$. The following calculation completes the proof:

$$\begin{aligned}&\|g(x_1) - g(x_2)\| = \|x_1 - x_2 - A^{-1}[f(x_1) - f(x_2)]\| = \|h(x_1) - h(x_2)\|\\&\le \|x_1 - x_2\| \sup\{\|Dh(z)\| : z \in [x_1, x_2]\} \le \|x_1 - x_2\|\tfrac{1}{2}.\end{aligned}$$ □

9-5.3. **Inverse Mapping Theorem** Let X be an open subset of E and $f : X \to E$ a continuously differentiable map. If the derivative $Df(a)$ at some $a \in X$ is invertible in $L(E)$, then there are open subsets V, W of E such that the following conditions hold:

(a) $a \in V \subset X$;

(b) $f : V \to W$ is a bijection;

(c) the local inverse map $f^{-1} : W \to V$ is continuously differentiable.

Furthermore for every $x \in V$ we have $D(f^{-1})(y) = [Df(x)]^{-1}$ where $y = f(x)$.

Proof. The same notation of Contraction Lemma is used. We claim that for all $x_1, x_2 \in \overline{\mathbb{B}}(a, \delta)$ we have

$$\|x_1 - x_2\| \le 2\|A^{-1}\|\ \|f(x_1) - f(x_2)\|.$$

In fact, let g be the contraction with $v = 0$ in last lemma. Then

$$g(x_1) - g(x_2) = x_1 - x_2 - A^{-1}[f(x_1) - f(x_2)]$$

i.e. $$x_1 - x_2 = g(x_1) - g(x_2) + A^{-1}[f(x_1) - f(x_2)].$$

Hence $$\begin{aligned}\|x_1 - x_2\| &\le \|g(x_1) - g(x_2)\| + \|A^{-1}\|\ \|f(x_1) - f(x_2)\|\\&\le \tfrac{1}{2}\|x_1 - x_2\| + \|A^{-1}\|\ \|f(x_1) - f(x_2)\|,\end{aligned}$$

or, $$\tfrac{1}{2}\|x_1 - x_2\| \le \|A^{-1}\|\ \|f(x_1) - f(x_2)\|$$

that is, $$\|x_1 - x_2\| \le 2\|A^{-1}\|\ \|f(x_1) - f(x_2)\|.$$

Therefore, f is injective on $\overline{\mathbb{B}}(a, \delta)$. Next, choose $\delta_1 > 0$ satisfying $\|A^{-1}\|\delta_1 < \frac{1}{2}\delta$. Let $W = \mathbb{B}[f(a), \delta_1]$ and $V = \mathbb{B}(a, \delta) \cap f^{-1}(W)$. Then V

is an open set in E and satisfies $a \in V \subset X$. Clearly $f : V \to W$ is injective. Take any $y \in W$. Let $v = A^{-1}[y - f(a)]$. Then we have

$$\|v\| \le \|A^{-1}\| \, \|y - f(a)\| \le \|A^{-1}\|\delta_1 \le \tfrac{1}{2}\delta.$$

Hence g with this v in last lemma is a contraction on $\overline{\mathbb{B}}(a, \delta)$. Since E is a Banach space, the closed ball $\overline{\mathbb{B}}(a, \delta)$ is complete. There is $x \in \overline{\mathbb{B}}(a, \delta)$ such that $g(x) = x$, i.e. $x + v - A^{-1}[f(x) - f(a)] = x$, or $A^{-1}[y - f(a)] - A^{-1}[f(x) - f(a)] = 0$. Hence $y = f(x)$. Observe

$$\|x - a\| \le 2\|A^{-1}\| \, \|f(x) - f(a)\| \le 2\|A^{-1}\| \, \|y - f(a)\| < 2\|A^{-1}\|\delta_1 \le \delta.$$

Thus $x \in \mathbb{B}(a, \delta) \cap f^{-1}(W) = V$. Therefore $f : V \to W$ is bijective. To show that f^{-1} is differentiable at any point $y_0 \in W$, let $x_0 = f^{-1}(y_0) \in V$. Since f is differentiable at x_0, for every $\varepsilon > 0$ there is $\delta_2 > 0$ such that $\mathbb{B}(x_0, \delta_2) \subset \mathbb{B}(a, \delta)$ and for every $x \in \mathbb{B}(x_0, \delta_2)$ we have

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \le \varepsilon\|x - x_0\|.$$

Take any $\|y - f(a)\| < \min\{\delta_1, \delta_2/(2\|A^{-1}\|)\}$. Write $y = f(x)$ for some $x \in V$. Then $\|x - a\| \le 2\|A^{-1}\|\|f(x) - f(a)\| \le \delta_2$. Now the following calculation:

$$\begin{aligned}
&\|f^{-1}(y) - f^{-1}(y_0) - [Df(x_0)]^{-1}(y - y_0)\| \\
&= \|x - x_0 - [Df(x_0)]^{-1}[f(x) - f(x_0)]\ \| \\
&\le \|Df(x_0)^{-1}\| \, \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \\
&\le \|Df(x_0)^{-1}\| \ \varepsilon \ \|x - x_0\| \\
&\le \|Df(x_0)^{-1}\| \ \varepsilon \ 2 \ \|A^{-1}\| \ \|f(x) - f(x_0)\| \\
&\le 2\varepsilon\|Df(x_0)^{-1}\| \ \|A^{-1}\| \ \|y - y_0\|
\end{aligned}$$

shows that f^{-1} is differentiable at y_0 and its derivative is given by

$$D(f^{-1})(y_0) = [Df(x_0)]^{-1}.$$

Since $D(f^{-1})(y_0) = \{Df[f^{-1}(y_0)]\}^{-1}$ is a composite of continuous maps, $D(f^{-1})$ is continuous on W. □

9-5.4. **Implicit Mapping Theorem** Let E, F be Banach spaces; M an open subset of $E \times F$; $f : M \to F$ a continuously differentiable map; $(a, b) \in M$ and $c = f(a, b)$. If the partial derivative $\partial_y f(a, b) : F \to F$ is invertible, then the equation $f(x, y) = c$ has a unique *local implicit solution* $y = g(x)$ near (a, b). More precisely, there are open balls X, Y with centers a, b respectively and a unique continuously differentiable map $g : X \to Y$ such that

(a) $X \times Y \subset M$ and $g(a) = b$;

(b) for all $(x, y) \in X \times Y$, $y = g(x)$ iff $f(x, y) = c$;

(c) $Dg = -(\partial_y f)^{-1}(\partial_x f)$ on X.

Proof. Define $h : M \subset E\times F \to E\times F$ by $h\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ f(x,y) \end{bmatrix}$ for all $(x,y) \in M$. Then at (a,b), the derivative of h and its inverse are given by

$$Dh = \begin{bmatrix} I_E & 0 \\ \partial_x f & \partial_y f \end{bmatrix} \quad \text{and} \quad (Dh)^{-1} = \begin{bmatrix} I_E & 0 \\ -(\partial_y f)^{-1}\partial_x f & (\partial_y f)^{-1} \end{bmatrix}.$$

By Inverse Mapping Theorem, there are open subsets V, W of $E \times F$ such that $(a,b) \in V \subset M$; $h : V \to W$ is bijective and $h^{-1} : W \to V$ is continuously differentiable. Now for $(a,b) \in V$, we have $h(a,b) = (a,c) \in W$. Choose open balls A, B with centers a, c respectively such that $A \times B \subset W$. By continuity of h, there are open balls Q, Y with centers a, b respectively such that $Q \times Y \subset M$ and $h(Q \times Y) \subset A \times B$. Now $Q \subset A$ because for every $x \in Q$; $(x,b) \in Q \times Y$; thus $h(x,b) = (x, f(x,b)) \in A \times B$ and so $x \in A$. Next, let $\pi : E \times F \to F$ be the projection $\pi(x,y) = y$. For every $x \in Q$, we have $(x,c) \in A \times B$ and so $k(x) = \pi h^{-1}(x,c)$ is well-defined with $k(a) = \pi h^{-1}(a,c) = \pi(a,b) = b$. Clearly $k : Q \to F$ is continuously differentiable. By continuity, there is an open ball X with center a such that $k(X) \subset Y$. Then $g = k|X : X \to Y$ is a continuously differentiable map with $g(a) = b$. To prove (b), for every $(x,y) \in X \times Y$, $y = g(x)$ iff $y = \pi h^{-1}(x,c)$ iff $(x,y) = h^{-1}(x,c)$ iff $h(x,y) = (x,c)$ iff $f(x,y) = c$. Finally using the notation of column vectors to differentiate $f\begin{bmatrix} x \\ g(x) \end{bmatrix} = f\begin{bmatrix} a \\ b \end{bmatrix}$, we have $[\,\partial_x f \quad \partial_y f\,]\begin{bmatrix} I_E \\ Dg \end{bmatrix} = (Df)D\begin{bmatrix} x \\ g(x) \end{bmatrix} = 0$, that is $\partial_x f + (\partial_y f)(Dg) = 0$. Since Dh is invertible, so is $\partial_y f$. Consequently $Dg = -(\partial_y f)^{-1}(\partial_x f)$ holds as required. Finally for uniqueness, let g_1 be another implicit solution on A. Then $Dg_1 = -(\partial_y f)^{-1}(\partial_x f) = Dg$ on A. Since the ball A is connected, $g_1 - g$ is constant on A. Since $g_1(a) = b = g(a)$, we have $g_1 = g$ on A. □

9-5.5. Let E, F be Banach spaces, X an open subset of E and $f : X \to F$ a differentiable map. Then for every $a \in X$ the map $g : X \to F$ defined by $g(x) = f(a) + Df(a)(x-a)$ is called the *linearization* of f at a. It is supposed to be a convenient approximation of f when x is near a.

9-5.6. **Exercise** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\begin{bmatrix} u \\ v \end{bmatrix} = f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \sin xy \\ y + e^{xy} \end{bmatrix}$.

(a) Evaluate $f(a)$ where $a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the linearization of f at a. Find the inverse map of this linearization.

(b) Show that f is locally invertible at a.

(c) Find the total derivative of the local inverse map f^{-1} at the point $f(a)$. Also find the linearization of f^{-1} when $\begin{bmatrix} u \\ v \end{bmatrix}$ is near $f(a)$. Compare the results obtained from (a) and (c).

(d) Define a vector map g to represent the system of equations: $u = x + y$ and $v = x - y$. Find an explicit formula for the composite map gf. Verify the matrix form of Chain Rule at the point a.

9-5.7. **Example** Let $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = \begin{bmatrix} rstuv - 1 \\ r^4 + s^5 + u^2 + v^3 - 4 \end{bmatrix}$$

where $x = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ and $y = \begin{bmatrix} u \\ v \end{bmatrix}$. Suppose $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Calculate the partial derivatives as follow

$$\partial_x f(a, b) = [\,\partial_r f \quad \partial_s f \quad \partial_t f\,] = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 0 \end{bmatrix}$$

and also

$$\partial_y f(a, b) = [\partial_u f \ \partial_v f] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

Since $det\, \partial_y f(a, b) = 1 \neq 0$, there is an implicit solution $y = g(x)$ for the equation $f(x, y) = 0$ when (x, y) near (a, b). Although we do not know the exact values, we may approximate $g(x)$ by its linearization as follow

$$\begin{bmatrix} u \\ v \end{bmatrix} = g(a) + Dg(a)(x - a) = b - (\partial_y f)^{-1}(\partial_x f)(x - a)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} r - 1 \\ s - 1 \\ t - 1 \end{bmatrix} = \begin{bmatrix} r + 2s - 3t + 1 \\ -2r - 3s + 2t + 4 \end{bmatrix}.$$

In terms of scalar equations, the approximate implicit solution is given by

$$u = r + 2s - 3t + 1 \text{ and } v = -2r - 3s + 2t + 4.$$

□

9-6 Local Properties of Differentiable Maps

9-6.1. Let f be a continuously differentiable function from an open subset X of $\mathbb{K}^n$ into $\mathbb{K}^m$. Properties f near a point $a \in X$ will be derived based on the corresponding properties of the derivative $Df(A)$. As a result of Surjection Theorem below, being interior point is invariant under coordinate transformation. Hence the interior and consequently the boundary of a differentiable manifold is well-defined in terms of charts. Most of the results of this section have been generalized to the context of Banach spaces. We restrict ourselves to finite dimensional case so that our students can see the link with linear algebra.

9-6.2. **Lemma** For every constant k, the set $M = \{x \in X : \, rank \; Df(x) \geq k\}$ is open.

Proof. Take any $a \in M$. Then $rank \; Df(x) \geq k$. It follows from linear algebra that there is a submatrix $S(a)$ of $Df(a)$ such that $det \; S(a) \neq 0$. Let $S(x)$ be the submatrix of $Df(x)$ with the same corresponding rows and columns of $S(a)$. Then $det \; S(x)$ is a continuous function of $x \in X$. Hence there is a ball $\mathbb{B}(a) \subset X$ such that for all $x \in \mathbb{B}(a)$, we have $det \; S(x) \neq 0$, i.e. $rank \; Df(x) \geq k$. Thus $\mathbb{B}(a) \subset M$. Therefore M is an open set. □

9-6.3. **Lemma** If a linear map $A : \mathbb{K}^n \to \mathbb{K}^m$ is injective then it is bounded below, i.e. there is a constant $\lambda > 0$ such that for all $x \in \mathbb{K}^n$ we have $\|Ax\| \geq \lambda \|x\|$.

Proof. Suppose to the contrary that for all j, there is $\|x_j\| = 1$ satisfying $\|Ax_j\| \leq \frac{1}{j}\|x_j\|$, i.e. $Ax_j \to 0$. By compactness of the unit sphere, there is a convergent subsequence $y_j \to y$. Then $Ay_j \to Ay$ by continuity of A. Thus $Ay = 0$. On the other hand, since $\|y_j\| = 1$ we get $\|y\| = 1$. Therefore A is not injective. □

9-6.4. **Injection Theorem** If the derivative $Df(a)$ is injective at a point $a \in X$, then there is a ball $\mathbb{B}(a) \subset X$ such that

(a) the derivative $Df(x)$ is injective for every $x \in \mathbb{B}(a)$;

(b) the map f is injective on $\mathbb{B}(a)$.

Proof. (a) Since $Df(a)$ is injective, we have ker $Df(a) = \{0\}$. Hence we have

$$rank \; Df(a) = n - \; nullity \; Df(a) = n.$$

There is $\delta > 0$ such that for all $x \in \mathbb{B}(a, \delta)$ we obtain $rank \; Df(x) \geq n$ which means $nullity \; Df(x) = n - \; rank \; Df(x) \leq 0$, i.e. ker $Df(x) = \{0\}$. Therefore $Df(x)$ is also injective.

(b) The proof is independent of (a). Since $Df(a)$ is injective, there is $\lambda > 0$ such that for all $x \in \mathbb{K}^n$ we get $\|Df(a)x\| \geq \lambda\|x\|$. By Uniform Mean-Value Theorem, for $\varepsilon = \frac{1}{2}\lambda$ there is $\delta > 0$ such that for all $x, y \in \mathbb{B}(a, \delta)$ we obtain

$$\|f(x) - f(y) - Df(a)(x - y)\| \leq \varepsilon\|x - y\|;$$

i.e.

$$\|Df(a)(x - y)\| - \|f(x) - f(y)\| \leq \tfrac{1}{2}\lambda\|x - y\|.$$

Now by the choice of λ, we get

$$\lambda\|x-y\| \le \|Df(a)(x-y)\| \le \|f(x)-f(y)\| + \tfrac{1}{2}\lambda\|x-y\|.$$

Thus $$\|x-y\| \le \tfrac{2}{\lambda}\|f(x)-f(y)\|, \forall\, x,y \in X.$$

Therefore if $f(x)=f(y)$ then $x=y$. Consequently, f is injective on $\mathbb{B}(a,\delta)$. □

9-6.5. **Interior Theorem** If $Df(a)$ is surjective where $a \in X$ then $f(a)$ is an interior point of $f(X)$.

Proof. Write $Df(a) = [\partial_1 f(a), \partial_2 f(a), \cdots, \partial_n f(a)]$ where $\partial_j f(a)$ is the j-th column of $Df(a)$. Since $Df(a)$ is surjective, $rank\ Df(a) = m$. Hence $Df(a)$ has m columns which are linearly independent. Without loss of generality, we may assume that the first m columns $\partial_1 f(a), \cdots, \partial_m f(a)$ of $Df(a)$ to be independent. Then the matrix $A = [\partial_1 f(a), \cdots, \partial_m f(a)]$ is an invertible square matrix. Define $B = [\partial_{m+1} f(a), \cdots, \partial_n f(a)]$. So, $Df(a) = [A, B]$. Let $b = f(a)$ and define $g(y) = \begin{bmatrix} A^{-1} \\ 0 \end{bmatrix}(y-b)+a$ for each $y \in \mathbb{K}^m$. Then $g : \mathbb{K}^m \to \mathbb{K}^n$ is a continuous map and $g(b) = a$. Hence $Y = g^{-1}(X)$ is an open subset of $\mathbb{K}^m$ containing b. Now $fg : Y \to \mathbb{K}^m$ is a continuously differentiable map satisfying $fg(b) = b$ and $D(fg)(b) = Df(a)Dg(b) = [A, B]\begin{bmatrix} A^{-1} \\ 0 \end{bmatrix} = I_m$. By Inverse Mapping Theorem, $b = fg(b)$ is an interior point of $fg(Y) = f(X)$. □

9-6.6. **Corollary** If $Df(x)$ is surjective for every $x \in X$ then $f(X)$ is open in $\mathbb{K}^m$.

9-6.7. **Surjection Theorem** If $Df(a)$ is surjective for some $a \in X$, then there is $\delta > 0$ such that the following conditions hold:
(a) $Df(x)$ is surjective for every $x \in \mathbb{B}(a,\delta)$;
(b) f carries every open subset of the open ball $\mathbb{B}(a,\delta)$ onto an open subset of $\mathbb{K}^m$.

Proof. Since $Df(a)$ is surjective, $rank\ Df(a) = m$. There is $\delta > 0$ such that for all $x \in \mathbb{B}(a,\delta)$ we have $rank\ Df(x) \ge m$ and consequently $Df(x)$ is surjective. Next take any open subset M of $\mathbb{B}(a,\delta)$ and any $x \in M$. Then $x \in \mathbb{B}(a,\delta)$ and thus $Df(x)$ is surjective. So, $f(x)$ is an interior point of $f(M)$. Therefore $f(M)$ is open in $\mathbb{K}^m$. □

9-6.8. **Lagrange Multipliers** Let X be an open subset of $\mathbb{R}^n$ and let $f, g_j : X \to \mathbb{R}$ be continuously differentiable functions for $j = 1, 2, \cdots, p < n$. Suppose $f(x)$ has a local maximum or minimum at $x = a \in X$ subject to the constraints $g_j(x) = 0$ for all j. Then there are constants $\lambda_1, \lambda_2, \cdots, \lambda_p$ such that all partial derivatives of the Lagrangian function: $L = f + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_p g_p$ vanish at a.

Proof. Let m be the rank of the $p \times n$-matrix $[\partial_j g_i(a)]$. Without loss of generality, we may assume that the first m rows are linearly independent. Define $g : X \to \mathbb{R}^m$ and $h : X \to \mathbb{R}^{m+1}$ by $g(x) = \begin{bmatrix} g_1(x) \\ \cdots \\ g_m(x) \end{bmatrix}$ and $h(x) = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$.
Then $Dh(a) = \begin{bmatrix} Df(a) \\ Dg(a) \end{bmatrix}$ is an $(m+1) \times n$ matrix. Suppose to the contrary that $rank\ Dh(a) = m+1$. By Surjection Theorem, there is $\delta > 0$ such that h carries open subsets of $\mathbb{B}(a, \delta)$ onto open subsets of $\mathbb{R}^{m+1}$. For any $\varepsilon > 0$, choose $0 < t < \min\{\varepsilon, \delta\}$. Then $h(a)$ belongs to the open subset $h[\mathbb{B}(a,t)]$ of $\mathbb{R}^{m+1}$. There is $\delta_1 > 0$ such that $\mathbb{B}[h(a), \delta_1] \subset h[\mathbb{B}(a,t)]$. Hence

$$h(x) = h(a) + \begin{bmatrix} \pm\frac{1}{2}\delta_1 \\ 0 \end{bmatrix}.$$

That is, $\|x - a\| \leq \varepsilon$; $f(x) = f(a) \pm \frac{1}{2}\delta_1$ and $g_j(x) = 0, \forall\ j = 1, 2, \cdots, m$. Therefore f cannot have a local maximum or a minimum. This contradiction shows $rank\ Dh(a) < m+1$. Since the rank of $Dg(a)$ is m, so is the rank of $Dh(a)$. Therefore the first row of

$$Dh(a) = \begin{bmatrix} Df(a) \\ Dg(a) \end{bmatrix} = \begin{bmatrix} Df(a) \\ Dg_1(a) \\ Dg_2(a) \\ \cdots\cdots \\ Dg_m(a) \end{bmatrix}$$

is a linear combination of the other rows. For some constants $\lambda_1, \lambda_2, \cdots, \lambda_m$, we have $Df(a) = -\lambda_1 Dg_1(a) - \lambda_2 Dg_2(a) - \cdots - \lambda_m Dg_m(a)$. Define $\lambda_j = 0$ for all $m < j \leq p$ and $L = f + \lambda_1 g_1 + \cdots \lambda_p f_p$. Then clearly it follows $\partial_j L(a) = 0$ for all $1 \leq j \leq p$. □

9-99. **References and Further Readings** : Cartan, Lang, Sagan, Berger, Ma-01, Dineen-81, Hamilton, Janos, Meyers and Wong-66.

Chapter 10

Polynomials and Higher Derivatives

10-1 Multilinear Maps on Banach Spaces

10-1.1. Polynomials are probably the simplest nonlinear maps. Higher derivatives allow us to approximate certain nonlinear maps by Taylor polynomials. In the context of vector spaces, polynomials are defined in terms of multilinear maps. Continuity of multilinear maps will be treated in the same way as linear maps.

10-1.2. Let $E_1, E_2, \cdots, E_n$ and F be vector spaces. A map f from the product space $\prod_{k=1}^n E_k$ into F is said to be *multilinear* if $f(x_1, x_2, \cdots, x_k, \cdots, x_n)$ is linear in each variable x_k, i.e. for all $\alpha, \beta \in \mathbb{K}$, all $x_j \in E_j$, $a_k, b_k \in E_k$, we have

$$\begin{aligned} &f(x_1, x_2, \cdots, \alpha a_k + \beta b_k, \cdots, x_n) \\ &= \alpha f(x_1, x_2, \cdots, a_k, \cdots, x_n) + \beta f(x_1, x_2, \cdots, b_k, \cdots, x_n). \end{aligned}$$

It is easy to prove that the set of all multilinear maps from $\prod_{k=1}^n E_k$ into F forms a vector space under pointwise operations. A scalar-valued multilinear map is also called a *multilinear form*.

10-1.3. Let $E_1, E_2, \cdots, E_n, F$ be normed spaces and let $f : \prod_{j=1}^n E_j \to F$ be a multilinear map. The *norm* of f is defined by

$$\|f\| = \sup\{\|f(x_1, x_2, \cdots, x_n)\| : \|x_j\| \leq 1, \forall\ 1 \leq j \leq n\}.$$

Note that f need not be continuous and its norm may be infinity. Because of $f(0, x_2, \cdots, x_n) = 0$, we have $\|f\| \geq 0$.

10-1.4. **Lemma** For all $x_j \in E_j$ we have

$$\|f(x_1, x_2, \cdots, x_n)\| \leq \|f\|\ \|x_1\|\ \|x_2\| \cdots \|x_n\|.$$

Proof. Consider the following cases. If some $x_j = 0$ then we get

$$\|f(x_1, x_2, \cdots, x_n)\| = 0 \leq \|f\|\ \|x_1\|\ \|x_2\| \cdots \|x_n\|.$$

If all $x_j \neq 0$ then we obtain

$$\begin{aligned}
&\|f(x_1, x_2, \cdots, x_n)\| \\
&= \left\| \|x_1\| \, \|x_2\| \cdots \|x_n\| f\left(\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \cdots, \frac{x_n}{\|x_n\|}\right)\right\| \\
&= \|x_1\| \, \|x_2\| \cdots \|x_n\| \left\| f\left(\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \cdots, \frac{x_n}{\|x_n\|}\right)\right\| \\
&\leq \|x_1\| \, \|x_2\| \cdots \|x_n\| \, \|f\|.
\end{aligned}$$

□

10-1.5. **Theorem** Let $f : \prod_{j=1}^{n} E_j \to F$ be a multilinear map. Then the following statements are equivalent.

(a) f is continuous on $\prod_{j=1}^{n} E_j$.

(b) f is continuous at the origin.

(c) $\|f\| < \infty$.

Proof. ($a \Rightarrow b \Rightarrow c$) It is obvious.

($c \Rightarrow b$) For every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\|\delta^n < \varepsilon$. Now for every $\|x_j\| \leq \delta$ in E_j, we have

$$\begin{aligned}
&\|f(x_1, x_2, \cdots, x_n) - f(0, 0, \cdots, 0)\| = \|f(x_1, x_2, \cdots, x_n)\| \\
&\leq \|f\| \, \|x_1\| \, \|x_2\| \cdots \|x_n\| \leq \|f\|\delta^n < \varepsilon.
\end{aligned}$$

Therefore f is continuous at the origin of $\prod_{j=1}^{n} E_j$.

($b \Rightarrow a$) Take any $a_j, h_j \in E_j$ for each $1 \leq j \leq n$. To show that f is continuous at $(a_1, a_2, \cdots, a_n)$, let $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $n\|f\|(\lambda + 1)^{n-1}\delta < \varepsilon$ where $\lambda = \sup\{\|a_j\| : 1 \leq j \leq n\}$. Now for all $\|h_j\| \leq \delta$ we get

$$\begin{aligned}
&\|f(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - f(a_1, a_2, \cdots, a_n)\| \\
&= \|f(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - f(a_1, a_2 + h_2, \cdots, a_n + h_n) \\
&\qquad + f(a_1, a_2 + h_2, \cdots, a_n + h_n) - f(a_1, a_2, \cdots, a_n + h_n) \\
&\qquad + \cdots\cdots + f(a_1, a_2, \cdots, a_{n-1}, a_n + h_n) - f(a_1, a_2, \cdots, a_{n-1}, a_n)\| \\
&= \|f(h_1, a_2 + h_2, \cdots, a_n + h_n) + f(a_1, h_2, \cdots, a_n + h_n) \\
&\qquad + \cdots\cdots + f(a_1, a_2, \cdots, a_{n-1}, h_n)\| \\
&\leq \|f(h_1, a_2 + h_2, \cdots, a_n + h_n)\| + \|f(a_1, h_2, \cdots, a_n + h_n)\| \\
&\qquad + \cdots\cdots + \|f(a_1, a_2, \cdots, a_{n-1}, h_n)\| \\
&\leq \|f\| \, \|h_1\| \, \|a_2 + h_2\| \cdots \|a_n + h_n\| + \|f\| \, \|a_1\| \, \|h_2\| \cdots \|a_n + h_n\| \\
&\qquad + \cdots\cdots + \|f\| \, \|a_1\| \, \|a_2\| \cdots \|a_{n-1}\| \, \|h_n\| \\
&\leq \|f\| \, \|h_1\|(\|a_2\| + \|h_2\|) \cdots (\|a_n\| + \|h_n\|) + \|f\| \, \|a_1\| \, \|h_2\| \cdots (\|a_n\| + \|h_n\|) \\
&\qquad + \cdots\cdots + \|f\| \, \|a_1\| \, \|a_2\| \cdots \|a_{n-1}\| \, \|h_n\|
\end{aligned}$$

$$\leq \|f\| \, \|h_1\|(\lambda+1)\cdots(\lambda+1) + \|f\|(\lambda+1)\|h_2\|\cdots(\lambda+1)$$
$$+\cdots\cdots+\|f\|(\lambda+1)(\lambda+1)\cdots(\lambda+1)\|h_n\|$$
$$= n\|f\|(\lambda+1)^{n-1}\delta < \varepsilon.$$

Therefore f is continuous at $(a_1, a_2, \cdots, a_n)$. □

10-1.6. **Theorem** The set $M(E_1, E_2, \cdots, E_n; F)$ of all continuous multilinear maps from $\prod_{j=1}^n E_j$ into F forms a normed space under pointwise operations. Furthermore, if F is a Banach space, then so is $M(E_1, E_2, \cdots, E_n; F)$. Since the proof is almost identical to the linear case, it is left as exercise. When $E_1 = \cdots = E_n = E$, write $L^n(E, F)$ instead of $M(E_1, E_2, \cdots, E_n; F)$. If F is the scalar field $\mathbb{K}$, we simply drop it. Hence $M(E_1, E_2, \cdots, E_n)$ and $L^n(E)$ denote the sets of all continuous multilinear forms on $\prod_{j=1}^n E_j$ and E^n respectively. Finally for convenience, define $L^0(E, F) = F$.

10-1.7. **Example** Let $f : C_\infty[0,1] \times C_1[0,1] \to \mathbb{K}$ be a map defined by $f(x, y) = \int_0^1 x(t)y(t)dt$. Clearly f is bilinear. Since $|f(x,y)| \leq \|x\|_\infty \|y\|_1 \leq 1$ for all $\|x\|_\infty \leq 1$ and $\|y\|_1 \leq 1$, we have $\|f\| \leq 1$. Thus f is continuous. Taking the constant function $x = y = 1$, we have $\|f\| = 1$.

10-1.8. **Exercise** Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For every $x = (x_1, x_2, \cdots)$ in ℓ_p and $y = (y_1, y_2, \cdots)$ in ℓ_q, let $f(x, y) = (x_1 y_1, x_2 y_2, \cdots)$. Show that $f : \ell_p \times \ell_q \to \ell_1$ is a continuous bilinear map. Find the norm of f.

10-1.9. **Exercise** For every $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ in $\mathcal{F}_\infty$, let $f(x, y) = x_1 y_1 + x_2 y_2 + \cdots$. Show that f is a bilinear form. Is it continuous?

10-1.10. **Exercise** Let $E_1, E_2, \cdots, E_n$ and F be Banach spaces and let $M(E_1, E_2, \cdots, E_n; F)$ denote the Banach space of all continuous multi-linear maps from the product space $\prod_{j=1}^n E_j$ into F. Prove that the formula

$$A(x_1 x_2 \cdots x_n) = (Ax_1)(x_2 \cdots x_n) = (Ax_1 \cdots x_{n-1})x_n$$

establishes unique isometric isomorphisms among the Banach spaces

$$M(E_1, E_2, \cdots, E_n; F) \simeq L[E_1, M(E_2, \cdots, E_n; F)] \simeq M[E_1, \cdots, E_{n-1}; L(E_n, F)].$$

10-1.11. **Exercise** Prove the following *Uniform Boundedness Theorem for Multilinear Maps*. Let $E_1, E_2, \cdots, E_n$ be Banach spaces and let F be a normed space. A family $\{f_i : i \in I\}$ of continuous multilinear maps from $\prod_{j=1}^n E_j$ into F is norm bounded, i.e. $\sup\{\|f_i\| : i \in I\} < \infty$ iff it is pointwise bounded, that is for each $(x_1, x_2, \cdots, x_n) \in \prod_{j=1}^n E_j$, we have $\sup\{\|f_i(x)\| : i \in I\} < \infty$.

10-1.12. **Exercise** Prove the following version of *Banach-Steinhaus Theorem for Multilinear Maps.* Let $E_1, E_2, \cdots, E_n$ be Banach spaces and let F be a normed space. Let $\{f_n : n \geq 1\}$ a sequence of continuous multilinear maps from $\prod_{j=1}^n E_j$ into F. If for each $x_1, x_2, \cdots, x_n \in \prod_{j=1}^n E_j$, the limit

$$g(x_1, x_2, \cdots, x_n) = \lim_{n\to\infty} f_n(x_1, x_2, \cdots, x_n)$$

exists, then g is a continuous multilinear map from $\prod_{j=1}^n E_j$ into F. Furthermore we have $\|g\| \leq \liminf_{n\to\infty} \|f_n\| \leq \sup_{n\geq 1} \|f_n\| < \infty$.

10-1.13. **Exercise** Prove that for Banach spaces, separate continuity implies joint continuity. More precisely, let $E_1, E_2, \cdots, E_n$ and F be Banach spaces and f a multilinear maps from $\prod_{j=1}^n E_j$ into F. If for each $1 \leq k \leq n$, the map $f(x_1, \cdots, x_k, \cdots, x_n)$ is continuous in $x_k \in E_j$, then $f : \prod_{j=1}^n E_j \to F$ is continuous.

10-2 Polynomials on Banach Spaces

10-2.1. Let E, F be Banach spaces. A multilinear map from E^n into F is also called an *n-linear map* on E. An n-linear map $A : E^n \to F$ is said to be *symmetric* if we have $A(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)}) = A(x_1, x_2, \cdots, x_n)$ for all $x_j \in E$ and all permutations π of order n, or equivalently,

$$A(x_1, \cdots, x_j, \cdots, x_k, \cdots, x_n) = A(x_1, \cdots, x_k, \cdots, x_j, \cdots, x_n), \forall\ 1 \leq j, k \leq n.$$

The set of all symmetric continuous n-linear maps from E^n into F will be denoted by $L_s^n(E, F)$.

10-2.2. A finite sequence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ of positive integers $\alpha_j \geq 0$ is called a *multi-index*. In this case, m is called the *length* of α. The *order* of α is defined by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and the *factorial* by $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_m!$. For each integer $n \geq |\alpha|$, the *multinomial coefficient* is defined by $\binom{n}{\alpha} = \frac{n!}{(n-|\alpha|)!\alpha!}$. For $m = 1$, it is reduced to binomial coefficients. For every symmetric n-linear map $A : E^n \to F$ we shall adopt the following abbreviation:

$$Ax_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} = A(x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots\cdots, x_m, \cdots, x_m)$$

where x_1 is repeated α_1-times, x_2 repeated α_2-times and x_m repeated α_m-times.

10-2.3. **Multinomial Theorem** For every given symmetric n-linear map $A : E^n \to F$ we have $A(x_1 + x_2 + \cdots + x_m)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} Ax_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$.

Proof. It follows by induction on m:

$$\begin{aligned}
&A(x_1 + x_2 + \cdots + x_k + x_{k+1})^n \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} A(x_1 + x_2 + \cdots + x_k)^j x_{k+1}^{n-j} && \text{; Binomial Theorem} \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \sum_{|\alpha|=j} \frac{j!}{\alpha!} Ax_1^{\alpha_1} \cdots x_k^{\alpha_k} x_{k+1}^{n-j} && \text{; inductive assumption} \\
&= \sum_{j=0}^{n} \sum_{|\alpha|=j} \frac{n!}{\alpha!(n-j)!} Ax_1^{\alpha_1} \cdots x_k^{\alpha_k} x_{k+1}^{n-j} \\
&= \sum_{|\beta|=n} \binom{n}{\beta} Ax_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k} x_{k+1}^{\beta_{k+1}} \ ; \ \beta = (\alpha, n-j).
\end{aligned}$$

□

10-2.4. For any symmetric n-linear map $A : E^n \to F$, let $\hat{A} : E \to F$ be defined by $\hat{A}(x) = Ax^n$. The map $\hat{A}$ is called an *n-homogeneous polynomial* from E into F. The following lemma when $x_0 = 0$ generalizes the corresponding formula in inner product spaces of linear algebras. See e.g. §§13-1.4c,6.3.

10-2.5. **Polarization Formula** Let $A : E^n \to F$ be a symmetric n-linear map. Then for all $x_0, x_1, \cdots, x_m \in E$, we have

$$A(x_1, x_2, \cdots, x_n) = \frac{1}{n!2^n} \sum_{\lambda_1, \lambda_2, \cdots, \lambda_n = \pm 1} \lambda_1 \lambda_2 \cdots \lambda_n \hat{A}(x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n).$$

Consequently if $\hat{A} = 0$ then $A = 0$.

Proof. By Multinomial Theorem, we have

$$\begin{aligned}
&\sum_{\lambda_j = \pm 1} \lambda_1 \lambda_2 \cdots \lambda_n \hat{A}(x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \\
&= \sum_{\lambda_j = \pm 1} \lambda_1 \lambda_2 \cdots \lambda_n A(x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)^n \\
&= \sum_{\lambda_j = \pm 1} \lambda_1 \lambda_2 \cdots \lambda_n \sum_{|\alpha|=n} \binom{n}{\alpha} Ax_0^{\alpha_0} (\lambda_1 x_1)^{\alpha_1} (\lambda_2 x_2)^{\alpha_2} \cdots (\lambda_n x_n)^{\alpha_n} \\
&= \sum_{|\alpha|=n} \sum_{\lambda_j = \pm 1} \binom{n}{\alpha} \lambda_1^{\alpha_1+1} \lambda_2^{\alpha_2+1} \cdots \lambda_n^{\alpha_n+1} Ax_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}
\end{aligned}$$

$$= \sum_{|\alpha|=n} \binom{n}{\alpha} \prod_{j=1}^{n} \left[1 + (-1)^{\alpha_j+1}\right] A x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

where the multi-index is of the form $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n)$. If $\alpha_j = 0$ for some $j \geq 1$ then $1 + (-1)^{\alpha_j+1} = 0$ and the whole term disappears. So we may assume that all $\alpha_1, \cdots, \alpha_n > 0$. Now the condition $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = n$ implies $\alpha_0 = 0$ and $\alpha_j = 1$ for all $j \geq 1$. Therefore we obtain

$$\sum_{\lambda_j=\pm 1} \lambda_1 \lambda_2 \cdots \lambda_n \hat{A}(x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$$

$$= \frac{n!}{1!1! \cdots 1!} \prod_{j=1}^{n} \left[\, 1 + (-1)^{1+1}\right] A x_1 x_2 \cdots x_n = n! 2^n A x_1 x_2 \cdots x_n. \qquad \square$$

10-2.6. **Lemma** Let $A : E^n \to F$ be a symmetric n-linear map which need not be continuous. Let $\|\hat{A}\| = \sup\{\|\hat{A}(x)\| : \|x\| \leq 1\}$. Then we have

(a) $\|\hat{A}(x)\| \leq \|\hat{A}\| \; \|x\|^n$.

(b) $\|\hat{A}\| \leq \|A\| \leq \dfrac{n^n}{n!} \|\hat{A}\|$.

Proof. If $x = 0$ in E, we have $\|\hat{A}(x)\| = 0 = \|\hat{A}\| \; \|x\|^n$. For $x \neq 0$ we have

$$\|\hat{A}x\| = \left\| \, \|x\|^n \hat{A}\left(\tfrac{x}{\|x\|}\right) \right\| = \|x\|^n \left\| \hat{A}\left(\tfrac{x}{\|x\|}\right) \right\| \leq \|x\|^n \|\hat{A}\|.$$

This proves (a). For part (b), observe that

$$\|\hat{A}\| = \sup\{\|\hat{A}(x)\| : \|x\| \leq 1\} = \sup\{\|A(x, x, \cdots, x)\| : \|x\| \leq 1\}$$
$$\leq \sup\{\|A(x_1, x_2, \cdots, x_n)\| : \|x_j\| \leq 1, \forall\, j\} = \|A\|.$$

On the other hand, take any $\|x_j\| \leq 1$ in E. By Polarization Formula, we get

$$\|A(x_1, x_2, \cdots, x_n)\|$$
$$\leq \frac{1}{n!2^n} \sum_{\lambda_j=\pm 1} \|\lambda_1 \lambda_2 \cdots \lambda_n \hat{A}(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)\|$$
$$\leq \frac{1}{n!2^n} \sum_{\lambda_j=\pm 1} \|\hat{A}\| \; \|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\|^n$$
$$\leq \frac{1}{n!2^n} \sum_{\lambda_j=\pm 1} \|\hat{A}\|(\, |\lambda_1| \; \|x_1\| + |\lambda_2| \; \|x_2\| + \cdots + |\lambda_n| \; \|x_n\|)^n$$
$$\leq \frac{1}{n!2^n} \sum_{\lambda_j=\pm 1} \|\hat{A}\| n^n = \frac{1}{n!2^n} \|\hat{A}\| n^n . 2^n = \frac{n^n}{n!} \|\hat{A}\|. \qquad \square$$

10-2.7. **Exercise** Let $A(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n$ for all $x_j \in \mathbb{K}$. Show that $A : \mathbb{K}^n \to \mathbb{K}$ is a symmetric multilinear form and that $\|A\| = \|\hat{A}\| = 1$.

10-2.8. **Exercise** For all column vectors $x_j = (x_j^1, x_j^2, \cdots, x_j^n)^t$ in $\mathbb{R}_1^n$, let

$$A(x_1, x_2, \cdots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)}^1 x_{\pi(2)}^2 \cdots x_{\pi(n)}^n$$

where S_n is the permutation group on $\{1, 2, \cdots, n\}$. Show that $A : \mathbb{K}_1^n \to \mathbb{K}$ is a symmetric multilinear form on $\mathbb{R}_1^n$. Derive that $\|A\| = \frac{1}{n!}$ and $\|\hat{A}\| = \frac{1}{n^n}$. Therefore the inequalities of last lemma are sharp.

10-2.9. **Theorem** For every symmetric n-linear map $A : E^n \to F$, the following statements are equivalent.

(a) A is continuous on E^n.

(b) $\hat{A}$ is continuous on E.

(c) $\hat{A}$ is bounded on the unit ball of E.

Proof. It follows immediately from §10-1.5. □

10-2.10. **Theorem** Let $P : E \to F$ be a *polynomial* given by

$$P(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n, \forall\, x \in E$$

where the *coefficients* $A_j : E^j \to F$ are symmetric j-linear maps.

(a) $P = 0$ iff all $A_j = 0$.

(b) P is continuous iff all A_j are.

Proof. ($a \Rightarrow$) Suppose $P = 0$. Then for each $x \in E$ and for all $\lambda \in \mathbb{K}$, we have $P(\lambda x) = 0$, i.e. $A_0 + A_1(\lambda x) + A_2(\lambda x)^2 + \cdots + A_n(\lambda x)^n = 0$, or

$$A_0 + \lambda A_1 x + \lambda^2 A_2 x^2 + \cdots + \lambda^n A_n x^n = 0.$$

For every $u \in F'$ we get $uA_0 + \lambda u A_1 x + \lambda^2 u A_2 x^2 + \cdots + \lambda^n u A_n x^n = 0$ which is a polynomial with scalar coefficients $uA_j x^j$. Since λ is arbitrary, we have $uA_j x^j = 0$. Because $u \in F'$ is arbitrary, we obtain $A_j x^j = 0$. Hence $A_j = 0$ by Polarization Formula.

($b \Rightarrow$) We shall prove by induction on n. Observe that

$$2^n P(x) - P(2x) = \sum_{j=0}^{n-1} (2^n - 2^j) A_j x^j$$

is a polynomial of degree at most $n-1$. Since the left hand side is continuous, it follows by induction that for each $0 \le j \le n-1$, the map $(2^n - 2^j)A_j$ is continuous, i.e. A_j is continuous. Next,

$$A_n x^n = P(x) - (A_0 + A_1 x + A_2 x^2 + \cdots + +A_{n-1} x^{n-1})$$

is continuous in $x \in E$. Therefore $\hat{A}_n$ is continuous and so is A_n.

The converses ($a \Leftarrow$) and ($b \Leftarrow$) are trivial. □

10-2.11. **Exercise** Prove that for every multilinear map $A \in L^n(\mathbb{K}, F)$, there is a unique vector $a \in F$ such that $A(x_1, x_2, \cdots, x_n) = ax_1x_2\cdots x_n$ for all $x_j \in \mathbb{K}$.

10-2.12. **Exercise** Prove that every polynomial $P : \mathbb{K} \to F$ is of the form: $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ for all $x \in \mathbb{K}$ where $a_j \in F$.

10-2.13. **Exercise** Prove that all multilinear maps and polynomials on finite dimensional Banach spaces are continuous.

10-2.14. **Exercise** Let E, F be Banach spaces and $P : E \to F$ a polynomial. Prove that if $P(\lambda x) = \lambda^n P(x)$ for all $\lambda \in \mathbb{K}$ and all $x \in E$, then P is an n-homogeneous polynomial.

10-3 Higher Derivatives

10-3.1. Let E, F be Banach spaces and X an open subset of E. Let $f : X \to F$ be a given map. Define $D^0 f = f$. Note that for all $n \geq 1$, $L^n(E, F)$ is a Banach space. Assume that $D^{n-1}f : X \to L^{n-1}(E, F)$ has been defined by induction. Then f is said to be *n-times differentiable* on X if $D^{n-1}f$ is differentiable on X. In this case, let $D^n f = D(D^{n-1}f)$. With the identification §10-1.10, $D^n f$ is a map from X into the Banach space $L^n(E, F)$. The original map f is called a *C^n-map* if $D^n f : X \to L^n(E, F)$ is continuous and in this case, f is also said to be *n-times continuously differentiable* on X. The map f is called a *C^∞-map* if it is a C^n-map for all $n = 1, 2, 3, \cdots$ and in this case, f is also said to be *smooth*. Clearly $D^n f$ is linear in f. Also if f is n-times differentiable then f is a C^k-map for all $k \leq n - 1$. Before we dig in, we need the following special case of higher chain rule.

10-3.2. **Lemma** If $f : X \to F$ is a C^n-map then for every continuous linear map $g : F \to G$ the composite gf is also a C^n-map. Furthermore for each $a \in X$ we have $D^n(gf)(a) = gD^n f(a)$.

Proof. We have proved the case for $n = 1$. Inductively, assume that it is true for n in order to prove the case for $n + 1$. For each $A \in L^n(E, F)$ let $\varphi(A) : E^n \to G$ be defined by $\varphi(A)h_1 \cdots h_n = g(Ah_1 \cdots h_n)$. Clearly $\varphi(A)$ is an n-linear map. Since $\|\varphi(A)h_1 \cdots h_n\| \leq \|g\|\ \|A\|\ \|h_1\| \cdots \|h_n\|$, φ is continuous. Hence the map $\varphi : L^n(E, F) \to L^n(E, G)$ is well-defined and linear. Because

$$\|\varphi(A)\| = \sup_{\|h_j\| \leq 1} \|\varphi(A)h_1 \cdots h_n\| \leq \sup_{\|h_j\| \leq 1} \|g\|\ \|A\|\ \|h_1\| \cdots \|h_n\| \leq \|g\|\ \|A\|,$$

φ is a continuous linear map. Now by induction we have

$$D^n(gf)(a) = g \circ D^n f(a) = (\varphi \circ D^n f)(a)$$

i.e. $D^n(gf) = \varphi \circ D^n f$. Applying the first order chain rule again, $D^n(gf)$ is continuously differentiable because both $\varphi, D^n f$ are. Now

$$D^{n+1}(gf)(a) = D(\varphi \circ D^n f)(a) = (D\varphi)D(D^n f)(a) = \varphi D^{n+1} f(a).$$

Hence $$D^{n+1}(gf)(a)h_0 = \varphi[D^{n+1} f(a)h_0].$$

Thus $$D^{n+1}(gf)(a)h_0 h_1 \cdots h_n = \varphi[D^{n+1} f(a)h_0]h_1 \cdots h_n$$

$$= g[D^{n+1} f(a)h_0 h_1 \cdots h_n] = [g \circ D^{n+1} f(a)]h_0 h_1 \cdots h_n.$$

Therefore $$D^{n+1}(gf)(a) = g \circ D^{n+1} f(a).$$ □

10-3.3. **Lemma** If f is a C^2-map then for all $h, k \in E$ and all $s, t \in \mathbb{K}$ we have $D^2 f(a)hk = \lim_{s,t\to 0} \frac{1}{st}[f(a+sh+tk) - f(a+sh) - f(a+tk) + f(a)]$.

Proof. Observe that

$$\begin{aligned}
&f(a+sh+tk) - f(a+sh) - f(a+tk) + f(a)\\
&= \int_0^1 Df(a+sh+\beta tk)tk\,d\beta - \int_0^1 Df(a+\beta tk)tk\,d\beta\\
&= \int_0^1 [Df(a+sh+\beta tk) - Df(a+\beta tk)]tk\,d\beta\\
&= \int_0^1 \left[\int_0^1 D^2 f(a+\alpha sh+\beta tk)sh\,d\alpha\right] tk\,d\beta\\
&= st\int_0^1 \int_0^1 D^2 f(a+\alpha sh+\beta tk)hk\,d\alpha\, d\beta
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{s,t\to 0} \frac{1}{st}[f(a+sh+tk) - f(a+sh) - f(a+tk) + f(a)]\\
&= \lim_{s,t\to 0} \int_0^1 \int_0^1 D^2 f(a+\alpha sh+\beta tk)hk\,d\alpha\, d\beta\\
&= \int_0^1 \int_0^1 \lim_{s,t\to 0} D^2 f(a+\alpha sh+\beta tk)hk\,d\alpha\, d\beta\\
&= \int_0^1 \int_0^1 D^2 f(a)hk\,d\alpha\, d\beta = D^2 f(a)hk.
\end{aligned}$$ □

10-3.4. **Symmetry Theorem** If f is a C^n-map then the n-linear map $D^n f(a)$ is symmetric.

Proof. Since the right hand side of last lemma is symmetric in h, k, we obtain $D^2 f(a)hk = D^2 f(a)kh$. This proves the case for $n = 2$. Next, for $n \geq 3$ to prove

$$D^n f(a)h_1 h_2 h_3 \cdots h_n = D^n f(a)h_1 h_3 h_2 \cdots h_n, \qquad \#1$$

let $H : L^{n-1}(E, F) \to F$ be defined by $H(A) = Ah_2 h_3 \cdots h_n$ and $g : X \to F$ by $g(x) = D^{n-1} f(x)h_2 h_3 \cdots h_n$. Then H is a continuous linear map and hence continuously differentiable. Since $D^{n-1} f : X \to L^{n-1}(E, F)$ is continuously differentiable, so is the composite $g = H \circ D^{n-1} f$. Therefore we have $Dg(a) = HD(D^{n-1} f)(a)$, i.e. $Dg(a)h_1 = HD^n f(a)h_1 = D^n f(a)h_1 h_2 h_3 \cdots h_n$. By induction, we have $g(x) = D^{n-1} f(x)h_3 h_2 \cdots h_n$. Repeating the above argument, we get $Dg(a)h_1 = [D^n f(a)]h_1 h_3 h_2 \cdots h_n$. Therefore #1 is proved. In a similar way we obtain

$$D^n f(a)h_1 \cdots h_j \cdots h_k \cdots h_n = D^n f(a)h_1 \cdots h_k \cdots h_j \cdots h_n$$

as long as $j, k \geq 2$. Next to show

$$D^n f(a)h_1 h_2 h_3 \cdots h_n = D^n f(a)h_2 h_1 h_3 \cdots h_n, \qquad \#2$$

let $g(x) = D^{n-2} f(x)h_3 \cdots h_n = HD^{n-2} f(x)$ where $H : L^{n-2}(E, F) \to F$ is a continuous linear map defined by $H(A) = Ah_3 \cdots h_n$. Since $g : X \to F$ is a C^2-map, we have $D^2 g(a)h_1 h_2 = D^2 g(a)h_2 h_1$. On the other hand,

$$Dg(x) = HD[D^{n-2} f(x)] = HD^{n-1} f(x).$$

Since $D^2 g(x) = HD^n f(x)$, we obtain $HD^n f(a)h_1 h_2 = HD^n f(a)h_2 h_1$ which is #2. Finally according to algebra, every permutation can be decomposed into a product $\pi_1 \pi_2 \cdots \pi_p$ where each π_j is either the transpose $(1, 2)$ or a transpose (j, k) with $j, k \geq 2$. This completes the proof. □

10-3.5. **Exercise** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$.

10-3.6. **Lemma** Every continuous n-linear map $f : E_1 \times E_2 \times \cdots \times E_n \to F$ is smooth. Furthermore at $(a_1, a_2, \cdots, a_n)$, the derivative is given by

$$\begin{aligned} &Df(a_1, a_2, \cdots, a_n)(h_1, h_2, \cdots, h_n) \\ &= f(h_1, a_2, \cdots, a_n) + f(a_1, h_2, \cdots, a_n) + \cdots + f(a_1, a_2, \cdots, h_n). \end{aligned}$$

Proof. The linear case ($n = 1$) has been done. For $n \geq 2$, consider $\partial_1 f(a_1, a_2, \cdots, a_n)h_1 = f(h_1, a_2, \cdots, a_n)$. Define $g(a_2, \cdots, a_n) : E_1 \to F$ by

$g(a_2, \cdots, a_n)h_1 = f(h_1, a_2, \cdots, a_n)$. Then $g(a_2, \cdots, a_n)$ is continuous linear. Hence $g : E_2 \times \cdots \times E_n \to F$ is well-defined. Because

$$\|g(a_2, \cdots, a_n)\| \leq \|f\| \; \|a_2\| \cdots \|a_n\|,$$

g is continuous $(n-1)$-linear. Next define $\pi : E_1 \times E_2 \times \cdots \times E_n \to E_2 \times \cdots \times E_n$ by $\pi(x_1, x_2, \cdots, x_n) = (x_2, \cdots, x_n)$. Then π is continuous linear. Since

$$\partial_1 f(a_1, a_2, \cdots, a_n) = (g \circ \pi)(a_1, a_2, \cdots, a_n),$$

$\partial_1 f(a_1, a_2, \cdots, a_n)$ is continuous jointly in $(a_1, a_2, \cdots, a_n)$. By characterization in terms of partial derivatives, f is continuously differentiable. Furthermore we have

$$Df(a_1, a_2, \cdots, a_n)(h_1, h_2, \cdots, h_n) = \sum_{j=1}^{n} \partial_j f(a_1, a_2, \cdots, a_n)h_j$$
$$= f(h_1, a_2, \cdots, a_n) + f(a_1, h_2, \cdots, a_n) + \cdots + f(a_1, a_2, \cdots, h_n). \qquad \square$$

10-3.7. **Exercise** The $(n+1)$-times derivative of a continuous n-linear map is zero.

10-3.8. **Exercise** All continuous polynomials are smooth. Furthermore if $f(x) = Ax^n$ where $A \in L_s^n(E, F)$, then we have $Df(a)h = nAa^{n-1}h$.

10-3.9. **Exercise** Let $C[0,1]$ be the Banach space of continuous real functions on $[0,1]$ under the sup-norm and $f : \mathbb{R} \to \mathbb{R}$ a smooth function. Prove that the function $g : C[0,1] \to \mathbb{R}$ given by $g(x) = \int_0^1 f[x(t)]dt$ is a smooth function and find formulas to express the first and second derivatives of g.

10-3.10. **Exercise** For every $x \in C[0,2]$, let $f(x) = \sin[x(1)]$. Show that f is a C^∞-function on $C[0,2]$.

10-4 C^n-Maps

10-4.1. In this section, we shall start with characterization of the C^n-maps in terms of coordinates of domain and target spaces. Then show that taking inverse of an operator is a smooth map and apply it to get the C^n-version of inverse and implicit mapping theorems. Finally, Newton's method of solving nonlinear equations will be given.

10-4.2. **Lemma** Let $F_1, F_2, \cdots, F_m$ be Banach spaces. Let P_j be the projection from $\prod_{k=1}^m F_k$ onto F_j and Q_j the natural injection from F_j into $\prod_{k=1}^m F_k$ given by $Q_j(y_j) = (0, \cdots, 0, y_j, 0, \cdots, 0)^t$ which is a column vector. Then we have $P_jQ_j = I_{F_j}$ and $\sum_{j=1}^m Q_jP_j = I_F$ where I_{F_j} and I_F denote the identity maps on F_j, F respectively.

10-4.3. **Theorem** Let $f : X \to \prod_{j=1}^m F_j$ be a given map. Then f is a C^n-map iff all coordinate maps $f_j = P_j f$ are C^n-maps. In this case, we have the following formulas: $D^n f_j(a) = P_j D^n f(a)$ and

$$D^n f(a) = \sum_{j=1}^m Q_j D^n f_j(a) = \begin{bmatrix} D^n f_1(a) \\ \cdots \\ D^n f_m(a) \end{bmatrix}.$$

Proof. Suppose that all f_j are C^n-maps. Then each $Q_j f_j$ is also a C^n-map. Hence $f = \left(\sum_{j=1}^m Q_j P_j\right) f = \sum_{j=1}^m Q_j f_j$ is a C^n-map. By linearity and §10-3.2, we obtain

$$D^n f(a) = \sum_{j=1}^m D^n(Q_j f_j)(a) = \sum_{j=1}^m Q_j D^n f_j(a).$$

The converse follows immediately as composite maps. □

10-4.4. **Higher Chain Rule** Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. If $f : X \to Y$ and $g : Y \to G$ are C^n-maps then so is the composite gf.

Proof. It follows from $n = 1$ that

$$D(gf)(x) = Dg[f(x)]Df(x) = [(Dg) \circ f](x)Df(x).$$

By induction, $x \to f(x) \to Dg[f(x)]$ is a C^{n-1}-map. Since $x \to Df(x)$ is a C^{n-1}-map, so is the map $x \to (Dg[f(x)], Df(x)) : X \to L(F,G) \times L(E,F)$ by last theorem. Since the composition $L(F,G) \times L(E,F) \to L(E,G)$ is continuous bilinear, it is a C^{n-1}-map. By induction, the composite $x \to D(gf)(x)$ is a C^{n-1}-map. Consequently gf is a C^n-map. □

10-4.5. **Lemma** Let $a = (a_1, \cdots, a_k)$ be a point in $\prod_{j=1}^k E_j$. For every x_j let $g_j(x_j) = (a_1, \cdots, a_{j-1}, x_j, a_{j+1}, \cdots, a_k)$. Then each g_j is a differentiable map. Furthermore, $Dg_j(x_j)$ is the natural injection and $D^2 g_j = 0$.

Proof. Let Q_j be the natural injection from E_j into $\prod_{p=1}^k E_p$. Then we have

$$g_j(x_j) = (a_1, \cdots, a_{j-1}, 0, a_{j+1}, \cdots, a_k) + Q_j(x_j).$$

Hence $Dg_j = DQ_j = Q_j$. Since this is a constant map, we get $D^2 g_j = 0$. □

10-4.6. **Theorem** Let X_j be an open subset of a Banach space E_j and f a given map on the product set $X = \prod_{j=1}^k X_j$ into a Banach space F. Then f is a C^{n+1}-map iff each partial derivative $\partial_j f : X \to L(E_j, F)$ exists and is a C^n-map.

Proof. Assume that f is a C^{n+1}-map. For $n = 0$, f is continuously differentiable. Hence $\partial_j f(a) = D(fQ_j)(a) = Df(a)Q_j$ is continuous in $a \in X$.

In general, since $Df(a)$ is a C^n-map, so is $\partial_j f(a) = Df(a)Q_j$ by Higher Chain Rule. Conversely, suppose all $\partial_j f : X \to L(E_j, F)$ are C^n-maps. We have proved that $Df(a) = \sum_{j=1}^k \partial_j f(a)P_j$ where P_j is the projection of $\prod_{i=1}^k E_i$ onto the j-th coordinate E_j. Since each $\partial_j f(a)$ is a C^n-map in $a \in X$, f is a C^{n+1}-map. □

10-4.7. **Lemma** Let $T(E, F)$ denote the set of all topological isomorphisms from a Banach space E onto a Banach space F. Then the map $f(A) = A^{-1}$ from $T(E,F)$ onto $T(F,E)$ is smooth. Furthermore for all $A \in T(E,F)$ and $H \in L(E,F)$, we have $Df(A)H = -A^{-1}HA^{-1}$. In general, the following holds for every $n \geq 1$, $D^n f(A)H^n = (-1)^n n! A^{-1}(HA^{-1})^n = (-1)^n n!(A^{-1}H)^n A^{-1}$.

Proof. Clearly $-A^{-1}HA^{-1}$ is a continuous linear map in H. To show that f is differentiable at $A \in T(E,F)$, by its continuity for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $\|H\| \leq \delta$ in $L(E,F)$ we have $\|A^{-1} - (A+H)^{-1}\| \leq \varepsilon / \|A^{-1}\|$. Since

$$\|f(A+H) - f(A) - (-A^{-1}HA^{-1})\| = \|[A^{-1} - (A+H)^{-1}]HA^{-1}\|$$
$$= \|[A^{-1} - (A+H)^{-1}]\| \; \|H\| \; \|A^{-1}\| \leq \varepsilon \|H\|,$$

f is differentiable and $Df(A)H = -A^{-1}HA^{-1}$. Next, for all $B, C \in L(F,E)$ and all $H \in L(E,F)$ define $g(B,C)H = -BHC$. Then the map

$$g(B,C) : L(E,F) \to L(F,E)$$

is continuous linear. Thus $g : L(F,E) \times L(F,E) \to L[L(E,F), L(F,E)]$ is a continuous bilinear map and thus g is a smooth map. It is easy to verify $Df(A)H = g(A^{-1}, A^{-1})H$. Hence $Df(A) = g(A^{-1}, A^{-1})$ is continuous in A. Therefore f is a C^1-map. Now assume $n > 1$. Since each step of

$$A \to (A^{-1}, A^{-1}) \to g(A^{-1}, A^{-1}) = Df(A)$$

is a C^{n-1}-map, Df is a C^{n-1}-map, i.e. f is a C^n-map. Therefore f is smooth. The general formula can be proved by induction. It is left as an exercise. □

10-4.8. Let X, Y be open subsets of Banach spaces E, F respectively. Then a bijection $f : X \to Y$ is called a *C^n-diffeomorphism* if both f and f^{-1} are C^n-maps. If f is a C^n-diffeomorphism for all n, then f is called a *C^∞-diffeomorphism* or simply a *diffeomorphism*. Local diffeomorphisms are defined in an obvious way.

10-4.9. **Lemma** Let X, Y be open subsets of E, F respectively and $f : X \to Y$ a bijection. If f is a C^n-map and if $Df(x)$ is invertible for each $x \in X$ then f is a C^n-diffeomorphism.

Proof. It follows from Inverse Mapping Theorem that $D(f^{-1})(y) = [Df(x)]^{-1}$ where $y = f(x)$. Assume inductively that f^{-1} is a C^{n-1}-map. By Higher Chain Rule, $D(f^{-1})(y) = [(Df) \circ f^{-1}(y)]^{-1}$ is a C^{n-1}-map. Hence f^{-1} is a C^n-map. Therefore f is a C^n-diffeomorphism. □

10-4.10. **C^n-Inverse Mapping Theorem** Let f be a C^n-map from an open subset X of E into F. If $Df(a)$ is a topological isomorphism from E into F then f is a local C^n-diffeomorphism at $a \in X$.

10-4.11. **C^n-Implicit Mapping Theorem** Let E, F be Banach spaces; M an open subset of $E \times F$; $f : M \to F$ a C^n-map; $(a,b) \in M$ and $c = f(a,b)$. If the partial derivative $\partial_y f(a,b) : F \to F$ is invertible, then the equation $f(x,y) = c$ has a unique local implicit C^n-solution $y = g(x)$ near (a,b).

Proof. For $n = 1$, it was done and the given formula holds. Now suppose f is a C^n-map for $n > 1$. Then both $\partial_x f, \partial_y f$ are C^{n-1}-maps and therefore $Dg = -(\partial_y f)^{-1}(\partial_x f)$ is a C^{n-1}-map. Consequently, g is a C^n-maps. □

10-4.12. **Newton's Method** Let X be an open subset of a Banach space E, $f : X \to E$ a C^2-map and $f(a) = 0$ where $a \in X$. If $Df(a)$ is a topological isomorphism then there is $\delta > 0$ such that for each $x_0 \in \mathbb{B}(a,\delta)$ the sequence $\{x_n\}$ defined by $x_{n+1} = x_n - [Df(x_n)]^{-1} f(x_n)$ converges to a.

Proof. Because $Df(a)$ is a topological isomorphism, $g(x) = x - [Df(x)]^{-1} f(x)$ is defined for all x near a. For each $h \in E$, observe

$$
\begin{aligned}
&Dg(x)h = h - D\{[Df(x)]^{-1} f(x)\}h \\
&= h - < D\{[Df(x)]^{-1}\}h, f(x) > - < [Df(x)]^{-1}, Df(x)h > \\
&= h - < -[Df(x)]^{-1} D[Df(x)]h[Df(x)]^{-1}, f(x) > -h \\
&= [Df(x)]^{-1} D^2 f(x)h\{[Df(x)]^{-1} f(x)\}.
\end{aligned}
$$

Since $Dg(a) = 0$, there is $\lambda > 0$ as in Contraction Lemma such that for all $0 < \delta < \lambda$, the map g carries $\overline{\mathbb{B}}(a,\delta)$ into itself. Replacing λ by smaller one, there is $L > 0$ such that for all $x \in \overline{\mathbb{B}}(a,\lambda)$ we have $\|f(x) - f(a)\| \le L\|x - a\|$ and $\|[Df(x)]^{-1}\|^2 \|D^2 f(x)\| \le L$. Now choose $0 < \delta < \lambda$ such that $L^2\delta < 1$. For all $x, y \in \overline{\mathbb{B}}(a,\delta)$, write $h = x - y$. Then the following calculation

$$
\begin{aligned}
&\|g(x) - g(y)\| = \left\| \int_0^1 Dg(z)h\,dt \right\| \qquad ; z = y + th \\
&\le \int_0^1 \|[Df(z)]^{-1}\|^2 \|D^2 f(z)\| \; \|f(x)\| \; \|h\| dt
\end{aligned}
$$

$$\leq \int_0^1 L\|f(x) - f(a)\| \, \|x - y\|dt \qquad ; f(a) = 0$$

$$\leq \int_0^1 L^2\|x - a\| \, \|x - y\|dt = L^2\delta\|x - y\|$$

shows that g is a contraction on the complete metric space $\overline{\mathbb{B}}(a, \delta)$. For each $x_0 \in \mathbb{B}(a, \delta)$, the sequence defined by $x_{n+1} = g(x_n)$ converges to the unique fixed point of g which is a by direct verification. □

10-5 Taylor's Expansion

10-5.1. Taylor's formula will be studied in this section. We also give numerical examples to interpret high derivatives in terms of matrices. Convenient tests to classify stationary points will be given. We hope that educationalists will support by making these examples as an integral part of undergraduate multivariate calculus.

10-5.2. **Taylor's Formula** Let X be an open *convex* subset of E and let $f : X \to F$ be a C^{n+1}-map. Then for all $a, x \in X$ we have $f(x) = T_n(x) + R_n(x)$ where the Taylor polynomial and the remainder are given by

$$T_n(x) = f(a) + \frac{1}{1!}Df(a)(x - a) + \frac{1}{2!}D^2 f(a)(x - a)^2 + \cdots + \frac{1}{n!}D^n f(a)(x - a)^n$$

and $$R_n(x) = \frac{1}{n!}\int_0^1 (1 - t)^n D^{n+1} f[(1 - t)a + tx](x - a)^{n+1} dt.$$

Proof. To simplify the notation, let $h = x - a$. For $n = 0$ it is reduced to Mean-Value Theorem. Integration by parts gives

$$R_{n-1}(x) = \frac{1}{(n-1)!}\int_0^1 (1 - t)^{n-1} D^n f(a + th)h^n dt$$

$$= \frac{1}{(n-1)!}\int_0^1 \left\{\frac{d}{dt}\left[-\frac{(1-t)^n}{n}\right]\right\} D^n f(a + th)h^n dt$$

$$= -\frac{(1-t)^n}{n!}D^n f(a + th)h^n\Bigg|_{t=0}^{1} + \frac{1}{n!}\int_0^1 (1 - t)^n \left\{\frac{d}{dt}[D^n f(a + th)h^n]\right\} dt$$

$$= \frac{1}{n!}D^n f(a)h^n + \frac{1}{n!}\int_0^1 (1 - t)^n D^{n+1} f(a + th)h^{n+1} dt$$

$$= \frac{1}{n!}D^n f(a)(x - a)^n + R_n(x).$$

Finally the proof is completed by induction as follow:

$$\begin{aligned} f(x) &= T_{n-1}(x) + R_{n-1}(x) \\ &= T_{n-1}(x) + \frac{1}{n!}D^n f(a)(x-a)^n + R_n(x) = T_n(x) + R_n(x). \end{aligned}$$ □

10-5.3. **Corollary** Let X be an open subset of E and $f : X \to F$ a C^n-map. Then for every $a \in X$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $\mathbb{B}(a,\delta) \subset X$ and for every $x \in \mathbb{B}(a,\delta)$, we have $\|f(x) - T_n(x)\| \le \varepsilon \|x-a\|^n$.

Proof. Since $D^n f$ is continuous at $a \in X$, for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mathbb{B}(a,\delta) \subset X$ and for all $x \in \mathbb{B}(a,\delta)$ we have $\|D^n f(x) - D^n f(a)\| \le n!\varepsilon$. Now take any $x \in \mathbb{B}(a,\delta)$ and let $h = x - a$. Then for all $t \in [0,1]$ we have $a + th \in \mathbb{B}(a,\delta)$. Therefore the proof is completed by the following estimation:

$$\begin{aligned} \|f(x) - T_n(x)\| &= \|T_{n-1}(x) + R_{n-1}(x) - T_n(x)\| \\ &= \left\| \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} D^n f(a+th)h^n dt - \frac{1}{n!}D^n f(a)h^n \right\| \\ &= \left\| \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} D^n f(a+th)h^n dt \right. \\ &\qquad \left. - \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} D^n f(a)h^n dt \right\| \\ &= \left\| \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} [D^n f(a+th) - D^n f(a)]h^n dt \right\| \\ &\le \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} \|D^n f(a+th) - D^n f(a)\| \|h\|^n dt \\ &\le \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} n!\varepsilon \|h\|^n dt \le \varepsilon \|h\|^n = \varepsilon \|x-a\|^n. \end{aligned}$$ □

10-5.4. **Corollary** Let X be an open convex subset of E and $f : X \to F$ a C^∞-map. Then for every $a \in X$ and every integer n, there is a C^∞-map $g : X \to L_s^n(E,F)$ such that for all $x \in X$, $f(x) = T_{n-1}(x) + g(x)(x-a)^n$ and $g(a) = \frac{1}{n!}D^n f(a)$.

Proof. From the remainder of Taylor's formula, the expression

$$g(x) = \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1} D^n f[(1-t)a + tx]dt$$

defines a map $g : X \to L_s^n(E,F)$. Then g is a C^∞-map by differentiation under the integral sign. The equalities follow from trivial calculation. □

10-5.5. Following §9-3.7 and also §9-1.2, identify the higher derivatives as matrices. We only deal with the case of two variables but the same matrix forms work for many variables. Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. Then for every higher differentiable map $f : X \times Y \to G$ and $(a, b) \in X \times Y$, we have in matrix form:

$$\begin{aligned} Df(a,b) &= [\partial_x f(a,b), \partial_y f(a,b)], \\ D^2 f(a,b) &= [\partial_x Df(a,b), \partial_y Df(a,b)] = [\partial_x^2 f, \partial_x\partial_y f, \partial_y\partial_x f, \partial_y^2 f], \\ D^3 f(a,b) &= [\partial_x D^2 f(a,b), \partial_y D^2 f(a,b)] \\ &= [\partial_x^3 f, \partial_x^2\partial_y f, \partial_x\partial_y\partial_x f, \partial_x\partial_y^2 f, \partial_y\partial_x^2 f, \partial_y\partial_x\partial_y f, \partial_y^2\partial_x f, \partial_y^3 f]. \end{aligned}$$

As a result of Symmetry Theorem, we have $D^{n+1} f = [D^n \partial_x f, D^n \partial_y f]$.

10-5.6. **Example** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x + x^2 + \sin(2x+y) \\ e^y + \cos(x-2y) \end{bmatrix}$.

Then the first derivative at the origin is given by

$$Df\begin{pmatrix} 0 \\ 0 \end{pmatrix} = [\partial_x f, \partial_y f] = \begin{bmatrix} 1 + 2x + 2\cos(2x+y) & \cos(2x+y) \\ -\sin(x-2y) & e^y + 2\sin(x-2y) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}.$$

The second derivative at the origin is

$$\begin{aligned} D^2 f\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= [\partial_x Df, \partial_y Df] \\ &= \begin{bmatrix} 2 - 4\sin(2x+y) & -2\sin(2x+y) & -2\sin(2x+y) & -\sin(2x+y) \\ -\cos(x-2y) & 2\cos(x-2y) & 2\cos(x-2y) & e^y - 4\cos(x-2y) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 2 & -3 \end{bmatrix}. \end{aligned}$$

The third derivative at the origin is

$$D^3 f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = [\partial_x D^2 f, \partial_y D^2 f] = \begin{bmatrix} -8 & -4 & -4 & -2 & -4 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The third Taylor's polynomial of f at the origin is given by

$$T_3\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} 0 \\ 0 \end{pmatrix} + Df\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2!}D^2 f\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}^2 + \frac{1}{3!}D^3 f\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}^3.$$

Now the approximate value of $f\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is supposed to be $T_3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ which is evaluated as follow

$$Df\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$D^2 f\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix},$$

$$D^3f\begin{pmatrix}0\\0\end{pmatrix}\begin{bmatrix}1\\-1\end{bmatrix}^3=\begin{bmatrix}-8&-4&-4&-2&-4&-2&-2&-1\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$$
$$=\begin{bmatrix}-4&-2&-2&-1\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-2&-1\\0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}.$$

Therefore,

$$T_3\begin{pmatrix}1\\-1\end{pmatrix}=\begin{bmatrix}0\\2\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}2\\-8\end{bmatrix}+\tfrac{1}{6}\begin{bmatrix}-1\\-1\end{bmatrix}=\tfrac{1}{6}\begin{bmatrix}17\\-19\end{bmatrix}.$$

10-5.7. **Exercise** Let E,F be Banach spaces, X an open subset of E and $f:X\to F$ a given map. Then f is said to be *differentiable* at $a\in X$ in the *direction* of $e\in E$ if $\lim_{t\to 0}\dfrac{f(a+te)-f(a)}{t}$ exists. In this case, the limit is called the *directional derivative* of f at a in direction of e.

(a) Prove that if f is differentiable at $a\in X$, then the directional derivative of f exists and it is given by $Df(a)e$.

(b) Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y}, & \text{if } x^2+y\neq 0,\\ 0, & \text{if } x^2+y=0.\end{cases}$$

Show that the directional derivative of f at $(0,0)$ in every direction exists but f is not differentiable at $(0,0)$.

10-5.8. Let $A:E^2\to\mathbb{K}$ be a continuous bilinear form. For each $h\in E$, define $A_h(k)=A(h,k),\forall\ k\in E$. Prove that $\varphi(h)=A_h$ defines a continuous linear map from E into the dual space E'. Estimate the norm of φ. We normally identify A as the element φ of $L(E,E')$ which consists of square matrices when $E=\mathbb{K}^n$. Note that this is a special case of §10-1.10.

10-5.9. Let X be an open subset of $\mathbb{R}^n$ and $f:X\to\mathbb{R}$ a twice continuously differentiable function. It is easy to verify that for all $a\in X$ and $x\in\mathbb{R}^n$, $D^2f(a)(x-a)^2$ is identical to

$$(x_1-a_1,\cdots,x_n-a_n)\begin{bmatrix}\partial_1^2 f & \partial_2\partial_1 f & & \partial_n\partial_1 f\\ \partial_1\partial_2 f & \partial_2^2 f & & \partial_n\partial_2 f\\ \cdots & \cdots & & \cdots\\ \partial_1\partial_n f & \partial_2\partial_n f & \cdots & \partial_n^2 f\end{bmatrix}\begin{bmatrix}x_1-a_1\\ x_2-a_2\\ \cdots\\ x_n-a_n\end{bmatrix}.$$

Therefore the second order Taylor's polynomial is of the form

$$T_2(x)=f(a)+Df(a)(x-a)+\tfrac{1}{2}(x-a)^tD^2f(a)(x-a)$$

where $D^2f(a)=[\partial_i\partial_j f(a)]$ is a real symmetric matrix. Suppose that a is a stationary point, i.e. $Df(a)=0$. In linear algebra, there is an orthogonal matrix M such that $MD^2f(a)M^t = diagonal(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is diagonal. Letting $u = (u_1, u_2, \cdots, u_n)^t = M(x - a)$, we have

$$T_2(x) = f(a) + \tfrac{1}{2}(\lambda_1 u_1^2 + \lambda_2 u_2^2 + \cdots + \lambda_n u_n^2).$$

Now we arrive the following conclusion:

a) If all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$, f has a local minimum.

b) If all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n < 0$, f has a local maximum.

c) If all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n \neq 0$ but some are positive while some are negative, f has a saddle point a.

d) If at least one eigenvalue $\lambda_j = 0$, this method fails to offer any information.

10-5.10. Note that the eigenvalues λ_j have nothing to do with the matrix M. They are roots of the characteristic polynomial $det[\lambda I - D^2f(a)] = 0$. In fact, we are not interest to actually find the eigenvalues. All we want to know is whether they are all positive, or all negative, or mixed, or some zeros. Combined with theory of equations, we have the following.

10-5.11. **Theorem** Let X be an open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ a twice continuously differentiable function. Let a be a stationary point of f and let $p(x) = \det[\lambda I - D^2f(a)]$ denote the characteristic polynomial of the second derivative. If the constant term $p(0) = 0$, then this test offers no information. Therefore we have to make sure that $p(0) \neq 0$ to start with.

a) If all coefficients are non-zero and positive, f has a local maximum at a.

b) If the coefficients are non-zero and have alternative signs, f has a local minimum at a.

c) Otherwise f has a saddle point a.

10-5.12. **Exercise** For each of the following expressions, find its stationary points and classify them as local maxima, local minima or saddle points.

a) $x^3 + y^3 + z^3 + 3xy + 3yz + 3zx$

b) $x^3 + y^2 + z^2 + 12xy + 2z$.

10-5.13. **Exercise** Let f be a continuous real function on the closed unit ball of $\mathbb{R}^n$. Suppose that f is differentiable on the open unit ball $\mathbb{B}$. Prove that if f is constant on the unit sphere, then $Df(a) = 0$ for some point $a \in \mathbb{B}$.

10-6 Higher Chain Formula and Higher Product Formula

10-6.1. **Lemma** Let E, F be Banach spaces and X an open subset of E. Let $f : X \to F$ be a C^{j+1}-map and for every $1 \leq k \leq j$, let $v_k : X \to E$ be a

C^1-map. Then $g : X \to F$ given by $g(x) = D^j f(x)v_1(x)\cdots v_j(x)$ is a C^1-map. Furthermore for every $h \in E$ we have

$$Dg(x)h = D^{j+1}f(x)v_1(x)\cdots v_j(x)h + \sum_{k=1}^{j} D^j f(x)v_1(x)\cdots[Dv_k(x)h]\cdots v_j(x).$$

Proof. Define a continuous multilinear map $\pi : X \to L^j(E,F) \times E^j$ by

$$\pi(x) = (D^j f(x), v_1(x), \cdots, v_j(x))^t \qquad \text{; transpose}$$

and a C^1-map $\varphi : L^j(E,F) \times E^j \to F$ by $\varphi(y_0, y_1, \cdots, y_j) = y_0 y_1 \cdots y_j$ respectively. Then the composite map $f = \varphi\pi$ is a C^1-map. It follows from Chain Rule that

$$[Dg(x)]h = D\varphi[\pi(x)]D\pi(x)h = [\partial_0\varphi, \partial_1\varphi, \cdots, \partial_j\varphi]\begin{bmatrix} D[D^j f(x)]h \\ Dv_1(x)h \\ \cdots \\ Dv_j(x)h \end{bmatrix}$$

$$= \varphi([D^{j+1}f(x)h], v_1(x), \cdots, v_j(x)) + \sum_{k=1}^{j}\varphi(D^j f(x), v_1(x), \cdots, [Dv_k(x)h], \cdots, v_j(x))$$

$$= D^{j+1}f(x)v_1(x)\cdots v_j(x)h + \sum_{k=1}^{j} D^j f(x)v_1(x)\cdots[Dv_k(x)h]\cdots v_j(x). \qquad \square$$

10-6.2. **Higher Chain Formula** Let E, F, G be Banach spaces and X, Y open subsets of E, F respectively. If $f : X \to Y$ and $g : Y \to G$ are C^n-maps then for all $(x,h) \in X \times E$ we have the follow formula for the composite map:

$$D^n(gf)(x)h^n = \sum_{j=1}^{n}\frac{1}{j!}\sum_{\alpha\in A(j,n)}\binom{n}{\alpha}D^j g[f(x)]\,[D^{\alpha_1}f(x)h^{\alpha_1}]\cdots\cdots[D^{\alpha_j}f(x)h^{\alpha_j}]$$

where $A(j,n)$ denotes the set of all multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_j)$ satisfying $|\alpha| = n$ and all $\alpha_k \geq 1$.

Proof. Let $T_j(\alpha) = D^j g[f(x)][D^{\alpha_1}f(x)h^{\alpha_1}]\cdots\cdots[D^{\alpha_j}f(x)h^{\alpha_j}]$ in order to simplify the notation. To prove the case for $n+1$ by induction, it follows from last lemma that

$$D^{n+1}gf(x)h^{n+1} = D[D^n(gf)(x)h^n]h = D\left[\sum_{j=1}^{n}\frac{1}{j!}\sum_{\alpha\in A(j,n)}\binom{n}{\alpha}T_j(\alpha)\right]h$$

$$= \sum_{j=1}^{n}\sum_{\alpha\in A(j,n)}\frac{n!}{j!\alpha!}\left[T_{j+1}(\alpha,1) + \sum_{k=1}^{j}T_j(\alpha+e_k)\right].$$

where $e_k = (\delta_{1k}, \delta_{2k}, \cdots, \delta_{jk})$ with $\delta_{ik} = 0$ for $i \neq k$ and $\delta_{ik} = 1$ for $i = k$. Now for $j = 1$, it must be $k = 1$. Hence $A(1,n)$ has only one multi-index $\alpha = (n)$ and therefore we have $T_1(n+1) = Dg[f(x)]D^{n+1}f(x)h^{n+1}$. On the other hand, for $j = n$, $A(n,n)$ has only one multi-index $u_n = (1,1,\cdots,1)$ and therefore we obtain $T_{n+1}(u_{n+1}) = D^{n+1}g[f(x)][Df(x)h]^{n+1}$. Consequently, we have

$$
\begin{aligned}
&D^{n+1}gf(x)h^{n+1}\\
&= T_1(n+1) + \sum_{j=1}^{n-1} \sum_{\alpha\in A(j,n)} \frac{n!}{j!\alpha!}\, T_{j+1}(\alpha,1)\\
&\qquad + \sum_{j=2}^{n} \sum_{\alpha\in A(j,n)} \frac{n!}{j!\alpha!} \sum_{k=1}^{j} T_j(\alpha+e_k) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{i=2}^{n} \sum_{\beta\in A(i-1,n)} \frac{n!}{(i-1)!\beta!} T_i(\beta,1) \ ;\ i = j+1 \text{ and } \alpha = \beta\\
&\qquad + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j,n)} \frac{n!(\alpha_k+1)}{j!\alpha!(\alpha_k+1)} T_j(\alpha+e_k) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{i=2}^{n} \sum_{\beta\in A(i-1,n)} \frac{n!}{(i-1)!\beta!}\frac{1}{i} \sum_{k=1}^{i} T_i(\beta_1,\cdots,\beta_{k-1},1,\beta_k,\cdots,\beta_{i-1})\\
&\qquad + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j,n)} \frac{n!(\alpha_k+1)}{j!\alpha!(\alpha_k+1)} T_j(\alpha+e_k) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{i=2}^{n}\sum_{k=1}^{i} \sum_{\beta\in A(i-1,n)} \frac{n!}{i!\beta!} T_i(\beta_1,\cdots,\beta_{k-1},1,\beta_k,\cdots,\beta_{i-1})\\
&\qquad + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j,n)} \frac{n!(\alpha_k+1)}{j!(\alpha+e_k)!} T_j(\alpha+e_k) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j-1,n)} \frac{n!}{j!\alpha!} T_j(\alpha_1,\cdots,\alpha_{k-1},1,\alpha_k,\cdots,\alpha_{j-1})\\
&\qquad + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j,n)} \frac{n!(\alpha_k+1)}{j!(\alpha+e_k)!} T_j(\alpha+e_k) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{j=2}^{n}\sum_{k=1}^{j} \sum_{\alpha\in A(j,n+1)} \frac{n!\alpha_k}{j!\alpha!} T_j(\alpha_1,\cdots,\alpha_k,\cdots,\alpha_j) + T_{n+1}(u_n,1)\\
&= T_1(n+1) + \sum_{j=2}^{n}(n+1) \sum_{\alpha\in A(j,n+1)} \frac{n!}{j!\alpha!} T_j(\alpha_1,\cdots,\alpha_k,\cdots,\alpha_j) + T_{n+1}(u_n,1)\\
&= \sum_{j=1}^{n+1} \sum_{\alpha\in A(j,n+1)} \frac{(n+1)!}{j!\alpha!} T_j(\alpha_1,\cdots,\alpha_k,\cdots,\alpha_j)
\end{aligned}
$$

$$= \sum_{j=1}^{n+1} \frac{1}{j!} \sum_{\alpha \in A(j,n+1)} \binom{n+1}{\alpha} D^j g[f(x)] \, [D^{\alpha_1} f(x) h^{\alpha_1}] \cdots\cdots [D^{\alpha_j} f(x) h^{\alpha_j}]. \;\square$$

10-6.3. **Higher Product Formula** Let E, F, G, H be Banach spaces and X an open subset of E. Let $f : X \to F$ and $g : X \to G$ be C^n-maps. Suppose $(u, v) \to \langle u, v \rangle$ is a continuous bilinear map from $F \times G$ into H. Then the composite map $x \to \langle f(x), g(x) \rangle$ from X into H is also a C^n-map. Furthermore, we have

$$D^n \langle f(x), g(x) \rangle h^n = \sum_{j=0}^{n} \binom{n}{j} \langle D^j f(x) h^i, D^{n-j} g(x) h^{n-j} \rangle, \forall\; h \in E.$$

Proof. Only the formula requires a verification by induction as follow:

$$D^n \langle f(x), g(x) \rangle h^n = D \left[D^{n-1} \langle f(x), g(x) \rangle h^{n-1} \right] h$$

$$= D \left[\sum_{j=0}^{n-1} \binom{n-1}{j} \langle D^j f(x) h^j, D^{n-1-j} g(x) h^{n-1-j} \rangle \right] h$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j} \langle D[D^j f(x) h^j] h, D^{n-1-j} g(x) h^{n-1-j} \rangle$$

$$+ \sum_{j=0}^{n-1} \binom{n-1}{j} \langle D^j f(x) h^j, D[D^{n-1-j} g(x) h^{n-1-j}] h \rangle$$

$$= \sum_{j=1}^{n} \binom{n-1}{j-1} \langle D^j f(x) h^j, D^{n-j} g(x) h^{n-j} \rangle \text{ ; replacing } j \text{ by } j-1$$

$$+ \sum_{j=0}^{n-1} \binom{n-1}{j} \langle D^j f(x) h^j, D^{n-j} g(x) h^{n-j} \rangle$$

$$= \sum_{j=0}^{n} \binom{n}{j} \langle D^j f(x) h^i, D^{n-j} g(x) h^{n-j} \rangle. \qquad \square$$

10-99. **References and Further Readings** : Abraham, Fraenkel, Barroso, Franzoni, Herve, Mujica, Abt, Araujo and Aron-01.

Chapter 11

Ordinary Differential Equations

11-1 Local Existence and Uniqueness

11-1.1. Let E be a *real* Banach space and Ω an open subset of $\mathbb{R} \times E$. Let $f : \Omega \to E$ be a *continuous* map and $(t_0, x_0) \in \Omega$ a given point. The *initial value problem* is to find an open interval J containing t_0 and a map $x(t)$ from J into E which satisfies the *differential equation*: $x'(t) = f[t, x(t)]$ for all $t \in J$ subject to the *initial condition*: $x(t_0) = x_0$. In this case, $x(t)$ is called a *solution*, J a *solution interval* and the graph of $x(t)$ a *solution curve* through (t_0, x_0). We also call f a *vector field* on Ω. Clearly every solution, if exists, is continuously differentiable on J. It is obvious that a map $x(t)$ is a solution on J to the given initial value problem iff

$$x(t) = x_0 + \int_{t_0}^{t} f[s, x(s)]ds, \qquad \forall\, t \in J.$$

Therefore the initial value problem is equivalent to existence of solution to an *integral equation* which can be formulated as the fixed point of an integral operator. The fixed point will be guaranteed by contraction in §11-1.6, direct calculation in §11-3.2 and topological method in §11-6.9. We shall use the value $x(t)$ to denote the map x itself.

11-1.2. **Theorem** If $f : \Omega \to E$ is a C^n-map then the solution $x(t)$ to the initial value problem $x' = f(t, x)$ and $x(t_0) = x_0$ where $(t_0, x_0) \in \Omega$ is a C^{n+1}-map.

Proof. It is trivial when $n = 0$. Suppose that $f(t, x)$ is a C^n-map in (t, x). Then it is a C^{n-1}-map and by induction $x(t)$ is a C^n-map in t. Hence $x'(t) = f[t, x(t)]$ is a C^n-map. Therefore $x(t)$ is a C^{n+1}-map. □

11-1.3. **Exercise** Prove that there are constants $M, \lambda > 0$ such that for all $|t - t_0| \le \lambda$ and $\|x - x_0\| \le \lambda$ we have $(t, x) \in \Omega$ and $\|f(t, x)\| \le M$.

11-1.4. Throughout this section, we shall assume that f is *locally Lipschitz* in x, i.e. for every $(t_0, x_0) \in \Omega$ there are constants $L, \lambda > 0$ such that if $|t - t_0| \le \lambda$,

$\|x - x_0\| \le \lambda$ and $\|y - x_0\| \le \lambda$ then both $(t, x), (t, y) \in \Omega$ satisfy

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|.$$

In this case, L is called a ***Lipschitz constant*** on the ***square***

$$\{(t, x) \in \mathbb{R} \times E : |t - t_0| \le \lambda, \|x - x_0\| \le \lambda\}.$$

We also assume that $\|f(t, x)\| \le M$ on the same square.

11-1.5. **Exercise** Prove that if the partial derivative $\partial_x f(t, x)$ exists and is continuous on Ω then f is locally Lipschitz.

11-1.6. **Local Existence Theorem** There are $\delta, \lambda > 0$ such that for every $|r - t_0| < \delta$ and $\|a - x_0\| < \delta$, the initial value problem $x' = f(t, x)$ and $x(r) = a$ has a solution $x(t)$ on the open interval $J = (t_0 - \delta, t_0 + \delta)$ with range $\|x - x_0\| \le \lambda$. Note that the solution interval J and the range are independent of the choice of the initial condition (r, a).

Proof. Let L, M, λ be given in §11-1.4 and $\delta = \min\left\{\dfrac{1}{1+2L}, \dfrac{\lambda}{1+2M}\right\}$. Let $C_\infty(\overline{J}, E)$ denote the Banach space of all continuous maps from the compact set $\overline{J} = [t_0 - \delta, t_0 + \delta]$ into E under the sup-norm. Let

$$\overline{\mathbb{B}} = \{x \in C_\infty(\overline{J}, E) : \|x(t) - x_0\| \le \lambda, \forall\, t \in \overline{J}\}.$$

Considering x_0 as a constant map, $\overline{\mathbb{B}}$ is a closed ball in $C_\infty(\overline{J}, E)$ and hence $\overline{\mathbb{B}}$ becomes a complete metric space. Now Suppose $|r - t_0| < \delta$ and $\|a - x_0\| < \delta$ are given. For each map $x \in \overline{\mathbb{B}}$ define

$$(Kx)(t) = a + \int_r^t f[s, x(s)]ds, \forall\, t \in \overline{J}.$$

Take any $t \in \overline{J}$ and $x \in \overline{\mathbb{B}}$. Then $|t - t_0| \le \delta < \lambda$ and $\|x(t) - x_0\| \le \lambda$. Since $(t, x(t))$ is in Ω, $f(t, x(t))$ and so $(Kx)(t)$ are well-defined. Because $(Kx)(t)$ is a continuous map in t, we have $Kx \in C(\overline{J}, E)$. Now for every $t \in \overline{J}$ we get

$$\|(Kx)(t) - x_0\| \le \|a - x_0\| + \left\|\int_r^t f[s, x(s)]ds\right\| \le \delta + \left|\int_r^t M ds\right| \le \delta + 2\delta M \le \lambda.$$

Hence $Kx \in \overline{\mathbb{B}}$. Consequently we have defined a map K from $\overline{\mathbb{B}}$ into itself. Next, for all $x, y \in \overline{\mathbb{B}}$ we have

$$\|(Kx)(t) - (Ky)(t)\,\| = \left\|\int_r^t \{f[s, x(s)] - f[s, y(s)]\}ds\right\|$$

$$\le \left|\int_r^t \|f[s, x(s)] - f[s, y(s)]\|ds\right|$$

$$\le \left|\int_r^t L\|x(s) - y(s)\|ds\right| \le (2\delta L)\|x - y\|_\infty$$

i.e. $$\|Kx - Ky\|_\infty \le \frac{2L}{1+2L}\|x-y\|_\infty.$$

Now the contraction K on the complete metric space $\overline{\mathbb{B}}$ has a fixed point, say x satisfying $Kx = x$. More precisely, for all $t \in \overline{\mathbb{B}}$, we have $x(t) = (Kx)(t)$, that is $x(t) = a + \int_\tau^t f[s, x(s)]ds$. In particular, the map x is a solution to our initial value problem on the open interval $(t_0 - \delta, t_0 + \delta)$. □

11-1.7. **<u>Gronwall's Inequality</u>** Let $k(t), v(t) \ge 0$ be continuous functions on an open interval J containing t_0. If there is a constant $C \ge 0$ satisfying

$$v(t) \le C + \left|\int_{t_0}^t k(s)v(s)ds\right|, \quad \forall\ t \in J$$

then we have an estimation for $v(t)$ as follow:

$$v(t) \le C \exp\left|\int_{t_0}^t k(s)ds\right|, \quad \forall\ t \in J.$$

Proof. Define

$$Q(t) = C + \int_{t_0}^t k(s)v(s)ds$$

for all $t \in J$. Then $Q(t)$ is differentiable on J. It is given that $v(t) \le Q(t)$ for all $t \ge t_0$ in J. Since $Q'(t) = k(t)v(t) \le k(t)Q(t)$, we have

$$\frac{d}{dt}\left\{Q(t)\exp\left[-\int_{t_0}^t k(s)ds\right]\right\} = [Q'(t) - k(t)Q(t)]\exp\left[-\int_{t_0}^t k(s)ds\right] \le 0.$$

Integrating from t_0 to t we obtain

$$Q(t)\exp\left[-\int_{t_0}^t k(s)ds\right] - Q(t_0)\exp\left[-\int_{t_0}^{t_0} k(s)ds\right] \le 0,$$

i.e. $$Q(t)\exp\left[-\int_{t_0}^t k(s)ds\right] - C \le 0,$$

or $$v(t) \le Q(t) \le C\exp\int_{t_0}^t k(s)ds, \forall\ t \ge t_0.$$

For $t \le t_0$, let $Q(t) = C - \int_{t_0}^t k(s)v(s)ds$ and the proof is completed by similar argument. □

11-1.8. **<u>Local Uniqueness Theorem</u>** If $x(t), y(t)$ are solutions to the initial value problem $x' = f(t, x)$ and $x(t_0) = x_0$ then there is $\delta > 0$ such that $x(t) = y(t)$ for all $|t - t_0| \le \delta$.

Proof. Since $x(t), y(t)$ are continuous in t, there is $0 < \delta \le \lambda$ such that both $x(t), y(t)$ are defined on $J = (t_0 - \delta, t_0 + \delta)$ and both $\|x(t) - x_0\| \le \lambda$ and

$\|y(t)-x_0\| \le \lambda$ hold on J. Hence $\|f[t,x(t)]-f[t,y(t)]\| \le L\|x(t)-y(t)\|$ holds. Since $x(t), y(t)$ are solutions to the initial value problem, we have

$$\|x(t)-y(t)\| = \left\| \left\{ x_0 + \int_{t_0}^{t} f[s,x(s)]ds \right\} - \left\{ x_0 + \int_{t_0}^{t} f[s,y(s)]ds \right\} \right\|$$
$$\le \left| \int_{t_0}^{t} \|f[s,x(s)] - f[s,y(s)]\| ds \right| \le \left| \int_{t_0}^{t} L\|x(s)-y(s)\| ds \right| .$$

If follows from Gronwall's Inequality that $\|x(t)-y(t)\| \le 0 \exp L|t-t_0| = 0$, that is, $x(t) = y(t)$ on J. □

11-1.9. **Exercise** Show that $x'(t) = 3x^{2/3}$ subject to $x(0) = 0$ has infinitely many solutions on $\mathbb{R}$. In fact, for each $\alpha > 0$, a solution is given by

$$x(t) = \begin{cases} 0, & \text{for } t < \alpha; \\ (t-\alpha)^3, & \text{for } t \ge \alpha. \end{cases}$$

11-1.10. Our Local Existence Theorem §11-1.6 ensures that two solutions $x(t), y(t)$ are defined on the same open interval $(t_0 - \delta, t_0 + \delta)$ as long as their initial conditions are near (t_0, x_0). This is important in order to be able to estimate $\|x(t) - y(t)\|$ in §11-5. Consider $dx/dt = 1/t$ on the real line. With initial condition $x(1) = 0$ we have $x(t) = \ell n\ t$ for $t > 0$ and with condition $y(-1) = 0$ we have $y(t) = \ell n\ (-t)$ for $t < 0$. Does it make sense to talk about $\|x(t) - y(t)\|$?

11-2 Integral Curves

11-2.1. In this section, local uniqueness will be used to provide global uniqueness and continuation. Because all our results include *infinite dimensional Banach spaces*, compactness has to be imposed as condition for the solution curve to approach to the boundary of Ω.

11-2.2. **Global Uniqueness Theorem** Let E be a *real* Banach space and Ω an open subset of $\mathbb{R} \times E$. Let $f : \Omega \to E$ be a *continuous* map which is locally Lipschitz in the second variable and let $(t_0, x_0) \in \Omega$ be a given point. If $x(t), y(t)$ are solutions on a common open interval J to the initial value problem $x' = f(t,x)$ and $x(t_0) = x_0$ then we have $x(t) = y(t)$ on J.

Proof. Suppose to the contrary that there is $u > t_0$ in J with $x(u) \neq y(u)$. Define $K = \{s \in J : x(t) = y(t) \text{ on } [t_0, s]\}$. Since $t_0 \in K$ and $u \notin K$, we have $v = \sup K \le u$. Choose a sequence $t_n \in K$ so that $t_0 \le t_n \uparrow v$. Since $x(t_n) = y(t_n)$ for all n, the continuity ensures $x(v) = y(v)$. By Local Uniqueness

Theorem, there is $\delta > 0$ such that $x(t) = y(t)$ on $(v - \delta, v + \delta) \subset J$. Thus $x(t) = y(t)$ on $[t_0, v + \frac{1}{2}\delta]$ which implies $v + \frac{1}{2}\delta \in K$. This contradiction to $v = \sup K$ shows $x(t) = y(t)$ for all $t \geq t_0$ in J. Similarly we can prove that $x(t) = y(t)$ for all $t \leq t_0$ in J. □

11-2.3. For every solution x to the initial value problem $x' = f(t, x)$ subject to $x(t_0) = x_0$, let J_x denote its solution interval. A solution $z(t)$ is said to be *maximal* if $J_x \subset J_z$ for every solution $x(t)$. The graph of a maximal solution is called the *integral curve* through (t_0, x_0).

11-2.4. **Continuation Theorem** There is a unique integral curve through each point $(t_0, x_0) \in \Omega$.

Proof. By definition, it is obvious that maximal solutions have the same solution interval. The uniqueness follows immediately from the Global Uniqueness Theorem. To prove the existence, let $\mathbb{F}$ be the family of all solutions to the initial value problem. Define $J = \cup\{J_x : x \in \mathbb{F}\}$. Since each J_x is an open connected set containing t_0, so is J, i.e. an open interval containing t_0. Now for each $t \in J$ there is $x \in \mathbb{F}$ so that $t \in J_x$. Define $z(t) = x(t)$. Suppose $y \in \mathbb{F}$ satisfies $t \in J_y$. Then both $x(t), y(t)$ are solutions to the initial value problem on the interval $J_x \cap J_y$. By Global Uniqueness Theorem, $x(s) = y(s)$ for all $s \in J_x \cap J_y$ and in particular $x(t) = y(t)$. Therefore $z(t)$ is independent of the choice of $x(t)$. Consequently, it is well-defined. Clearly $z(t)$ is a solution of the initial value problem on the interval J. □

11-2.5. From now on, by a solution to an initial value problem, we always mean the maximal solution and the solution interval is always the maximal one.

11-2.6. **Boundary Theorem** Suppose that the boundary $\partial\Omega$ of Ω is non-empty and that $d(t)$ denotes the distance from $x(t)$ to $\partial\Omega$ for each t in the solution interval J_x. If the graph $\{(t, x(t)) : t_0 \leq t \in J_x\}$ is contained in a compact subset C of $\mathbb{R} \times E$ then $d(t) \to 0$ as t tends to the right endpoint of J_x. Similar result holds for the left endpoint of J_x.

Proof. Without loss of generality, we may assume that $|t| \leq \|(t, x)\|$ and $\|x\| \leq \|(t, x)\|$ for all $(t, x) \in \mathbb{R} \times E$. Since the C is bounded, there is $M > 0$ such that for all $t_0 \leq t \in J_x$ we have $\|(t, x(t))\| \leq M$ which gives $|t| \leq M$. Hence $u = \sup J_x < \infty$. Now suppose to the contrary that $d(t_n) \geq \varepsilon$ and $t_n \uparrow u$

for some $\varepsilon > 0$ and $t_0 \leq t_n \in J_x$. Since C is compact, without loss of generality, we may assume that $(t_n, x(t_n))$ converges to (u, v) for some $v \in E$. Thus (u, v) is a closure point of Ω. We claim that (u, v) is a boundary point of Ω. In fact, suppose to the contrary that $(u, v) \in \Omega$. Since f is locally Lipschitz at (u, v), there is $\delta > 0$ such that for every $|r - u| < \delta$ and $\|z - v\| < \delta$ there is a solution on the open interval $(u - \delta, u + \delta)$ to the initial value problem $y' = f(t, y)$ and $y(r) = z$. Choose large n such that $|t_n - u| < \delta$ and $\|x(t_n) - v\| < \delta$. Then the solution curve $x(t)$ can be extended to the interval $[t_0, u + \delta)$ which contradicts the maximality of $x(t)$. Therefore (u, v) is a boundary point of Ω. Finally the contradiction: $\varepsilon \leq d(t_n) \leq \|(t_n, x(t_n)) - (u, v)\| \to 0$ as $n \to \infty$ establishes $d(t) \to 0$ as $t \uparrow u$. □

11-2.7. **Example** The solution curve to $x' = \frac{-t}{x}$ subject to $x(0) = 1$ on the open set $\Omega = \{(t, x) \in \mathbb{R}^2 : x > 0\}$ approaches to the boundary of Ω. Without compactness, the solution curve to $x' = x$ subject to $x(0) = 1$ on the open set $\Omega = \{(t, x) \in \mathbb{R}^2 : x > -1\}$ never comes close to the boundary of Ω.

11-3 Linear Equations

11-3.1. The nonlinear equation $x' = x^2$ is defined for all t and yet its solution subject to $x(0) = 1$ is not defined at $t = 1$. For a linear equation of the form $x' = Ax + b$, the solution is defined on the whole domain interval of A and b. The following theorem can be proved by contraction §9-4.2. We follow the direct approach because it offers more information on the approximate solutions.

11-3.2. **Theorem** Let $A(t), b(t)$ be a continuous map of t in an open interval J into $L(E), E$ respectively. Then for every $t_0 \in J$ and $x_0 \in E$, the initial value problem $x'(t) = A(t)x(t) + b(t)$ subject to $x(t_0) = x_0$ has a unique solution on the whole interval J. Furthermore the exact solution $x(t)$ can be approximated by $y_n(t)$ defined inductively by $y_0(t) = x_0$ and

$$y_{n+1}(t) = x_0 + \int_{t_0}^{t} \left\{ A(s)y_n(s) + b(s) \right\} ds$$

with an error bound

$$\|x(t) - y_n(t)\| \leq \frac{1}{(n+1)!} M L^n |t - t_0|^{n+1} e^{L|t-t_0|}$$

where $L = \sup \|A(s)\|$ and $M = \sup \|A(s)x_0 + b(s)\|$ for all s between t_0 and t.

Proof. The uniqueness follows from the general theory. To prove the existence of solution on the whole interval J rather than a neighborhood of t_0, we shall give a direct proof. Take any $\alpha < t_0 < \beta$ in J. Define

$$K = \sup\{\|A(s)\| : \alpha \le s \le \beta\}$$

and

$$N = \sup\{\|A(s)x_0 + b(s)\| : \alpha \le s \le \beta\}.$$

We claim that the given approximate solution satisfies the following inequality:

$$\|y_{n+1}(t) - y_n(t)\| \le \frac{1}{(n+1)!} N K^n |t - t_0|^{n+1}, \forall\, t \in [\alpha, \beta].$$

In fact, for $n = 0$ we have

$$\|y_1(t) - y_0(t)\| = \left\| \int_{t_0}^{t} \Big\{ A(s)x_0 + b(s) \Big\} ds \right\| \le \left| \int_{t_0}^{t} \Big\| A(s)x_0 + b(s) \Big\| ds \right| \le N|t - t_0|.$$

Next for $n > 0$,

$$\|y_{n+1}(t) - y_n(t)\| \le \left\| \int_{t_0}^{t} A(s)\big[y_n(s) - y_{n-1}(s)\big] ds \right\|$$

$$\le \left| \int_{t_0}^{t} \|A(s)\|\ \|y_n(s) - y_{n-1}(s)\| ds \right|$$

$$\le \left| \int_{t_0}^{t} K\ \frac{1}{n!} N K^{n-1} |t - t_0|^n ds \right| = \frac{1}{(n+1)!} N K^n |t - t_0|^{n+1}.$$

Thus for all $p > 0$, we obtain

$$\|y_{n+p}(t) - y_n(t)\|$$

$$\le \|y_{n+1}(t) - y_n(t)\| + \|y_{n+2}(t) - y_{n+1}(t)\| + \cdots + \|y_{n+p}(t) - y_{n+p-1}(t)\|$$

$$\le \frac{NK^n|t - t_0|^{n+1}}{(n+1)!} + \frac{NK^{n+1}|t - t_0|^{n+2}}{(n+2)!} + \frac{NK^{n+2}|t - t_0|^{n+3}}{(n+3)!} + \cdots$$

$$\le \frac{NK^n|t - t_0|^{n+1}}{(n+1)!} \left[1 + \frac{K|t - t_0|}{n+2} + \frac{K^2|t - t_0|^2}{(n+3)(n+2)} + \cdots \right]$$

$$\le \frac{NK^n|t - t_0|^{n+1}}{(n+1)!} e^{K|t - t_0|} \le \frac{NK^n(\beta - \alpha)^{n+1}}{(n+1)!} e^{K(\beta - \alpha)}$$

Hence the approximate solutions $\{y_n\}$ is uniformly Cauchy on $[\alpha, \beta]$ and therefore it converges uniformly to some map $x(t)$ which is continuous on $[\alpha, \beta]$. Since α, β are arbitrary, $x(t)$ is defined and continuous on J. Letting $n \to \infty$ in the defining equation of y_n, it follows from uniform continuity that $x(t) = x_0 + \displaystyle\int_{t_0}^{t} \big\{A(s)x(s) + b(s)\big\} ds$ which gives a solution to the given initial value problem on (α, β). Since α, β are arbitrary, $x(t)$ is also a solution on the whole interval J. Finally, letting $p \to \infty$ we have for each $t \in [\alpha, \beta]$,

$$\|x(t) - y_n(t)\| \le \frac{NK^n|t - t_0|^{n+1}}{(n+1)!} e^{K|t - t_0|}.$$

If $t > t_0$, letting $\alpha \uparrow t_0$ and $\beta \downarrow t$ we obtain a better estimation in terms of M, L as stated in the theorem. Similar touch-up for $t < t_0$ completes the proof of existence. □

11-3.3. **Lemma** Let $c(t), k(t), v(t) \geq 0$ be continuous functions in t on an open interval J containing t_0. Suppose that $c(t)$ is decreasing for $t \leq t_0$ and increasing for $t \geq t_0$. If

$$v(t) \leq c(t) + \left|\int_{t_0}^{t} k(s)v(s)ds\right|, \quad \forall\, t \in J$$

then $v(t)$ satisfies the following estimation:

$$v(t) \leq c(t) \exp\left|\int_{t_0}^{t} k(s)ds\right|, \quad \forall\, t \in J.$$

Proof. For any small number $\delta > 0$, we have $v(t) \leq \delta + c(t) + \left|\int_{t_0}^{t} k(s)v(s)ds\right|$,

that is, $\dfrac{v(t)}{\delta + c(t)} \leq 1 + \left|\int_{t_0}^{t} k(s)\dfrac{v(s)}{\delta + c(t)}ds\right| \leq 1 + \left|\int_{t_0}^{t} k(s)\dfrac{v(s)}{\delta + c(s)}ds\right|$.

By Gronwall's Inequality, we have $\dfrac{v(t)}{\delta + c(t)} \leq \exp\left|\int_{t_0}^{t} k(s)ds\right|$, or $v(t) \leq [\delta + c(t)] \exp\left|\int_{t_0}^{t} k(s)ds\right|$. Since $\delta > 0$ is arbitrary, the result follows. □

11-3.4. **Theorem** The solution $x(t)$ of $x' = Ax + b$ subject to $x(t_0) = x_0$ satisfies the following estimations on its solution interval J:

(a) $\|x(t) - x_0\| \leq \left|\int_{t_0}^{t} \|A(s)x_0 + b(s)\|ds\right| \exp\left|\int_{t_0}^{t} \|A(s)\|ds\right|$

(b) $\|x(t)\| \leq \left\{\|x_0\| + \left|\int_{t_0}^{t} \|b(s)\|ds\right|\right\} \exp\left|\int_{t_0}^{t} \|A(s)\|ds\right|$

Proof. From $x(t) = x_0 + \int_{t_0}^{t} [A(s)x(s) + b(s)]\,ds$, we have

$$x(t) - x_0 = \int_{t_0}^{t} A(s)[x_0 + b(s)]ds + \int_{t_0}^{t} [A(s)x(s) - x_0]ds,$$

i.e. $\quad \|x(t) - x_0\| \leq \left|\int_{t_0}^{t} \|A(s)x_0 + b(s)\|ds\right| + \left|\int_{t_0}^{t} \|A(s)\|\ \|x(s) - x_0\|ds\right|.$

Part (a) follows from last lemma. Part (b) is obtained by working with

$\|x(t)\| \leq \left\{\|x_0\| + \left|\int_{t_0}^{t} \|b(s)\|ds\right|\right\} + \left|\int_{t_0}^{t} \|A(s)\|\ \|x(s)\|ds\right|.$ □

11-3.5. In order to present rich matured results, we shall restrict ourselves to $E = \mathbb{R}^n$. The equation $x' = Ax$ is called the ***associated homogeneous equation*** of $x' = Ax + b$. Clearly the set of all solutions to $x' = Ax$ forms a vector subspace of all maps from J into $\mathbb{R}^n$. Elements of the solution space of $x' = Ax$ are called ***complementary solutions***. If x_p is any particular solution of $x' = Ax + b$, then a general solution can be expressed in the form $x = x_p + x_c$ for some complementary solution x_c.

11-3.6. Let $y_1, y_2, \cdots, y_n$ be solutions of $x' = Ax$. Then the matrix-valued map $Y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]$ for $t \in J$ satisfies the differential equation $Y'(t) = A(t)Y(t)$. The determinant $W(t) = detY(t)$ is called the ***Wronskian*** of $y_1, y_2, \cdots, y_n$.

11-3.7. **Lemma** $W(t) = W(t_0)exp\left\{\int_{t_0}^{t} tr[A(s)]ds\right\}$ where the trace $tr[A(s)]$ is the sum of the diagonal entries of the matrix $A(s)$.

Proof. Write $A(t) = [a_{ij}(t)]$ and let r_i denote the i-th row of $Y(t)$. Equating the entries of $Y'(t) = A(t)Y(t)$, we have $r_i' = \sum_{j=1}^{n} a_{ij}r_j$. Observe that

$$W'(t) = \sum_{i=1}^{n} \det \begin{bmatrix} r_1 \\ \cdots \\ r_i' \\ \cdots \\ r_n \end{bmatrix} = \sum_{i=1}^{n} \det \begin{bmatrix} r_1 \\ \cdots \\ a_{i1}r_1 + \cdots + a_{ii}r_i + \cdots + a_{in}r_n \\ \cdots \\ r_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \det \begin{bmatrix} r_1 \\ \cdots \\ a_{ii}r_i \\ \cdots \\ r_n \end{bmatrix} = \sum_{i=1}^{n} a_{ii} \det \begin{bmatrix} r_1 \\ \cdots \\ r_i \\ \cdots \\ r_n \end{bmatrix} = \sum_{i=1}^{n} a_{ii} W(t) = W(t)tr[A(t)]$$

i.e.
$$\frac{d}{dt}\left\{W(t)exp\left[-\int_{t_0}^{t} tr[A(s)]ds\right]\right\} = 0.$$

Integrating from t_0 to t, we obtain $W(t)exp\left\{-\int_{t_0}^{t} tr[A(s)]ds\right\} - W(t_0) = 0$ which gives the required result. □

11-3.8. **Corollary** If $Y(t_0)$ is invertible then $Y(t)$ is invertible for each $t \in J$.

Proof. It follows from $detY(t) = [detY(t_0)]exp\left\{\int_{t_0}^{t} tr[A(s)]ds\right\} \neq 0$. □

11-3.9. **Exercise** Let $y_1, y_2, \cdots, y_n$ be solutions of $x' = Ax$ on the open interval J. Show that if the column vectors $y_1(t), y_2(t), \cdots, y_n(t)$ are linearly independent at some point of J then they are linearly independent at every point of J.

11-3.10. The matrix-valued map $Y(t)$ is said to be *fundamental* if it is invertible, or equivalently its columns are linearly independent. It is called a *principal fundamental* matrix of $x' = Ax$ at t_0 if $Y(t_0) = I$, the identity matrix. Principal fundamental matrix exists and is unique because it is the solution to the linear differential equation: $X' = AX$ subject to $X(t_0) = I$ in the Banach space of $n \times n$ matrices. The matrix $X(t)$ is also called the *transition matrix* of $x' = Ax$ from t to t_0.

11-3.11. **Exercise** Verify that if $Y(t)$ is a fundamental matrix of the homogeneous equation $x' = Ax$ then $X(t) = Y(t)[Y(t_0)]^{-1}$ is the principal fundamental matrix.

11-3.12. **Exercise** Show that a fundamental matrix $Y(t)$ completely determines the coefficient matrix $A(t)$. In fact, we have $A = Y'Y^{-1}$.

11-3.13. **Variation of Parameters** If $X(t)$ is the principal fundamental matrix at t_0 then the unique solution of the initial value problem $x' = Ax + b$ subject to $x(t_0) = x_0$ is given by

$$x(t) = X(t)\left\{x_0 + \int_{t_0}^{t} X(s)^{-1} b(s)ds\right\}.$$

Proof. It can be verified by direct substitution. Historically, the columns $x_1(t), x_2(t), \cdots, x_n(t)$ of $X(t)$ were first found by solving the homogeneous equation $x_j' = Ax_j$ and $x_j(t_0) = e_j$ where $\{e_1, e_2, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Let $p_1(t), p_2(t), \cdots, p_n(t)$ be unknown parameters so that $x = p_1x_1 + p_2x_2 + \cdots + p_nx_n$ is the required solution of $x' = Ax + b$. Let p be the vector function with coordinates p_j. Then substitution of $x = Xp$ into $x' = Ax + b$ produces $X'p + Xp' = AXp + b$, i.e. $AXp + Xp' = AXp + b$, or $p' = X^{-1}b$ which is integrated to $p(t) - p(t_0) = \int_{t_0}^{t} X(s)^{-1}b(s)ds$. Since $X(t_0) = I$, we have $x_0 = X(t_0)p(t_0) = p(t_0)$, i.e. $p(t) = x_0 + \int_{t_0}^{t} X(s)^{-1}b(s)ds$ which gives the required formula. □

11-3.14. **Corollary** The dimension of the solution space of $x' = Ax$ is n.

Proof. Let $x_1, x_2, \cdots, x_n$ be the columns of a principal fundamental matrix $X(t)$ of $x' = Ax$ at t_0. Then they are linearly independent by evaluation at t_0. They also span the solution space by last theorem. The proof is complete. □

11-4 Exponential Functions of Matrices

11-4.1. The solution to a first order scalar differential equation $x' = ax$ with constant coefficient $a \in \mathbb{R}$ is an exponential function $x = ce^{at}$. Exponential functions of operators have been introduced in §8-8.15.

11-4.2. **Fulmer's Method** To find e^{At} where A is a square matrix independent of t, consider the characteristic polynomial $p(\lambda)$ of A. It is easy to verify that $p(\frac{d}{dt})e^{At} = p(A)e^{At} = 0$. Hence every entry $z(t)$ of e^{At} satisfies the differential equation with constant coefficients: $p(\frac{d}{dt})z = 0$. Therefore e^{At} has the same form as the general solution of $z(t)$ except the arbitrary constants are replaced by constant matrices which can be calculated by evaluation of derivatives of e^{At} at $t = 0$.

11-4.3. **Example** To find e^{At} where A is given by

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 2 & 1 & 0 & -2 \\ 2 & 2 & -1 & -2 \\ 3 & 1 & 0 & -2 \end{bmatrix},$$

consider its characteristic polynomial: $p(\lambda) = det(\lambda I - A) = (\lambda+1)^2(\lambda-1)^2$. The eigenvalues of A are $-1, -1, 1, 1$. The general solution of the scalar differential equation: $\left(\frac{d}{dt}+1\right)^2\left(\frac{d}{dt}-1\right)^2 z = 0$ is of the form $z = (\alpha t + \beta)e^{-t} + (\gamma t + \delta)e^t$ where $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Hence an expression for e^{At} is

$$e^{At} = (Et + F)e^{-t} + (Gt + H)e^t$$

where E, F, G, H are constant matrices. Taking derivatives, we have

$$Ae^{At} = (-Et + E - F)e^{-t} + (Gt + G + H)e^t,$$

$$A^2e^{At} = (Et - 2E + F)e^{-t} + (Gt + 2G + H)e^t,$$

$$A^3e^{At} = (-Et + 3E - F)e^{-t} + (Gt + 3G + H)e^t.$$

Letting $t = 0$, they become

$$I = F + H,$$

$$A = E - F + G + H,$$

$$A^2 = -2E + F + 2G + H,$$

$$A^3 = 3E - F + 3G + H.$$

Rewrite into matrix notation as follow:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 3 & -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix}.$$

Hence the matrices E, F, G, H can be found by

$$\begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ -2 & 1 & 2 & 1 \\ 3 & -1 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & -3 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 2 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix}.$$

Therefore we obtain

$$e^{At} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} e^{-t} + \left\{ \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} t + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\} e^t$$

$$= \begin{bmatrix} (t+1)e^t & te^t & 0 & -te^t \\ -e^{-t}+e^t & e^t & 0 & e^{-t}-e^t \\ -e^{-t}+e^t & -e^{-t}+e^t & e^{-t} & e^{-t}-e^{-t} \\ -e^{-t}+(t+1)e^t & te^t & 0 & e^{-t}-te^t \end{bmatrix}.$$

11-4.4. **Theorem** Consider the homogeneous differential equation:

$$x'(t) = A(t)x(t)$$

for t in an open interval J containing t_0. If $A(t)$ commutes with $\int_{t_0}^t A(s)ds$ for every $t \in J$ then the principal fundamental matrix at t_0 is given by

$$X(t) = exp \int_{t_0}^t A(s)ds, t \in J.$$

Proof. Clearly $X(t_0) = I$. Direct calculation completes the proof as follow:

$$X'(t) = \left\{ exp \int_{t_0}^t A(s)ds \right\} A(t) = A(t)X(t). \qquad \square$$

11-4.5. **Corollary** Let A be a constant matrix. Then the principal fundamental matrix for the homogeneous equation: $x'(t) = Ax(t)$ is given by $e^{A(t-t_0)}$. The unique solution of the initial value problem $x' = Ax + b$ subject to $x(t_0) = x_0$ is given by

$$x(t) = e^{At} \left\{ e^{-At_0}x_0 + \int_{t_0}^t e^{-As}b(s)ds \right\}.$$

11-4.6. **Corollary** For every square matrix A, we have $det\ e^A = e^{trA}$.

Proof. We may assume that A is constant. By §11-3.7, we have

$$det\ e^{A(t-t_0)} = det\ e^{A(t_0-t_0)} exp \left\{ \int_{t_0}^t trA ds \right\} = e^{(trA)(t-t_0)}.$$

The result follows by letting $t = 1$ and $t_0 = 0$. $\square$

11-5 Global Dependence on Initial Conditions

11-5.1. Let E be a *real* Banach space and Ω an open subset of $\mathbb{R} \times E$. Let $f : \Omega \to E$ be a *continuous* map which is locally Lipschitz in the second variable. Consider the initial value problem $x' = f(t, x)$ subject to $x(u) = a$ where $(u, a) \in \Omega$. In this section, we shall show that the solution depends smoothly on the initial condition (u, a) and parameters.

11-5.2. To investigate global properties of integral curves, we shall work with tubes rather than just small neighborhoods of the initial point. Let $y : (\alpha, \beta) \to E$ be a continuous map. Then the set

$$\{(t, z) \in \mathbb{R} \times E : \alpha \le t \le \beta, \|z - y(t)\| < \rho\}$$

is called the ***tube*** of ***radius*** ρ along the curve $y(t)$ between α, β. The function g in the following lemma is the maximum of norms of the derivatives, partial derivatives of f when we deal with the smooth case later in this section. For the time being, it is considered as a constant function and hence it should be ignored.

11-5.3. **Global Bound Lemma** Let $g : \Omega \to \mathbb{R}$ be a continuous function. For each continuous curve $y : [\alpha, \beta] \to \Omega$, there exist a tube $U \subset \Omega$ along $y(t)$ and constants $L, M > 0$ such that for all $(t, a), (t, b) \in U$, we have $\|f(t, a)\| \le M$, $g(t, a) \le M$ and $\|f(t, a) - f(t, b)\| \le L\|a - b\|$. In this case, L is called a ***Lipschitz constant*** of f on U and M an ***upper bound*** of f, g on U.

Proof. Since f is locally Lipschitz and locally bounded, for each $s \in [\alpha, \beta]$ there exist $\delta_1(s) > 0$ and $L(s), M(s) > 0$ such that

$$R(s) = \{(t, a) : |t - s| < \delta_1(s), \|a - y(s)\| < \delta_1(s)\} \subset \Omega$$

and for all $(t, a), (t, b) \in R(s)$, we have $\|f(t, a) - f(t, b)\| \le L(s)\|a - b\|$ and $\|f(t, a)\| \le M(s)$. By continuity of g, we may assume that $g(t, a) \le M(s)$ when $\delta_1(s)$ is replaced by a smaller one if necessary. Since $y(t)$ is continuous at s, there is $0 < \delta_2(s) < \frac{1}{2}\delta_1(s)$ such that for all $|t - s| < \delta_2(s)$ we obtain $\|y(t) - y(s)\| < \frac{1}{2}\delta_1(s)$. By compactness, write

$$[\alpha, \beta] \subset \bigcup_{j=1}^{n} \Big(s_j - \delta_2(s_j), s_j + \delta_2(s_j)\Big).$$

Define $\rho = \min\{\delta_2(s_j) : 1 \le j \le n\}$, $L = \max\{L(s_j) : 1 \le j \le n\}$ and $M = \max\{M(s_j) : 1 \le j \le n\}$. Let U be the tube of radius ρ along $y(t)$ between α, β. Finally take any $(t, a), (t, b) \in U$. Then $|t - s_j| < \delta_2(s_j)$ for some j. Thus $\|y(t) - y(s_j)\| < \frac{1}{2}\delta_1(s_j)$. Since $(t, a) \in U$, we get

$$\|a - y(t)\| < \rho < \delta_2(s_j) < \tfrac{1}{2}\delta_1(s_j).$$

Combined together, we get $\|a - y(s_j)\| < \delta_1(s_j)$. Since $|t - s_j| < \delta_1(s_j)$, we have $(t, a) \in R(s_j)$. Similarly, $(t, b) \in R(s_j)$. Therefore we obtain

$$\|f(t,a) - f(t,b)\| \le L(s_j)\|a - b\| \le L\|a - b\|, \ \|f(t,a)\| \le M(s_j) \le M(s)$$

and $g(t, a) \le M(s_j) \le M(s)$. □

11-5.4. **Global Continuity of Initial Condition** Given a differential equation $x' = f(t, x)$ on Ω, let $x(t)$ be an integral curve passing through $(u, a) \in \Omega$ and let $\alpha < \beta$ be two points in the solution interval. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $|v - u| < \delta$ and $\|b - a\| < \delta$, the solution interval of the integral curve $y(t)$ passing through (v, b) contains (α, β) on which $\|x(t) - y(t)\| \le \varepsilon$. More precisely, if L is a Lipschitz constant and M an upper bound of $f(t, x)$ on a tube of radius ρ along $x(t)$ between α, β then we can choose δ small enough so that for every $t \in (\alpha, \beta)$ we have

$$\|x(t) - y(t)\| \le (\|a - b\| + M|v - u|)\exp(L|t - u|) < \min\{\varepsilon, \rho\}.$$

Proof. Without loss of generality we may assume $\alpha < \alpha_1 < u < \beta_1 < \beta$ where α_1, β_1 are arbitrary constants close to α, β respectively. By continuity of $x(t)$, for each $t \in [\alpha_1, \beta_1]$ there is $\nu(t) > 0$ so that for each $|s - t| \le 3\nu(t)$ we have $\alpha < s < \beta$ and $\|x(s) - x(t)\| \le \frac{1}{3}\rho$. By compactness, write

$$[\alpha_1, \beta_1] \subset \bigcup_{j=1}^{n} \Big(t_j - \nu(t_j), t_j + \nu(t_j)\Big).$$

Pick $\lambda > 0$ so that $\lambda < \frac{1}{3}\rho$ and also $\lambda \le \nu(t_j), \forall\ 1 \le j \le n$. For each fixed $t \in [\alpha_1, \beta_1]$, consider the point (s, y) in the square $|s - t| \le \lambda$ and $\|y - x(t)\| \le \lambda$. Select j with $|t - t_j| < \nu(t_j)$. Then $|s - t_j| \le |s - t| + |t - t_j| \le 2\nu(t_j) < 3\nu(t_j)$. Since $\|y - x(s)\| \le \|y - x(t)\| + \|x(t) - x(t_j)\| + \|x(t_j) - x(s)\| \le \lambda + \frac{1}{3}\rho + \frac{1}{3}\rho < \rho$, we obtain $(s, y) \in U$. It follows from the Local Existence Theorem, the solution interval of the integral curve through (s, y) contains the interval $(t - \ell, t + \ell)$ where

$$\ell = \min\left\{\frac{1}{1 + 2L}, \frac{\lambda}{1 + 2M}\right\}$$

is independent of t. Now choose

$$0 < \delta < \frac{\min\{\varepsilon, \lambda, \rho\}}{(1 + M)\exp[L(\beta - \alpha)]}.$$

Fix any initial condition $|v - u| < \delta$ and $\|b - a\| < \delta$. The solution interval J_y of the solution curve $y(t)$ through (v, b) contains the open interval $(u - \ell, u + \ell)$.

Since $\|f[t, y(t)]\| \le M$ and $\|f[t, x(t)] - f[t, y(t)]\| \le L\|x(t) - y(t)\|$ for all $|t - u| < \ell$, we have

$$\|x(t) - y(t)\| = \left\| \left\{ a + \int_u^t f[s, x(s)]\, ds \right\} - \left\{ b + \int_v^t f[s, y(s)] ds \right\} \right\|$$
$$\le \|a - b\| + \left\| \int_v^u f[s, y(s)]\, ds \right\| + \left\| \int_u^t \{f[s, x(s)] - f[s, y(s)]\} ds \right\|$$
$$\le \|a - b\| + M|u - v| + \left| \int_u^t L\|x(s) - y(s)\| ds \right|.$$

By Gronwall's Inequality, we obtain

$$\|x(t) - y(t)\| \le (\|a - b\| + M|u - v|) \exp(L|t - u|) \qquad \#1$$
$$\le \delta(1 + M) \exp[L(\beta - \alpha)] < \min\{\varepsilon, \lambda, \rho\}.$$

Finally let S be the set of real numbers z satisfying $u < z \le \beta_1$, $[u, z] \subset J_y$ and also satisfying the inequality #1 on $[u, z]$. Clearly $u + \frac{1}{2}\ell \in S$. Let $\mu = \sup S$. Suppose to the contrary that $\mu < \beta_1$. Let $s = \mu - \frac{1}{4}\ell$. Then by #1, $\|x(s) - y(s)\| < \lambda$. Now the solution interval J_y of the solution curve through $(s, y(s))$ contains the open interval $(s - \ell, s + \ell)$, i.e. $\mu + \frac{1}{2}\ell \in S$ which is a contradiction. Therefore $\mu \ge \beta_1$, i.e. for all $t \in [u, \beta_1)$, inequality #1 holds. Similarly it also holds for $t \in (\alpha_1, u]$. Since α_1, β_1 are arbitrary, the proof is complete. □

11-5.5. For each $(u, a) \in \Omega$, let $x_{(u,a)}(t)$ be the solution of the initial value problem $x' = f(t, x)$ subject to $x(u) = a$ and let $J(u, a)$ denote its maximal solution interval. Then $\Omega_f = \cup\{J(u, a) \times (u, a) : (u, a) \in \Omega\}$ is a subset of $\mathbb{R} \times \Omega$. Define a map $\varphi : \Omega_f \to E$ by $\varphi(t, u, a) = x_{(u,a)}(t)$ for every $(t, u, a) \in \Omega_f$. The rest of this paragraph can be skipped if you only want to read the statements of theorems. However, it will provide common notations for several proofs in this section. Let $(r, u, a) \in \Omega_f$ and $\varepsilon > 0$ be given. Then r belongs to the solution interval $J(u, a)$ of $x_{(u,a)}(t)$. Select $\alpha, \beta \in J(u, a)$ such that $\alpha < \min\{r, u\}$ and $\max\{r, u\} < \beta$. Choose $\delta > 0$ according to §11-5.4. We may assume $\delta < \min\{r - \alpha, \beta - r\}$.

11-5.6. **Corollary** The set Ω_f is open in $\mathbb{R} \times \Omega$ and the map $\varphi : \Omega_f \to E$ is continuous.

Proof. The notation of §11-5.5 is used. By continuity of $x_{(u,a)}(t)$ at $t = r$, we may demand $\|x_{(u,a)}(r) - x_{(u,a)}(s)\| < \varepsilon$ for every $s \in \mathbb{R}$ satisfying $|r - s| < \delta$. Take any $s \in \mathbb{R}$ and $(v, b) \in \Omega$ such that $|r - s| < \delta$, $|u - v| < \delta$ and $\|a - b\| < \delta$. Since the solution interval $J(v, b)$ contains the interval $[\alpha, \beta]$, we have $s \in J(v,b)$, i.e. $(s,v,b) \in \Omega_f$. Therefore Ω_f is open. Furthermore, the continuity of φ is established by the following estimation

$$\|\varphi(r,u,a) - \varphi(s,v,b)\| \le \|x_{(u,a)}(r) - x_{(u,a)}(s)\| + \|x_{(u,a)}(s) - x_{(v,b)}(s)\| \le 2\varepsilon. \quad \square$$

11-5.7. Let $f : \Omega \to E$ be a continuously differentiable map. We shall prove that φ is also continuously differentiable on Ω_f by considering its partial derivatives separately. Letting $g(t,x) = \|Df(t,x)\| + \|\partial_t f(t,x)\| + \|\partial_x f(t,x)\|$ in §11-5.3, we assume that $g(t,x) \le M$ on a tube of radius ρ along $x_{(u,a)}(t)$ between α, β.

11-5.8. **Lemma** The solution $x_{(u,a)}(t)$ is differentiable with respect to the initial location a. Furthermore, its derivative satisfies the linear differential equation

$$\partial_t \partial_a x_{(u,a)}(t) = \partial_x f[t, x_{(u,a)}(t)] \partial_a x_{(u,a)}(t)$$

subject to $$\partial_a x_{(u,a)}(u) = I.$$

Proof. The notations of §11-5.5,7 will be used and write $x_a(t)$ instead of $x_{(u,a)}(t)$. Then $x_a(t)$ is governed by

$$x_a(t) = a + \int_u^t f[s, x_a(s)]ds.$$

Take any $\|h\| < \delta$ in E. The integral curve $x_{a+h}(t)$ through $(r, a+h)$ is governed by the integral equation

$$x_{a+h}(t) = a + h + \int_u^t f[s, x_{a+h}(s)]ds$$

and its solution interval contains (α, β). Since $\partial_x f[t, x_a(t)] \in L(E)$, the composite $\partial_x f[t, x_a(t)]z$ is in $L(E)$ for each $z \in L(E)$. Because the map $\partial_x f : \Omega \to L(E)$ is continuous, $A(t)z = \partial_x f[t, x_a(t)]z$ defines a continuous map $A : (\alpha, \beta) \to L[L(E), L(E)]$. The linear equation $z' = A(t)z$ subject to $z(u) = I$ has a unique solution on the whole interval (α, β). In fact, it is governed by the integral equation:

$$z(t) = I + \int_u^t \partial_x f[s, x_a(s)]z(s)ds.$$

Define $\triangle(h) = x_{a+h}(t) - x_a(t) - z(t)h$ for every $t \in (\alpha, \beta)$. Clearly, we have

$$\triangle(h) = \int_u^t \Big\{ f[s, x_{a+h}(s)] - f[s, x_a(s)] - \partial_x f[s, x_a(s)]z(s)h \Big\} ds.$$

It follows from integral mean-value theorem that

$$\begin{aligned}&f[s, x_{a+h}(s)] - f[s, x_a(s)]\\&= \int_0^1 \partial_x f[s, (1-\theta)x_a(s) + \theta x_{a+h}(s)][x_{a+h}(s) - x_a(s)]d\theta.\end{aligned}$$

Letting $\pi(s, \theta, h) = \partial_x f[s, (1-\theta)x_a(s) + \theta x_{a+h}(s)]$, we have

$$\begin{aligned}\triangle(h) = &\int_u^t \int_0^1 \{\pi(s,\theta,h) - \pi(s,\theta,0)\}[x_{a+h}(s) - x_a(s)]d\theta ds\\&+ \int_u^t \partial_x f[s, x_a(s)] \triangle(s) ds.\end{aligned}$$

Since $\pi : [\alpha, \beta] \times [0,1] \times \mathbb{B}(0,\delta) \to L(E)$ is continuous and $[\alpha,\beta] \times [0,1]$ is compact, it follows from §2-7.6 that there is $0 < \delta_1 < \delta$ such that for all $(s,\theta) \in [\alpha,\beta] \times [0,1]$ and for all $\|h\| < \delta_1$ in E, we have

$$\|\pi(s,\theta,h) - \pi(s,\theta,0)\| < \varepsilon.$$

By §11-5.4, we get $\|x_{a+h}(s) - x_a(s)\| \le \|h\| e^{L(\beta-\alpha)}$. Since $\|\partial_x f[s, x_a(s)]\| \le M$, we obtain

$$\|\triangle(h)\| \le (\beta - \alpha)\varepsilon \|h\| e^{L(\beta-\alpha)} + \left| \int_u^t M \|\triangle(s)\| ds \right|.$$

By Gronwall's inequality, we have

$$\|x_{a+h}(t) - x_a(t) - z(t)h\| = \|\triangle(h)\| \le (\beta-\alpha)e^{(L+M)(\beta-\alpha)}\varepsilon\|h\|, \forall\ \|h\| < \delta_1.$$

Therefore $x_a(t)$ is differentiable in a. Furthermore we have $\partial_a x_a(t) = z(t)$. This completes the proof. □

11-5.9. **Lemma** The solution $x_{(u,a)}(t)$ is differentiable with respect to the initial time u. Furthermore, its derivative satisfies the linear differential equation

$$\partial_t \partial_u x_{(u,a)}(t) = \partial_x f[t, x_{(u,a)}(t)] \partial_u x_{(u,a)}(t)$$

subject to $$\partial_u x_{(u,a)}(u) = -f(u,a).$$

Proof. The notations of §11-5.5,7 will be used and write $x_u(t)$ instead of $x_{(u,a)}(t)$. Then $x_u(t)$ is governed by

$$x_u(t) = a + \int_u^t f[s, x_u(s)] ds.$$

Take any $|\lambda| < \delta$ in $\mathbb{R}$. The integral curve $x_{u+\lambda}(t)$ through $(u+\lambda, a)$ is governed by

$$x_{u+\lambda}(t) = a + \int_{u+\lambda}^t f[s, x_{u+\lambda}(s)] ds$$

and its solution interval contains (α, β). Since $\partial_x f[t, x_u(t)] \in L(E)$ is a continuous map of $t \in (\alpha, \beta)$, the linear equation $z' = A(t)z$ subject to $z(u) = -f(u, a)$ has a unique solution on the whole interval (α, β). In fact, it is governed by

$$z(t) = -f(u,a) + \int_u^t \partial_x f[s, x_u(s)]z(s)ds.$$

For fixed $t \in (\alpha, \beta)$, define $\triangle(\lambda) = \frac{1}{\lambda}[x_{u+\lambda}(t) - x_u(t)] - z(t)$. The proof is complete if we can prove $\lim \triangle(\lambda) = 0$ as $\lambda \to 0$ in $\mathbb{R}$. So let $\varepsilon > 0$ be given. From the above integral equations,

$$\triangle(\lambda) = \int_u^t \frac{f[s, x_{u+\lambda}(s)] - f[s, x_u(s)]}{\lambda} ds - \frac{1}{\lambda}\int_u^{u+\lambda} f[s, x_{u+\lambda}(s)]ds$$
$$+ f(u,a) - \int_u^t \partial_x f[s, x_u(s)]z(s)ds.$$

By integral mean-value theorem, we have

$$\frac{f[s, x_{u+\lambda}(s)] - f[s, x_u(s)]}{\lambda}$$
$$= \int_0^1 \partial_x f[s, (1-\theta)x_u(s) + \theta x_{u+\lambda}(s)] \frac{x_{u+\lambda}(s) - x_u(s)}{\lambda} d\theta.$$

Letting $\pi(s, \theta, \lambda) = \partial_x f[s, (1-\theta)x_u(s) + \theta x_{u+\lambda}(s)]$, rewrite

$$\triangle(\lambda) = \int_u^t \int_0^1 \left[\pi(s,\theta,\lambda) - \pi(s,\theta,0)\right] \frac{x_{u+\lambda}(s) - x_u(s)}{\lambda} d\theta ds$$
$$+ \frac{1}{\lambda}\int_u^{u+\lambda} \Big\{ f(u,a) - f[s, x_{u+\lambda}(s)] \Big\} ds + \int_u^t \partial_x f[s, x_u(s)] \triangle (s) ds.$$

Since $\pi : [\alpha, \beta] \times [0, 1] \times (-\delta, \delta) \to L(E)$ is continuous and $[\alpha, \beta] \times [0, 1]$ is compact, it follows from §2-7.6 that there is $0 < \delta_1 < \delta$ such that for all $(s, \theta) \in [\alpha, \beta] \times [0, 1]$ and for all $|\lambda| < \delta_1$ in E, we have

$$\|\pi(s,\theta,\lambda) - \pi(s,\theta,0)\| < \varepsilon.$$

By §11-5.4, we get

$$\|x_{u+\lambda}(s) - x_u(s)\| \leq M|\lambda|e^{L(\beta-\alpha)}.$$

This takes care of the first term. Next, by continuity of f at (u, a), there is $\delta_2 >$ such that $\|f(u, a) - f(v, b)\| < \varepsilon$ whenever $|u - v| < \delta_2$ and $\|a - b\| < \delta_2$. Also from the continuity of $\varphi : \Omega_f \to E$ at (u, u, a), there is $\delta_3 > 0$ such that for all $|u - s| < \delta_3$ and $|u - v| < \delta_3$ we have $\|\varphi(u, u, a) - \varphi(s, v, a)\| < \delta_2$, i.e. $\|a - x_v(s)\| < \delta_2$. We may assume $\delta_3 < \delta_1$. Hence for every $|\lambda| < \delta_3$ we obtain

$$\left| \frac{1}{\lambda}\int_u^{u+\lambda} \Big\{ f(u,a) - f[s, x_{u+\lambda}(s)] \Big\} ds \right| < \varepsilon.$$

Finally since $\|\partial_x f[s, x_u(s)]\| \le M$ for all $\alpha < s < \beta$, we have

$$\| \triangle(\lambda)\| \le \left[(\beta-\alpha)Me^{L(\beta-\alpha)}+1\right]\varepsilon + \left|\int_u^t M\|\triangle(\lambda)\|ds\right|.$$

It follows from Gronwall's inequality that

$$\| \triangle(\lambda)\| \le \left[(\beta-\alpha)Me^{L(\beta-\alpha)}+1\right]e^{M(\beta-\alpha)}\varepsilon,$$

i.e. $\lim \triangle(\lambda) = 0$ as $\lambda \to 0$. □

11-5.10. **Global Smoothness Theorem** If $f : \Omega \to E$ is a C^n-map for some $n \ge 1$, then $\varphi : \Omega_f \to E$ is also a C^n-map.

Proof. By induction, assume that the partial derivatives $\partial_t\varphi(t,u,a)$, $\partial_u\varphi(t,u,a)$ and $\partial_a\varphi(t,u,a)$ are C^{n-1}-maps. Now the partial derivative $\varphi_t(t,u,a) = f[t,\varphi(t,u,a)]$ is a C^{n-1}-map. Next, since the integrands of the following equations

$$\partial_u\varphi(t,u,a) = -f(u,a) + \int_u^t \partial_x f[s, \partial_u\varphi(s,u,a)]ds$$

and
$$\partial_a\varphi(t,u,a) = I + \int_u^t \partial_x f[s,\varphi(s,u,a)]\partial_a\varphi(s,u,a)ds$$

are C^{n-1}-maps, so are the partial derivatives $\partial_u\varphi(t,u,a), \partial_a\varphi(t,u,a)$ by §9-3.9. Therefore φ is a C^n-map. □

11-5.11. **Corollary** If $f : \Omega \to E$ is smooth then so is $\varphi : \Omega_f \to E$.

11-5.12. Finally, we shall study equations with parameters. Let E, F be real Banach spaces and let $\Omega, \mathbb{P}$ be open subsets of $\mathbb{R} \times E, F$ respectively. Let $f : \Omega \times \mathbb{P} \to E$ be a C^n-map where $n \ge 1$. For any $(u,a,p) \in \Omega \times \mathbb{P}$, let $x_{(u,a,p)}(t)$ be the integral curve of the initial-value problem $x' = f(t,x,p)$ subject to $x(u) = a$ and let $J(u,a,p)$ be the solution interval of $x_{(u,a,p)}(t)$. On the set $\Omega_f = \bigcup\{J(u,a,p) \times (u,a,p) : (u,a,p) \in \Omega \times \mathbb{P}\}$, define $\varphi(t,u,a,p) = x_{(u,a,p)}(t)$.

11-5.13. **Lemma** Let $\alpha < \beta$ be two given points in $J(u,a,p)$, then there is $\delta > 0$ such that for all $|v-u| < \delta$ in $\mathbb{R}$, $\|b-a\| < \delta$ in E and $\|q-p\| < \delta$ in F, the solution interval $J(v,b,q)$ contains both α, β.

Proof. Consider the initial-value problem on $\Omega \times \mathbb{P} \subset \mathbb{R} \times (E \times F)$ given by $z' = (f(t,z), 0)$ subject to $z(u) = (a,p)$ where $z = (x,y) \in E \times F$. Clearly $z(t) = (x_{(u,a,p)}(t), p)$ is a solution by direction substitution and hence the only solution by uniqueness. Observe that $x_{(u,a,p)}(t), z(t)$ have the same solution interval. Now the result follows from §11-5.4. □

11-5.14. **Theorem** The set Ω_f is open in $\mathbb{R} \times E \times F$ and the map $\varphi : \Omega \to E$ is a C^n-map. Furthermore, if f is smooth, so is φ.

Proof. It follows immediately by applying §11-5.6 to $z(t)$ of last lemma. □

11-6 Solutions without Uniqueness

11-6.1. Let E be a Banach space and Ω an open subset of $\mathbb{R} \times E$. Let $f : \Omega \to E$ be a continuous map and let $(t_0, x_0) \in \Omega$ be given. Consider the initial value problem $x' = f(t, x)$ subject to $x(t_0) = x_0$. In general, the problem has no solution in infinite dimensional Banach space as shown in [Godunov]. The main result of this section is a generalization of Peano's Theorem which demands additional condition that f is locally compact. We start with certain preparation in a general framework.

11-6.2. Let M be a non-empty subset of a normed space E. Then the intersection of all balanced set containing M is called the ***balanced hull*** of M. It is denoted by $ba(M)$.

11-6.3. **Theorem** The balanced hull of a compact set is compact.

Proof. Let M be a non-empty compact set in E and $\mathbb{B}$ the closed unit ball of $\mathbb{K}$. Since the scalar multiplication $f : \mathbb{K} \times M \to E$ given by $f(\lambda, x) = \lambda x$ is continuous, the image $f(\mathbb{B} \times M)$ of a compact set is compact. It is easy to show that $f(\mathbb{B} \times M)$ is the balanced hull of M. □

11-6.4. **Theorem** The convex hull of a precompact set M is precompact.

Proof. Let $\varepsilon > 0$ be given. Write $\mathbb{B} = \mathbb{B}(0, \varepsilon)$ for simplicity. There is a finite subset A of M such that $M \subset \bigcup_{a \in A} \mathbb{B}(a, \varepsilon) = A + \mathbb{B}$. Since the convex hull $co(A)$ is compact, there is a finite set V such that $co(A) \subset \bigcup_{v \in V} \mathbb{B}(v, \varepsilon) = V + \mathbb{B}$. Because $co(A) + \mathbb{B}$ is a convex set containing M, we have

$$co(M) \subset co(A) + \mathbb{B} \subset V + 2\mathbb{B} = \bigcup_{v \in V} \mathbb{B}(v, 2\varepsilon).$$

Consequently, $co(M)$ is precompact. □

11-6.5. **Exercise** Prove that the convex hull and the closure of a balanced set is balanced. What is the natural definition of the closed convex balanced hull of a non-empty subset of a normed space?

11-6.6. **Theorem** The closed convex balanced hull of a relatively compact set M in a Banach space E is compact.

Proof. The balanced hull $ba(\overline{M})$ of the compact set $\overline{M}$ is compact and hence its subset $ba(M)$ is precompact. The convex hull $co[ba(M)]$ is precompact. Therefore the closed convex balanced hull $\overline{co}[ba(M)]$ is closed precompact in the Banach space E. Consequently, it is compact. □

11-6.7. **Exercise** Prove that if M is a balanced set, then for all $|\alpha| \leq |\beta|$ in $\mathbb{K}$, we have $\alpha M \subset \beta M$.

11-6.8. Let E be a Banach space and Ω an open subset of $\mathbb{R} \times E$. Suppose that $f : \Omega \to E$ is a continuous map which is locally compact, i.e. every point of Ω has a ball B contained in Ω such that $f(B)$ is relatively compact in E. Note that when E is finite dimensional, every continuous map on Ω is automatically locally compact.

11-6.9. **Generalized Peano's Theorem** The initial value problem $x' = f(t,x)$ for a locally compact field f subject to the initial condition $x(t_0) = x_0$ has at least one local solution for each $(t_0, x_0) \in \Omega$.

Proof. Without loss of generality, we may assume $t_0 = 0 \in \mathbb{R}$ and $x_0 = 0 \in E$. There is $q > 0$ such that the set

$$Q = \{(t,x) \in \mathbb{R} \times E : |t| \leq q \text{ and } \|x\| \leq q\}$$

is contained in Ω and $f(Q)$ is relatively compact in E. Then the closed convex balanced hull K of $f(Q)$ is compact. There is $\lambda > \|z\|$ for all $z \in K$. Let $0 < r < \frac{q}{\lambda}$ and $J = [-r, r]$. Let $C_\infty(J, E)$ be the Banach space of all continuous maps from the compact interval J into E. Then the closed ball $X = \overline{\mathbb{B}}(0, q)$ in $C_\infty(J, E)$ is a closed convex set. For every $x \in X$, the integral

$$(Ax)(t) = \int_0^t f[s, x(s)]ds$$

of continuous map is defined for each $t \in J$. Because

$$\|(Ax)(t_2) - (Ax)(t_1)\| \leq \left\| \int_{t_1}^{t_2} f[s, x(s)]ds \right\| \leq |t_2 - t_1|\lambda$$

the set $\{Ax : x \in X\}$ is equicontinuous on J. Since $(Ax)(t) \in tK \subset rK$, we obtain $\|(Ax)(t)\| \leq r\lambda < q$, i.e. $Ax \in X$. Since $(AX)(t) \subset rK$ which is relatively compact, it follows from Ascoli's Theorem that AX is a relatively compact subset of X. Because X is closed, AX is contained in a compact subset of X. To show that $A : X \to X$ is continuous, take any $x \in X$ and any $\varepsilon > 0$. For every $s \in J$, by continuity of f at $[s, x(s)]$ there is $\alpha(s) > 0$ such that for all $|t - s| \leq \alpha(s)$ and $\|u - f[x(s)]\| \leq 2\alpha(s)$ we have

$\|f(t,u) - f[s,x(s)]\| \le \varepsilon$. Since $x(t)$ is continuous at s, there is $0 < \delta(s) < \alpha(s)$ such that for all $|t-s| \le \delta(s)$, we get $\|x(t) - x(s)\| \le \alpha(s)$. By compactness of J, write $J \subset \bigcup_{j=1}^{n} \big(s_j - \delta(s_j), s_j + \delta(s_j)\big)$. Let $\delta = \min\{\delta(s_j) : 1 \le j \le n\}$. Now take any $y \in \mathbb{B}(x,\delta)$. Then $\|x(t) - y(t)\| < \delta$ for all $t \in J$. Choose any $t \in J$. Then $|t - s_j| < \delta(s_j)$ for some j. Now $\|x(t) - x(s_j)\| \le \alpha(s_j)$ and $|t - x_j| \le \delta(s_j) \le \alpha(s_j)$ give $\|f[t,x(t)] - f[s_j, x(s_j)]\| \le \varepsilon$. Furthermore, since $\|y(t) - x(s_j)\| \le \|y(t) - x(t)\| + \|x(t) - x(x_j)\| \le 2\alpha(s_j)$, we also get $\|f[t,x(t)] - f[s_j,x(s_j)]\| \le \varepsilon$, or $\|f[t,x(t)] - f[t,y(t)]\| \le 2\varepsilon$ for all $t \in J$. Therefore

$$\|(Ax - Ay)(t)\| = \left\| \int_0^t \{f[t,x(s)] - f[t,y(s)]\} ds \right\| \le |t|(2\varepsilon) \le 2r\varepsilon,$$

or $\|Ax - Ay\| \le 2r\varepsilon$. Consequently, $A : X \to X$ is a compact map. It has a fixed point, say x, i.e.

$$x(t) = \int_0^t f[s,x(s)]ds$$

for all $t \in J$. Clearly, it is a solution to $x' = f(t,x)$ subject to $x(0) = 0$. ☐

11-6.10. It can be proved by Zorn's Lemma that maximal solution always exists. Because local uniqueness is not available, $x(t)$ may branch out to different directions.

11-6.11. **Exercise** Show that the solution $x(t)$ to a locally compact initial value problem $x' = f(t,x)$ need not be continuous map of the initial condition $x(t_0) = x_0$. See §11.1-9.

11-99. **References and Further Readings** : Arnold, Godunov, Astala, Barbu, Deimling-77,78, Cullen, Yeh, Hadzic, Li and Lobanov.

Chapter 12

Compact Linear Operators

12-1 Basic Properties

12-1.1. A square matrix $B = [b_{ij}]$ can be regarded as a function of two variables i, j in a finite discrete set $N = \{1, 2, \cdots, n\}$. The image $y = Bx$ can be written in terms of coordinates as $(Bx)_i = \sum_{j=1}^{n} b_{ij}x_j$ for $x = (x_1, x_2, \cdots, x_n)$. As shown in §12-1-9, a trivial way to consider an infinite dimensional analogue would be to change N by the closed unit interval $[0, 1]$. Then B is replaced by a continuous function k on the unit square and the summation by an integral. It turns out that this kind of integral operators are compact linear operators on $C_\infty[0, 1]$. It motivates the study of this chapter. However, the interesting topics of linear integral equations and determinants in infinite dimensional spaces are beyond our scope. More examples on compact linear maps will be given in §§12-1.14, 14-5.5.

12-1.2. Let E, F be normed spaces and V the closed unit ball of E. A linear map $f : E \to F$ is said to be *compact* if $f(V)$ is a relatively compact subset of F. The set of all compact linear maps from E into F is denoted by $K(E, F)$.

12-1.3. **Lemma** Let $f : E \to F$ a linear map. Then the following statements are equivalent:

(a) f is a compact linear map.

(b) For every bounded sequence $\{x_n\}$ in E, there is a convergent subsequence of $\{f(x_n)\}$ in F.

(c) f carries every bounded set into a relatively compact set.

Proof. Let $\{x_n\}$ be any bounded sequence in E. There is $\lambda > 0$ such that $\|x_n\| \leq \lambda$ for all n. Now all $f(x_n/\lambda)$ belongs to the relatively compact set $f(V)$. There is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\{f(y_n/\lambda)\}$ converges. Hence $\{f(y_n)\}$ also converges. This proves $(a \Rightarrow b)$. The rest is left as an exercise. □

12-1.4. **Theorem** If $f : E \to F$ a compact linear map, then for every weakly convergent sequence $x_n \to a$ in E, we have $\|f(x_n) - f(a)\| \to 0$ in F. Furthermore if E is reflexive, then the converse is also true.

Proof. Assume that f is a compact linear map. Suppose to the contrary that there is $\varepsilon > 0$ and a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\|f(y_n) - f(a)\| \geq \varepsilon$. Since $y_n \to a$ weakly, $\{y_n\}$ is a bounded sequence. By compactness of f, there is a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\|f(z_n) - b\| \to 0$ for some $b \in F$. Hence $f(z_n) \to b$ weakly. By continuity of f, $f(z_n) \to f(a)$ weakly. Thus $b = f(a)$, i.e. $\varepsilon \leq \|f(z_n) - f(a)\| = \|f(z_n) - b\| \to 0$ which is a contradiction. Therefore every compact linear map carries weakly convergent sequence into a strongly convergent sequence. Conversely, if E is reflexive, then every bounded sequence in E has a weakly convergent subsequence which is carried to a strongly convergent sequence in F and hence f is compact. □

12-1.5. **Theorem** $K(E, F)$ is a vector subspace of $L(E, F)$.

Proof. Let $f, g : E \to F$ be a compact linear map. Since $f(V)$ is relatively compact, it is bounded and hence f is continuous, i.e. $K(E, F) \subset L(E, F)$. Next, let $\alpha, \beta \in \mathbb{K}$ be given scalars. Since $f(V), g(V)$ are relatively compact, so is $\alpha f(V) + \beta g(V)$ which contains $(\alpha f + \beta g)(V)$. Therefore $\alpha f + \beta g$ is also a compact linear map. □

12-1.6. **Theorem** If F is a Banach space, then $K(E, F)$ is closed in $L(E, F)$.

Proof. Let g be a closure point of $K(E, F)$ in $L(E, F)$. Since F is complete, it suffices to show that $g(V)$ is precompact. Let $\varepsilon > 0$ be given. There is $f \in K(E, F)$ such that $\|f - g\| \leq \frac{1}{3}\varepsilon$. Since $f(V)$ is precompact, there is a finite subset J of V such that for every $x \in V$ we have $\|f(x) - f(a)\| \leq \frac{1}{3}\varepsilon$ for some $a \in J$. Now observe

$$\begin{aligned} &\|g(x) - g(a)\| \leq \|g(x) - f(x)\| + \|f(x) - f(a)\| + \|f(a) - g(a)\| \\ &\leq \|g - f\| \; \|x\| + \tfrac{1}{3}\varepsilon + \|g - f\| \; \|a\| \leq \tfrac{1}{3}\varepsilon \cdot 1 + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon \cdot 1 = \varepsilon. \end{aligned}$$

Therefore $g(V)$ is precompact. Consequently, $K(E, F)$ is closed in $L(E, F)$. □

12-1.7. Let E, F be vector spaces and $f : E \to F$ a linear map. Then f is said to be *finite dimensional* or of *finite rank* if $f(E)$ is a finite dimensional vector subspace of F. For the rest of this section, let E, F be normed spaces and $f : E \to F$ a continuous linear map.

12-1.8. **Theorem** If f is finite dimensional, then it is compact.

Proof. By continuity, $f(V)$ is bounded in F. Since $f(E)$ is finite dimensional, $f(V)$ is bounded in a finite dimensional subspace $f(E)$ of F. Hence $f(V)$ is relatively compact in $f(E)$. Since $f(E)$ is closed in F, $f(V)$ is also relatively compact in F. □

12-1.9. **Example** Let $E = C^r_\infty[0,1]$ and $k : [0,1] \times [0,1] \to \mathbb{R}$ a continuous function. Each $x \in E$ is a continuous function on $[0,1]$. Let $f(x)$ be a function on $[0,1]$ given by $f(x)(s) = \int_0^1 k(s,t)x(t)dt$. Then $f : E \to E$ is a compact linear map.

Proof. It is obvious that $f(x)$ is linear in x. Let V be the closed unit ball of E consisting of $x \in E$ with $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. It is required to prove that $f(V)$ is relatively compact in $C^r_\infty[0,1]$. It suffices to show that $f(V)$ is uniformly bounded and equicontinuous on $[0,1]$. Since k is continuous on the compact space $[0,1] \times [0,1]$, it is bounded, i.e. there is $\lambda > 0$ such that for all $s,t \in [0,1]$, we have $|k(s,t)| \le \lambda$. Now for any $x \in V$,

$$|f(x)(s)| = \left|\int_0^1 k(s,t)x(t)dt\right| \le \int_0^1 |k(s,t)|\,|x(t)|dt \le \lambda.$$

Hence $f(V)$ is uniformly bounded. Next by uniform continuity of k on the compact space $[0,1] \times [0,1]$, for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $(s,t),(s',t') \in [0,1] \times [0,1]$, $\|(s,t) - (s',t')\| \le \delta$ implies $|k(s,t) - k(s',t')| \le \varepsilon$. Now take any $s, s_0 \in [0,1]$ with $|s - s_0| \le \delta$. Then for every $x \in V$, we have

$$|f(x)(s) - f(x)(s_0)| = \left|\int_0^1 [k(s,t) - k(s_0,t)]x(t)dt\right|$$

$$\le \int_0^1 |k(s,t) - k(s_0,t)|\,|x(t)|dt \le \varepsilon.$$

Hence $f(V)$ is equicontinuous on $[0,1]$. This completes the proof. □

12-1.10. **Theorem** Let E, F, G be normed spaces. Suppose $f : E \to F$ and $g : F \to G$ are continuous linear maps. If f or g is compact, then the composite gf is also compact.

Proof. Firstly assume that f is compact. Then the closure $\overline{f(V)}$ is compact in F and hence so is $g[\overline{f(V)}]$. Its subset $gf(V)$ is relatively compact and hence gf is compact. Next assume that g is compact. Take any bounded subset B of E. By continuity of f, $f(B)$ is bounded and hence $gf(B) = g[f(B)]$ is relatively compact in G. Therefore gf is again compact. □

12-1.11. **Theorem** If f is compact, then so is its transpose f^t.

Proof. Let V, U be the closed unit balls of E, F' respectively. Since f is compact, the closure X of $f(V)$ is compact in F. There is $\lambda > 0$ such that $\|x\| < \lambda$ for all $x \in X$. Consider U as a subset of $C_\infty(X)$. Take any $u \in U$. Since $|u(x)| \leq \|u\|\|x\| \leq \lambda$, for all $x \in X$, we have $\|u\|_\infty \leq \lambda$. Therefore U is uniformly bounded on X. Next, for all $x, y \in X$, we get

$$|u(x) - u(y)| = |u(x-y)| \leq \|u\| \; \|x - y\| \leq \|x - y\|.$$

Hence U is also equicontinuous on X. Therefore U is relatively compact in $C_\infty(X)$ and consequently, it is precompact. Let $\varepsilon > 0$ be given. There is a finite subset J of U such that for each $u \in U$, we have

$$\|u - w\|_\infty = \sup\{|(u-w)(x)| : x \in X\} \leq \varepsilon$$

for some $w \in J$. Hence for each $b \in V$, we have

$$| < f^t(u-w), b > | = | < u - w, f(b) > |$$
$$\leq \sup\{|(u-w)(x)| : x \in X\} = \|u - w\|_\infty \leq \varepsilon.$$

Taking supremum over $b \in V$, we obtain $\|f^t(u) - f^t(w)\| \leq \varepsilon$. Therefore $f^t(U)$ is precompact in the Banach space E'' and consequently $f^t(U)$ is relatively compact. This proves that f^t is a compact linear map. □

12-1.12. **Theorem** If F is a Banach space and if the transpose f^t is compact, then f itself is a compact linear map.

Proof. It follows from last theorem that $f^{tt} : E'' \to F''$ is compact. Let V, V'' be the closed unit balls of E, E'' respectively. Then $f^{tt}(V'')$ is relatively compact and hence it is precompact. Since f^{tt} is an extension of f and $J(V) \subset V''$ where $J : E \to E''$ is the natural embedding, $J[f(V)]$ is precompact. Therefore $f(V)$ is precompact in the Banach space F and so it is relatively compact. Consequently, f is a compact linear map. □

12-1.13. **Exercise** Prove that if the identity map of a normed space E is compact then E must be finite dimensional.

12-1.14. **Exercise** Let $a = (a_1, a_2, a_3, \cdots)$ be a bounded sequence and let $1 \leq p \leq \infty$. Prove the following statements:

(a) For every $x \in \ell_p$, $f(x) = (a_1x_1, a_2x_2, a_3x_3, \cdots) \in \ell_p$.

(b) $f : \ell_p \to \ell_p$ is a continuous linear map.

(c) f is compact iff $a_n \to 0$ as $n \to \infty$.

12-1.15. **Exercise** Let E, F be Banach spaces, X an open subset of E and $g : X \to F$ a differentiable map. Prove that if g is *locally compact*, i.e. every point $x \in X$ has a ball $\mathbb{B}(x) \subset X$ such that $g[\mathbb{B}(x)]$ is relatively compact in F, then every derivative $Dg(x)$ is a compact linear map.

12-2 Riesz-Schauder Theory

12-2.1. In this section, we shall work with one given compact operator A and its *displacement operator* $T = I - A$. The closed unit ball of E will be denoted by V. The close relationship between compact operators and finite dimensional vector subspaces will be unfolded.

12-2.2. **Theorem** $\ker(T)$ is a finite dimensional vector subspace of E.

Proof. Since $\ker(T)$ is closed in E, the closed unit ball W of $\ker(T)$ is closed in E. For any $x \in W$, we have $\|x\| \leq 1$ and $Tx = 0$, i.e. $x = Ax \in A(V)$. Hence $W \subset A(V)$. Since A is a compact operator, W is a closed subset of the compact set $\overline{A(V)}$. Therefore W is compact. Consequently, $\ker(T)$ is finite dimensional. □

12-2.3. **Lemma** For each integer $n \geq 1$, we have
(a) $\ker(T^{n-1}) \subset \ker(T^n)$;
(b) If $\ker(T^{n-1}) = \ker(T^n)$, then $\ker(T^n) = \ker(T^{n+1})$.

Proof. (a) For any $x \in \ker(T^{n-1})$, we have $T^n x = T(T^{n-1}x) = T0 = 0$, i.e. $x \in \ker(T^n)$.
(b) Take any $x \in \ker(T^{n+1})$. Then $T^n(Tx) = 0$, i.e. $Tx \in \ker(T^n) = \ker(T^{n-1})$, or $T^n x = T^{n-1}(Tx) = 0$. Thus $x \in \ker(T^n)$. This proves $\ker(T^{n+1}) \subset \ker(T^n)$. The reverse inequality follows from (a). □

12-2.4. **Lemma** There is an integer $n \geq 1$ such that $\ker(T^{n-1}) = \ker(T^n)$.

Proof. Suppose to the contrary that for all $n \geq 1$, $\ker(T^{n-1}) \subsetneq \ker(T^n)$. There is $x_n \in \ker(T^n)$ such that $\|x_n\| = 1$ and $d(x_n, \ker T^{n-1}) \geq \frac{1}{2}$. Then for all $p \geq 1$, we have

$$T^{n+p-1}(Tx_{n+p} + Ax_n) = T^{n+p}x_{n+p} + AT^{n+p-1}x_n = 0,$$

i.e. $Tx_{n+p} + Ax_n \in \ker(T^{n+p-1})$. Now

$$\|Ax_{n+p} - Ax_n\| = \|x_{n+p} - (Tx_{n+p} + Ax_n)\| \geq d(x_{n+p}, \ker T^{n+p-1}) \geq \tfrac{1}{2}$$

shows that $\{Ax_n\}$ cannot have convergent subsequence. This contradiction to the compactness of A completes the proof. □

12-2.5. **Theorem** There is an integer k such that $\ker(T^{n-1}) \neq \ker(T^n)$ for all $n \leq k$ and $\ker(T^n) = \ker(T^{n+1})$ for all $n \geq k$. This integer is called the *ascent* of T.

Proof. The smallest integer k satisfying last lemma is the solution. □

12-2.6. **Lemma** There is $\lambda > 0$ satisfying the following condition: for each $y \in T(E)$, there is at least one solution $x \in E$ for $Tx = y$ and $\|x\| \leq \lambda\|y\|$.

Proof. Suppose to the contrary that for each $n \geq 1$, there is $y_n \in T(E)$ such that for every $x \in E$, $Tx = y_n$ implies $\|x\| > n\|y_n\|$. Choose any $x_n \in E$ satisfying $Tx_n = y_n$. Since $y_n \neq 0$, x_n does not belong to the closed set $\ker(T)$ and hence $d_n = d(x_n, \ker T) > 0$. There is $z_n \in \ker(T)$ such that $\|x_n - z_n\| \leq 2d_n$. Let $b_n = \dfrac{x_n - z_n}{\|x_n - z_n\|}$. Now $T(x_n - z_n) = Tx_n = y_n$ implies $\|x_n - z_n\| > n\|y_n\|$, or

$$\|Tb_n\| = \frac{\|T(x_n - z_n)\|}{\|x_n - z_n\|} = \frac{\|y_n\|}{\|x_n - z_n\|} < \frac{1}{n}.$$

Therefore $Tb_n \to 0$ as $n \to \infty$. On the other hand, since $\{b_n\}$ is bounded and A is a compact operator, there is a subsequence $\{b_{n(j)}\}$ and some $b \in E$ such that $Ab_{n(j)} \to b$ in E as $j \to \infty$. Consequently $b_{n(j)} = Tb_{n(j)} + Ab_{n(j)} \to b$. Observe that $Tb = \lim Tb_{n(j)} = 0$, i.e. $b \in \ker(T)$. Now

$$\|b_n - b\| = \left\|\frac{x_n - z_n}{\|x_n - z_n\|} - b\right\| = \frac{\|x_n - (z_n - \|x_n - z_n\|b)\|}{\|x_n - z_n\|} \geq \frac{d_n}{\|x_n - z_n\|} \geq \frac{1}{2}$$

contradicts to $b_{n(j)} \to b$. This completes the proof. □

12-2.7. **Theorem** $\operatorname{Im}(T)$ is closed in E.

Proof. Let y be a closure point of $T(E)$ in E. There is a sequence $\{y_n\}$ in $T(E)$ convergent to y. Choose $\lambda > 0, x_n \in E$ such that $Tx_n = y_n$ and $\|x_n\| \leq \lambda\|y_n\|$. Since $y_n \to y$, $\{y_n\}$ is a bounded sequence and hence so is $\{x_n\}$. Because A is a compact operator, there is a subsequence $\{x_{n(j)}\}$ such that $\{Ax_{n(j)}\}$ converges. Hence $x_{n(j)} = Tx_{n(j)} + Ax_{n(j)} \to x$ for some $x \in E$. Then we have $y = \lim y_{n(j)} = \lim Tx_{n(j)} = Tx$. Therefore $y \in T(E)$. Consequently, $\operatorname{Im}(T)$ is closed in E. □

12-2.8. **Theorem** $\operatorname{Im}(T) = (\ker T^t)^\perp$ and $\operatorname{Im}(T^t) = (\ker T)^\perp$.

Proof. It is because $\operatorname{Im}(T)$ is closed. □

12-2.9. **Fredholm Alternatives** The following statements are equivalent:

(a) T is bijective.

(b) T is surjective.

(c) T is injective.

Proof. It is equivalent to the proposition: $\text{Im}(T) = E$ iff $\ker(T) = \{0\}$.

($\Rightarrow$) Suppose to the contrary that $\text{Im}(T) = E$ and $\ker(T) \neq \{0\}$. Then there is a non-zero $a \in E$ but $Ta = 0$. By $\text{Im}(T) = E$, there is $x_1 \in E$ such that $a = Tx_1$. Inductively, there is $x_n \in E$ such that $x_{n-1} = Tx_n$. Now for each n, we have $T^n x_n = T^{n-1}(Tx_n) = T^{n-1}x_{n-1} = \cdots = Tx_1 = a \neq 0$, i.e. $x_n \notin \ker(T^n)$. Next, $T^{n+1}x_n = T(T^n x_n) = Ta = 0$, i.e. $x_n \in \ker(T^{n+1})$. This shows $\ker(T^n) \neq \ker(T^{n+1})$ for all n which is a contradiction.

($\Leftarrow$) Suppose $\ker(T) = \{0\}$. Then $\text{Im}(T^t) = E'$, i.e. $T^t = I - A^t$ is surjective. Since A^t is a compact operator on E', we can apply the first part to A^t. Hence T^t is injective, i.e. $\ker(T^t) = \{0\}$. Consequently, we have

$$\text{Im}(T) = (\ker\ T^t)^{\perp} = E. \qquad \square$$

12-2.10. **Theorem** Both $\ker(T)$ and $\ker(T^t)$ have the same finite dimension.

Proof. We have prove that $\ker(T)$ is finite dimensional. Let $a_1, a_2, \cdots, a_n$ be a basis of $\ker(T)$. There are $u_1, u_2, \cdots, u_n \in E'$ such that $u_i(a_j) = \delta_{ij}$ for all $i, j = 1, 2, \cdots, n$. We claim

$$\dim\ \ker(T^t) \leq \dim\ \ker(T). \qquad \#1$$

Suppose this is false. There are $n + 1$ independent vectors $v_1, v_2, \cdots, v_n, v_{n+1}$ in $\ker(T^t)$. Choose $b_1, b_2, \cdots, b_{n+1} \in E$ such that $v_i(b_j) = \delta_{ij}$ for all $i, j = 1, 2, \cdots, n + 1$. Define $Bx = Ax + \sum_{j=1}^n u_j(x)b_j, \forall\ x \in E$. Then B is a compact operator in E. We also claim

$$\ker(I - B) = \{0\}. \qquad \#2$$

Indeed, assume $x \in \ker(I - B)$. Then $Tx = (I - A)x = \sum_{j=1}^n u_j(x)b_j$. By $v_j \in \ker(T^t)$, we have

$$\begin{aligned} 0 &= < T^t v_i, x > = < v_i, Tx > \\ &= \left\langle v_i, \sum_{j=1}^n u_j(x)b_j \right\rangle = \sum_{j=1}^n u_j(x)v_i(b_j) = \sum_{j=1}^n u_j(x)\delta_{ij} = u_i(x). \end{aligned}$$

Hence $Tx = \sum_{j=1}^n u_j(x)b_j = 0$, or $x \in \ker(T)$. Since $a_1, a_2, \cdots, a_n$ is a basis of $\ker(T)$, write $x = \sum_{j=1}^n \alpha_j a_j$ for some scalar α_j. Now observe

$$0 = u_i(x) = \sum_{j=1}^n \alpha_j u_i(a_j) = \alpha_j$$

for all j, i.e. $x = 0$. Consequently #2 is proved. As a result, $\text{Im}(I - B) = E$. In particular, we get $b_{n+1} \in \text{Im}(I - B)$. There is $x \in E$ such that $b_{n+1} = (I - B)x$. Consider the calculation:

$$\begin{aligned} 1 &= v_{n+1}(b_{n+1}) = v_{n+1}(I - B)x \\ &= v_{n+1}\left[(I - A)x - \sum_{j=1}^{n} u_j(x)b_j\right] = < v_{n+1}, (I - A)x > - \sum_{j=1}^{n} u_j(x)v_{n+1}(b_j) \\ &= < T^t v_{n+1}, x > - \sum_{j=1}^{n} u_j(x)\delta_{n+1,j} = < 0, x > - \sum_{j=1}^{n} u_j(x) \cdot 0 = 0. \end{aligned}$$

This contradiction establishes #1. For the reverse inequality, applying #1 to A^t, we have $\dim\ \ker(T^t) \geq \dim\ \ker(T^{tt}) \geq \dim\ \ker(T)$. This completes the proof. □

12-2.11. **Theorem** Both $\text{Im}(T)$ and $\text{Im}(T^t)$ have the same codimension. Furthermore, $\text{codim Im}(T) = \dim\ \ker(T^t) = \dim\ \ker(T) = \text{codim Im}(T^t)$ holds.

Proof. Since $(E/\text{Im } T)' = (\text{Im } T)^{\perp} = (\ker T^t)^{\perp\perp} = \ker(T^t)$, the space $(E/\text{Im } T)'$ is finite dimensional and hence it has the same dimension as $E/\text{Im } T$.

From $\quad \text{codim Im}(T) = \dim\ E/\text{Im } T = \dim\ (E/\text{Im } T)' = \dim\ \ker(T^t)$

and

$$\begin{aligned} &\text{codim Im}(T^t) = \dim(E/\text{Im } T^t) = \dim(E/\text{Im } T^t)' \\ &= \dim(\text{Im } T^t)^{\perp} = \dim(\ker T)^{\perp\perp} = \dim\ \ker(T), \end{aligned}$$

the proof is complete.

12-2.12. **Theorem** Let k be the ascent of T. Then we have

$$\text{Im}(T) \supsetneq \text{Im}(T^2) \supsetneq \cdots \supsetneq \text{Im}(T^k) = \text{Im}(T^n), \forall\ n > k.$$

As a result, we also call k the *descent* of T.

Proof. Clearly $\text{Im}(T^{n-1}) \supset \text{Im}(T^n)$. For all $n \leq k$, we have

$$\text{codim Im}(T^{n-1}) = \dim\ \ker(T^{n-1}) \neq \dim\ \ker(T^n) = \text{codim Im}(T^n).$$

Hence $\text{Im}(T^{n-1}) \neq \text{Im}(T^n)$. Next, for all $n > k$, we get

$$\text{codim Im}(T^n) = \dim\ \ker(T^n) = \dim\ \ker(T^k) = \text{codim Im}(T^k).$$

Therefore $\text{Im}(T^n) = \text{Im}(T^k)$. □

12-2.13. **Exercise** Let A be a compact operator on a Banach space E and $T = I - A$ its displacement operator. Prove that if T is a topological isomorphism, its inverse is also a displacement of some compact operator. Prove that composites of displacement operators are displacement operators.

12-2.14. **Exercise** Prove that $E = \text{Im}(T^k) \oplus \ker(T^k)$ where k is the ascent of T.

12-2.15. **Exercise** Prove that both $\mathrm{Im}(T^k)$ and $\ker(T^k)$ are closed invariant subspace of T, i.e. both are closed vector subspaces, $T[\mathrm{Im}(T^k)] \subset \mathrm{Im}(T^k)$ and $T[\ker(T^k)] \subset \ker(T^k)$.

12-2.16. **Exercise** Prove that the restriction $T|\mathrm{Im}(T^k)$ is an isomorphism.

12-2.17. **Exercise** Prove that the restriction $T|\ker(T^k)$ is nilpotent, i.e. $T|\ker(T^k)^p = 0$ for some integer p.

12-2.18. **Exercise** Prove that there is a topological isomorphism J and a finite dimensional operator K on E such that $T = J + K$.

12-2.19. **Exercise** Let E, F be Banach spaces and $A : E \to F$ a compact linear map. Prove that for every topological isomorphism $J : E \to F$, the set $(J - A)(E)$ is closed in F.

12-3 Spectrum of a Compact Operator

12-3.1. In this section, we shall prove that the spectrum of a compact operator is a finite set or a null sequence.

12-3.2. **Theorem** Let A be a compact operator on a Banach space E. Then every non-zero spectral value of A is an eigenvalue.

Proof. Suppose that λ is not an eigenvalue of A. Then $I - \lambda^{-1}A$ is injective. By Fredholm Alternative, $I - \lambda^{-1}A$ is invertible and hence $\lambda I - A$ is invertible. Consequently, λ is not a spectral value. □

12-3.3. **Lemma** For every $\varepsilon > 0$, the set $S = \{\lambda \in \sigma(A) : |\lambda| \geq \varepsilon\}$ is finite.

Proof. Suppose to the contrary that S is an infinite set. There is a sequence $\{\lambda_n\}$ of distinct points in $\sigma(A)$ with all $|\lambda_n| \geq \varepsilon$. Since each λ_n is an eigenvalue of A, there is non-zero $x_n \in E$ such that $Ax_n = \lambda_n x_n$. Let H_n be the vector space spanned by $x_1, x_2, \cdots, x_n$ and let $H_0 = \{0\}$. Then all H_n are closed in E. Since $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, $x_1, x_2, \cdots, x_n$ are linearly independent. Hence $H_{n-1} \subsetneq H_n$. Choose $y_n \in H_n$ such that $\|y_n\| = 1$ and $d(y_n, H_{n-1}) \geq \frac{1}{2}$. Observe that

$$\left\| \frac{y_n}{\lambda_n} \right\| = \frac{\|y_n\|}{|\lambda_n|} \leq \frac{1}{\varepsilon}.$$

Since A is a compact operator, the sequence $\left\{ A\left(\frac{y_n}{\lambda_n}\right) \right\}$ must have a convergent subsequence. To show that no subsequence can be Cauchy, take any $n > m$. Write $y_n = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Then we get

$$y_n - A\left(\frac{y_n}{\lambda_n}\right) = \sum_{j=1}^{n-1} \alpha_j \left(1 - \frac{\lambda_j}{\lambda_n}\right) x_j.$$

Hence deduce that

$$z = y_n - A\left(\frac{y_n}{\lambda_n}\right) + A\left(\frac{y_m}{\lambda_m}\right) \in H_{n-1}.$$

Therefore

$$\left\| A\left(\frac{y_n}{\lambda_n}\right) - A\left(\frac{y_m}{\lambda_m}\right) \right\| = \|y_n - z\| \geq d(y_n, H_{n-1}) \geq \frac{1}{2}.$$

This contradiction shows that S must be finite. □

12-3.4. **Theorem** The spectrum of A is a countable set of scalars. If the spectrum is an infinite set, its points converge to $0 \in \mathbb{K}$.

Proof. Let $D_n = \{\lambda \in \sigma(A) : |\lambda| \geq \frac{1}{n}\}$. Then all D_n are finite. Since $\sigma(A) \subset \{0\} \cup \bigcup_{n=1}^{\infty} D_n$, $\sigma(A)$ is countable. Next, if $\sigma(A)$ is an infinite set, clearly it converges to $0 \in \mathbb{K}$ because all except those in D_n belong to the ball $\mathbb{B}(0, \frac{1}{n})$. □

12-3.5. **Theorem** If E is *infinite dimensional*, then we have $0 \in \sigma(A)$.

Proof. Suppose to the contrary that zero is not a spectral value of A. Then $\lambda = 0$ is a resolvent value. Hence $A = -(\lambda I - A)$ is invertible. Since A is compact, so is $I = A^{-1}A$. Let V be the closed unit ball of E. Then $V = I(V)$ is a closed relatively compact set in E and hence it is compact. Consequently, V is finite dimensional. □

12-4 Existence of Invariant Subspaces

12-4.1. Normally we start with a nonlinear function $y = f(x)$ and then approximate f by linear maps through derivatives or affine maps and hopefully we can get some information of f from its linear approximations. It was Lomonosov that applied nonlinear technique to solve linear problem.

12-4.2. Let T be an operator on a complex Banach space E. For finite dimensional E, we can identify T as a square matrix from which all eigenvalues are provided by its characteristic equations. Each eigenspace is an invariant subspace of T and consequently T can be expressed as direct sum of relatively simple submatrices. For example, it is well-known in linear algebra that every square matrix is similar to a Jordan form. For infinite dimensional case, it seems to be natural to ask whether what classes of operators has non-trivial invariant subspaces. One class including the compact operators will be presented in this section.

12-4.3. Let E be a *complex* Banach normed space, $L(E)$ the set of all operators on E and $T \in L(E)$. A vector subspace M of E is called
(a) an *invariant subspace* of T if $T(M) \subset M$,
(b) a *hyperinvariant subspace* if for every operator $S \in L(E)$ commuting with T, i.e. $ST = TS$, we have $S(M) \subset M$ and
(c) an *invariant subspace* of a subalgebra $\mathcal{A}$ of $L(E)$ if $T(M) \subset M$ for every $T \in \mathcal{A}$.

12-4.4. A subspace M of E is said to be *trivial* if either $M = \{0\}$ or $M = E$. A subalgebra $\mathcal{A}$ of $L(E)$ is said to be *transitive* if every invariant subspace of $\mathcal{A}$ is trivial. In this section, we assume that the dimension of E is non-zero.

12-4.5. **Lemma** Let $\mathcal{A}$ be a transitive algebra of operators on E. Then for every non-zero compact operator K, there is an operator $T \in \mathcal{A}$ such that the compact operator KT has a non-zero fixed point.

Proof. Let V denote the *open* unit ball of E. Since the closure $\overline{K(V)}$ is compact, it is bounded. There is $\lambda \in \mathbb{R}$ such that $\|K(V)\| < \lambda$. By $K \neq 0$, there is $a \in E$ satisfying $\|K(a)\| > \lambda$. Let $X = K(a) + \overline{K(V)}$. Then X is a non-empty compact convex set which does not contain the origin. Next, we claim that $X \subset \bigcup_{T\in\mathcal{A}} T^{-1}(a+V)$. Suppose to the contrary that there is $y \in X$ but $y \notin T^{-1}(a+V)$ for all $T \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, $\mathcal{A}y = \{T(y) : T \in \mathcal{A}\}$ forms a vector subspace of E and so does its closure $M = \overline{\mathcal{A}y}$. Observe that for each $S \in \mathcal{A}$, we get $S(M) \subset S(\overline{\mathcal{A}y}) \subset \overline{S(\mathcal{A}y)} = \overline{\{STy : T \in \mathcal{A}\}} \subset \overline{\mathcal{A}y} = M$ because $\mathcal{A}$ is an algebra. Therefore M is an invariant subspace of $\mathcal{A}$. By transitivity of $\mathcal{A}$, we have either $M = \{0\}$ or $M = E$. Suppose $M = \{0\}$. Then $T(y) = 0$ for all $T \in \mathcal{A}$. Since $E \neq \{0\}$, take any $y \neq 0$ in E. Then $\mathbb{C}y = \{\alpha y : \alpha \in \mathbb{C}\}$ is a non-trivial invariant subspace of $\mathcal{A}$, contrary to the transitivity of $\mathcal{A}$. Next, suppose $M = E$. Since $T(y) \notin a + V$ for all $T \in \mathcal{A}$, the set $\mathcal{A}y$ is contained in the closed set $E \setminus (a+V)$, i.e. $M \subset E \setminus (a+V)$, contrary to $M = E$. Therefore $\{T^{-1}(a + V) : T \in \mathcal{A}\}$ is an open cover of the compact set X. We can select a finite number of $T_j \in \mathcal{A}$ such that $X \subset \bigcup_{j=1}^{n} N_j$ where $N_j = T_j^{-1}(a + V)$. Let $\{\alpha_j\}$ be a partition of unity on X subordinated to $\{N_j : 1 \leq j \leq n\}$. Define $f(x) = \sum_{j=1}^{n} \alpha_j(x)KT_j(x)$ for every $x \in X$. Then

$f : X \to E$ is a continuous map. Note that if $\alpha_j(x) \neq 0$, then $x \in N_j$, or $T_j(x) \in a + V$, i.e. $KT_j(x) \in K(a+V) \subset \overline{\{K(a+V)\}} = X$. Therefore $f(X) \subset X$. Now the continuous map $f : X \to X$ has a fixed point, say $b \in X$. Since $0 \notin X$, we have $b \neq 0$. From $f(b) = b$, we get $\sum_{j=1}^{n} \alpha_j(b)KT_j(b) = b$, i.e. $KT(b) = b$ where $T = \sum_{j=1}^{n} \alpha_j(b)T_j \in \mathcal{A}$. □

12-4.6. **Lomonosov's Theorem** If a non-scalar operator A commutes with a non-zero compact operator K, then A has a non-trivial hyperinvariant subspace.

Proof. Let $\mathcal{A} = \{T \in L(E) : AT = TA\}$. Clearly, $\mathcal{A}$ is a subalgebra of $L(E)$. Suppose to the contrary that every hyperinvariant subspace of A is trivial. Then $\mathcal{A}$ is transitive. There exist $T \in \mathcal{A}$ and $b \in E$ such that $KT(b) = b \neq 0$ and $b \in H = \ker(KT - I)$. Since K is compact, so is KT and thus H is finite dimensional. For every $x \in H$, we have $KT(x) = x$, or $KTA(x) = AKT(x)$, i.e. $A(x) \in H$. Therefore H is a finite dimensional invariant subspace of A. It follows that A has a non-zero eigenvector in H, i.e. there is $\lambda \in \mathbb{C}$ such that $M = \ker(A - \lambda I) \neq \{0\}$. Since A is non-scalar, we have $M \neq E$. So, M is non-trivial closed vector subspace of E. Now take any $x \in M$ and $T \in \mathcal{A}$. Then $AT(x) = TA(x) = T(\lambda x) = \lambda T(x)$, i.e. $T(x) \in M$. Hence M is a non-trivial hyperinvariant subspace of A. This contradiction establishes the proof. □

12-5 Fredholm Operators

12-5.1. Let E, F be Banach spaces. A continuous linear map $f : E \to F$ is called a *Fredholm* map if $\ker(f)$ is of finite dimension and the image $f(E)$ is of finite codimension. All Fredholm maps in this book are linear although they were initiated by the study of nonlinear maps whose derivatives are Fredholm. Readers may skip this section without any discontinuity.

12-5.2. Let f be a Fredholm map from E into F. Any topological complement $E \ominus \ker(f)$ of the finite dimensional subspace $\ker(f)$ in E is called a *mirror* of f. Any topological complement $F \ominus f(E)$ of the finite codimensional subspace $f(E)$ is called a *deficit* of f. Define $\text{nullity}(f) = \dim \ker(f)$, $\text{defect}(f) = \dim F/f(E)$ and $\text{index}(f) = \text{nullity}(f) - \text{defect}(f)$. Clearly, we have $\text{defect}(f) = \dim \text{deficit}(f)$.

12-5.3. **Lemma** (a) $f(E)$ is closed in F.
(b) The restriction of f is a topological isomorphism $\text{mirror}(f) \simeq F(E)$.

(c) If the index of f is zero, then $\ker(f)$ and $\text{deficit}(f)$ are of the same finite dimension and consequently $E = \text{mirror}(f) \oplus \ker(f)$ is topologically isomorphic to $F = f(E) \oplus \text{deficit}(f)$.
(d) If A is a compact operator on E, then $T = I - A$ is a Fredholm operator with index zero.
(e) If E is finite dimensional, then all linear maps are Fredholm and they have the same index $\dim E - \dim F$. Hence Fredholm maps are interesting only if E is infinite dimensional.

Proof. (a) See §7-4.14.
(b) By §7-4.6c, $\text{mirror}(f) = E \ominus \ker(f) \simeq E/\ker(f)$. By §§7-2.5b, 6-9.3; $E/\ker(f) \simeq f(E)$.
(d) See §12-2.11.
(e)

$$\begin{aligned} \text{index}(f) &= \text{nullity}(f) - \text{defect}(f) \\ &= [\dim \text{mirror}(f) + \dim \ker(f)] - [\dim f(E) + \dim \text{deficit}(f)] \\ &= \dim E - \dim F. \end{aligned}$$

□

12-5.4. **Theorem** Let E, F, G be Banach spaces and let $f : E \to F$, $g : F \to G$ be Fredholm maps. Then the composite gf is also a Fredholm map. Furthermore we have $\text{index}(gf) = \text{index}(g) + \text{index}(f)$.

Proof. Let $A = f(E) \cap \ker(g)$, $B = f(E) \ominus A$ and $C = \ker(g) \ominus A$. Then $f(E) = A \oplus B$. Since $f(E)$ is closed and B is finite dimensional, $f(E) + C$ is closed and of finite codimension. Let $D = F \ominus [f(E) + C]$, $M = \text{mirror}(f)$, $P = (f|M)^{-1}(A)$ and $Q = (f|M)^{-1}(B)$. Since $f|M$ is a topological isomorphism, we obtain $M = P \oplus Q$ and $E = \ker(f) \oplus P \oplus Q$. Take any $x = p + q + r$ where $p \in P$, $q \in Q$ and $r \in \ker(f)$. Since $f(p) \in A \subset \ker(g)$, we have $gf(p) = 0$ and also $f(r) = 0$. Observe that $f|Q$ is an isomorphism. From $B \cap \ker(g) = \{0\}$, $g|B$ is an isomorphism. Thus $gf(x) = gf(q) = 0$ iff $q = 0$, i.e. $x = p + r$. Hence $\ker(gf) = P \oplus \ker(f)$ is finite dimensional. Next, since $f(E) \cap D = \{0\}$ we have

$$\begin{aligned} g(F) &= g[f(E) + C + D] = gf(E) \oplus g(D) = g(A) \oplus g(B) \oplus g(D) \\ &= g(B) \oplus g(D) = gf(Q) \oplus g(D). \end{aligned}$$

Therefore $\text{deficit}(gf) = g(D) \oplus \text{deficit}(g)$ is also finite dimensional. Consequently, gf is a Fredholm map. Furthermore we obtain

$$\begin{aligned} \text{index}(gf) &= \text{nullity}(gf) - \text{defect}(gf) \\ &= \dim \ker(gf) - \dim \text{deficit}(gf) \\ &= [\dim P + \dim \ker(f)] - [\dim g(D) + \dim \text{deficit}(g)] \\ &= [\dim A + \text{nullity}(f)] - [\dim(D) + \text{defect}(g)] \end{aligned}$$

$$= [\dim A + \dim C + \text{nullity}(f)] - [\dim C + \dim D + \text{defect}(g)]$$
$$= [\dim \ker(g) + \text{nullity}(f)] - [\text{defect}(f) + \text{defect}(g)]$$
$$= \text{index}(g) + \text{index}(f). \qquad \square$$

12-5.5. A continuous linear map $f : E \to F$ is ***pseudo-invertible*** if there is a continuous linear map $g : F \to E$ such that $fgf = f$. In this case, g is called a ***pseudo-inverse*** of f. Clearly both fg, gf are idempotents. Also observe that $\ker(gf) = \ker(f)$ and $fg(F) = f(E)$.

12-5.6. **Theorem** Let $f : E \to F$ be a Fredholm map. Then there is a Fredholm map $g : F \to E$ which is also a pseudo-inverse of f such that both $I - fg$ and $I - gf$ are of finite rank. Furthermore, if $\text{index}(f) = 0$ then we may assume that g is invertible.

Proof. Let $M = \text{mirror}(f)$ and $D = \text{deficit}(f)$. Define $\xi : D \to \ker(f)$ as any isomorphism if $\text{index}(f) = 0$ and define $\xi = 0$ otherwise. Next define

$$g : F = f(E) \oplus D \to E = M \oplus \ker(f)$$

by $g|f(E) = (f|M)^{-1}$ and $g|D = \xi$. Clearly g is a continuous linear map and it is a topological isomorphism if $\text{index}(f) = 0$. It is simple to verify $fgf = f$ on E. Since $\ker(g) \subset D$ and $\text{deficit}(g) \subset \ker(f)$, g is a Fredholm map. Next, $f(I - gf)E = (f - fgf)E = 0$ gives $(I - gf)E \subset \ker(f)$. Finally, observe that $(I - fg)F = (I - fg)[f(E) \oplus D] = (I - fg)D$. Therefore both $I - gf$ and $I - fg$ are of finite rank. This completes the proof. $\square$

12-5.7. A Banach space L which is also an algebra. It is called a ***Banach algebra*** if for all $A, B \in L$ we have $\|AB\| \leq \|A\|\ \|B\|$. If L has a multiplicative identity I, we also demand $\|I\| = 1$. Most results of $L(E)$ could have been developed in the general framework of Banach algebra.

12-5.8. **Exercise** Prove that the set of all invertible elements in L is an open set. See §8-6.4.

12-5.9. **Exercise** Prove that if K is a closed two-sided ideal of a Banach algebra L, then the quotient algebra L/K is also a Banach algebra. See §7-2.4.

12-5.10. Let $K(E)$ denote the set of all compact operators on E. It follows from §12-1.6,10 that $K(E)$ is a closed two-sided ideal of the algebra $L(E)$ of all operators on E. Let φ denote the quotient map from $L(E)$ onto the quotient Banach algebra $L(E)/K(E)$ which is also called the ***Calkin algebra***.

12-5.11. **Atkinson's Theorem** An operator $f \in L(E)$ is Fredholm iff $\varphi(f)$ is invertible in the quotient algebra $L(E)/K(E)$.

Proof. We shall denote the identity map on E and $L(E)/K(E)$ by the same symbol I. Let f be a Fredholm operator on E. There is a Fredholm operator g on E such that $I - fg$ and $I - gf$ are of finite rank and hence they are compact operators. Now $0 = \varphi(I - fg) = I - \varphi(f)\varphi(g)$ gives $\varphi(f)\varphi(g) = I$. Similarly, $\varphi(g)\varphi(f) = I$. Therefore $\varphi(f)$ is invertible in $L(E)/K(E)$ with inverse $\varphi(g)$. Conversely, suppose that $g \in L(E)$ satisfies $\varphi(f)\varphi(g) = \varphi(g)\varphi(f) = I$ on $L(E)/K(E)$. Then $h = I - fg$ and $k = I - gf$ are compact operators on E. Therefore $\ker(f) \subset \ker(gf) = \ker(I - k)$ which is finite dimensional. Also from $f(E) \supset (fg)E = (I - h)E$ which is of finite codimension. Therefore $\ker(f)$ is finite dimensional and $f(E)$ is finite codimensional. Consequently, f is a Fredholm map. □

12-5.12. **Theorem** The set $\mathcal{F}(E)$ of all Fredholm operators is open in $L(E)$ and the map $\text{index} : \mathcal{F}(E) \to \mathbb{Z}$ is a continuous function.

Proof. Let V denote the open set of all invertible elements in the quotient algebra $L(E)/K(E)$. By continuity of the quotient map $\varphi : L(E) \to L(E)/K(E)$, the set $\mathcal{F}(E) = \varphi^{-1}(V)$ is open in $L(E)$. Now take any Fredholm operator f on E. Choose a pseudo-inverse g such that $h = I - fg$ and $k = I - gf$ are compact operators. Suppose that $p \in L(E)$ satisfies $\|f - p\| < \|g\|^{-1}$. Then $\|fg - pg\| < 1$. Hence $q = I - (fg - pg)$ is invertible in $L(E)$ and so $\text{index}(q) = 0$. From $qf = f - fgf + pgf = pgf$, we have

$$\text{index}(q) + \text{index}(f) = \text{index}(p) + \text{index}(g) + \text{index}(f)$$

or, $\text{index}(p) = -\text{index}(g)$. Therefore $\text{index}(f)$ is locally constant. Consequently, $\text{index}(f)$ is a continuous function of f. □

12-5.13. **Theorem** For every Fredholm operator f and every compact operator k on E, $f + k$ is a Fredholm operator. Furthermore we have $\text{index}(f + k) = \text{index}(f)$.

Proof. Let $\varphi : L(E) \to L(E)/K(E)$ be the quotient map. Then $\varphi(f+k) = \varphi(f)$ is invertible in $L(E)/K(E)$. Hence $f + k$ is a Fredholm map. Next, since the function $\xi(t) = \text{index}(f + tk)$ from $[0, 1]$ into $\mathbb{Z}$ is continuous, it must be a constant, i.e. $\text{index}(f) = \xi(0) = \xi(1) = \text{index}(f + k)$. □

12-5.14. **Corollary** An operator f is a sum of an invertible operator and a compact operator on E iff $\text{index}(f) = 0$.

Proof. Let $f = g + k$ where g is an invertible operator and k is a compact operator. Then we have $\text{index}(f) = \text{index}(g + k) = \text{index}(g) = 0$. Conversely, suppose $\text{index}(f) = 0$. Let g be an invertible Fredholm pseudo-inverse of f. Then from $f = fgf$, we have $\varphi(f) = \varphi(f)\varphi(g)\varphi(f)$. Because $\varphi(f)$ is invertible in $L(E)/K(E)$, we have $\varphi(f)\varphi(g) = \varphi(g)\varphi(f) = I$, or $\varphi(f) = \varphi(g)^{-1} = \varphi(g^{-1})$. Hence $k = f - g^{-1}$ is compact, or $f = g^{-1} + k$ is the sum of an invertible operator and a compact operator. □

12-99. **References and Further Readings** : Jorgens, Schechter, Ringrose, Collins, Vala, Dieudonne-85, Johnson-79, Szankowski, Kaneko, Konig-86, Ma-86, Sikorski, Simon-79, Corduneanu, Lomonosov, Simonivc, Sadovnichii, Olagunju, Pietsch, Schmitt, Caradus, Enflo, Wu and Han.

Chapter 13

Operators on Hilbert Spaces

13-1 Complex Inner Product Spaces

13-1.1. This chapter is a natural extension of linear algebra of matrices to infinite dimensional spaces and should be read concurrently with Chapter 3 on Banach spaces. Most of the elementary properties hold for *real* Hilbert spaces even though we restrict ourselves to complex case in order to prepare the background so that you can start to work on C^*-algebras and analytic functions of operators elsewhere as soon as possible.

13-1.2. Let H be a vector space over the *complex* field $\mathbb{C}$. A function $(x,y) \to < x,y >$ from $H \times H$ into $\mathbb{C}$ is called an *inner product* on H if for all $x,y,z \in H$ and for all $\lambda \in \mathbb{C}$, we have
(a) $< x+y,z > = < x,z > + < y,z >$, additive;
(b) $< \lambda x,y > = \lambda < x,y >$, homogeneous;
(c) $< x,y > = < y,x >^-$, conjugate-symmetric;
(d) $< x,x > \geq 0$, positive;
(e) if $< x,x > = 0$, then $x = 0$; non-degenerate.

13-1.3. A vector space equipped with a given inner product is called an *inner product space*. In an inner product space H, write $\|x\| = \sqrt{< x,x >}, \forall\, x \in H$. We shall use the notation $i^2 = -1$ consistently in the subsequent context.

13-1.4. **Theorem** Let H be an inner product space. For all $x,y,z \in H$ and for all complex numbers α, β, we have
(a) Conjugate linear: $< z, \alpha x + \beta y > = \alpha^- < z,x > + \beta^- < z,y >$.
(b) Parallelogram Law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
(c) Polarization Formula:

$$4 < x,y > = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = \sum_{n=0}^{3} i^n \|x+i^n y\|^2.$$

13-1.5. **Cauchy-Schwartz Inequality** For all $x,y \in H$, we have

$$| < x,y > | \leq \|x\|\ \|y\|.$$

Furthermore $| < x, y > | = \|x\| \; \|y\|$ iff $x = \lambda y$ or $y = \lambda x$ for some $\lambda \in \mathbb{C}$.

Proof. If $y = 0$, then both sides are zero and the result follows immediately. Assume $y \neq 0$. Then $< y, y > \neq 0$. Let $\lambda = \dfrac{< x, y >}{< y, y >}$. Then we have

$$\begin{aligned} 0 \leq \; & < x - \lambda y, x - \lambda y > \\ = & < x, x > - \lambda^{-} < x, y > - \lambda < y, x > + \lambda\lambda^{-} < y, y > \\ = & < x, x > - \frac{< x, y >< y, x >}{< y, y >}. \qquad \#1 \end{aligned}$$

Therefore we have $| < x, y > |^2 = < x, y >< y, x > \leq < x, x >< y, y >$. Finally, #1 is zero iff $| < x, y > | = \|x\| \; \|y\|$ iff $x - \lambda y = 0$. □

13-1.6. **Theorem** Every inner product space H is a normed space under the norm $\|x\| = \sqrt{< x, x >}, \forall \, x \in H$.

Proof. We shall verify only the triangular inequality and leave the others as an exercise. Let $x, y \in H$ be given. Then we have

$$\begin{aligned} & \|x + y\|^2 = < x + y, x + y > = < x, x > + < x, y > + < y, x > + < y, y > \\ & = < x, x > + 2Re < x, y > + < y, y > \leq < x, x > + 2| < x, y > | + < y, y > \\ & \leq \|x\|^2 + 2\|x\| \; \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2, \end{aligned}$$

i.e.

$$\|x + y\| \leq \|x\| + \|y\|. \qquad \square$$

13-1.7. As a result, topological properties such as convergent sequences and continuity, are available in H. An inner product space which is also a complete normed space is called a *Hilbert space*.

13-1.8. **Exercise** Prove that in a normed space, if the Parallelogram Law holds, then the polarization formula defines an inner product which induces the original norm.

13-1.9. **Theorem** The inner product $(x, y) \to < x, y >$ from the product normed space $H \times H$ into $\mathbb{C}$ is continuous.

Proof. Suppose $x_n \to a$ and $y_n \to b$ in H. Then we have

$$\begin{aligned} & | < x_n, y_n > - < a, b > | \\ & = | < x_n - a, y_n - b > | + | < x_n - a, b > | + | < a, y_n - b > | \\ & \leq \|x_n - a\| \; \|y_n - b\| + \|x_n - a\| \; \|b\| + \|a\| \; \|y_n - b\| \to 0 \text{ as } n \to \infty. \qquad \square \end{aligned}$$

13-1.10. **Example** For all $x, y \in \ell_2$, let $< x, y > = x_1 y_1^{-} + x_2 y_2^{-} + \cdots$ where y_j^{-} denotes the complex conjugate of y_j. It is trivial to verify that ℓ_2 is an inner product space. Since we know that ℓ_2 is complete, it is a Hilbert space. Note that the notation is different from normed spaces. Here we have to take the complex conjugate.

13-1.11. **Exercise** A bisequence is a sequence in $\mathbb{K}$ indexed by all integers rather then by positive integers only. A bisequence can be represented by

$$x = (\cdots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \cdots).$$

Prove that the set b_2 of all bisequences x satisfying $\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$ forms a Hilbert space under the coordinatewise operations and the inner product given by $< x, y > = \sum_{j=-\infty}^{\infty} x_j y_j^{-}$.

13-1.12. **Exercise** Let $P_2[-\pi, \pi]$ denote the vector space of all continuous functions f on $[-\pi, \pi]$ satisfying $f(-\pi) = f(\pi)$. With $< x, y > = \int_{\pi}^{\pi} x(t)y(t)^{-}dt$, show that $P_2[-\pi, \pi]$ is an inner product apace but not a Hilbert space.

13-1.13. **Exercise** Are $\mathbb{C}_1^2$ and $\mathbb{C}_\infty^2$ inner product spaces?

13-2 Orthogonality in Inner Product Spaces

13-2.1. Let H be an inner product space. Two vectors x, y in H are said to be *orthogonal* if $< x, y > = 0$. A vector x is said to be *orthogonal to a subset* B if x is orthogonal to every vector in B. A subset B of H is said to be *orthogonal* if distinct vectors in B are orthogonal. A subset of H is said to be *orthonormal* if it is an orthogonal set consisting of unit vectors only.

13-2.2. **Pythagora's Theorem** If $x \perp y$ in H, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

13-2.3. **Exercise** Let $x, y \in H$. Prove that $x \perp y$ iff we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 = \|x + iy\|^2.$$

13-2.4. **Theorem** Every orthogonal set B of non-zero vectors is linearly independent.

Proof. Let $b_1, b_2, \cdots, b_n$ be distinct vectors in B. Assume $\sum_{j=1}^{n} \lambda_j b_j = 0$ for some $\lambda_j \in \mathbb{C}$. Then for each k, we have

$$0 = \left\langle \sum_{j=1}^{n} \lambda_j b_j, b_k \right\rangle = \sum_{j=1}^{n} \lambda_j < b_j, b_k > = \lambda_k < b_k, b_k >.$$

Since $b_k \neq 0$, we have $\lambda_k \neq 0$. Therefore $b_1, b_2, \cdots, b_n$ are linearly independent. Consequently, B is linearly independent. □

13-2.5. **Theorem** Let $e_1, e_2, \cdots, e_n$ be an orthonormal set and suppose a vector $x \in H$ is given.

(a) For all complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, we have

$$\left\| x - \sum_{j=1}^{n} \lambda_j e_j \right\|^2 = \|x\|^2 + \sum_{j=1}^{n} |< x, e_j > - \lambda_j|^2 - \sum_{j=1}^{n} |< x, e_j >|^2.$$

(b) $\sum_{j=1}^{n} |< x, e_j >|^2 \leq \|x\|^2$; Bessel's inequality.

Proof. Part (a) follows from direct routine calculation. Part (b) follows from (a) by choosing $\lambda_j = < x, e_j >$. □

13-2.6. **Theorem** Let B be an orthonormal set in an inner product space H. Suppose that a vector $x \in H$ is given. Then

(a) the set $\{b \in B :< x, b > \neq 0\}$ is countable; and

(b) $\sum_{b \in B} |< x, b >|^2 \leq \|x\|^2$.

Proof. (a) Let $D_n = \left\{ b \in B : |< x, b >|^2 > \frac{1}{n}\|x\|^2 \right\}$. Then D_n cannot have more than n elements. Therefore $B = \bigcup_{n=1}^{\infty} D_n$ is countable.

(b) It follows immediately from Part (b) of last theorem. □

13-2.7. **Orthonormalization Process** Let $\{x_n\}$ be a sequence, finite or infinite, of linearly independent vectors in an inner product space H. Then there is an orthonormal sequence $\{e_n\}$ in H such that for each k, both $\{x_1, x_2, \cdots, x_k\}$ and $\{e_1, e_2, \cdots, e_k\}$ generate the same vector subspace of H.

Proof. Let $G(z_1, z_2, \cdots, z_k)$ denote the vector subspace spanned by the vectors $z_1, z_2, \cdots, z_k \in H$. Let $e_1 = x_1/\|x_1\|$. Suppose $e_1, e_2, \cdots, e_k$ have been constructed by induction. Let $a_{k+1} = x_{k+1} - \sum_{k=1}^{k} < x_{k+1}, e_j > e_j$. Suppose to the contrary that $a_{k+1} = 0$. Then $x_{k+1} \in G(e_1, e_2, \cdots, e_k) = G(x_1, x_2, \cdots, x_k)$ gives the linear dependence of $x_1, x_2, \cdots, x_k, x_{k+1}$ which is a contradiction. Hence $a_{k+1} \neq 0$. Define $e_{k+1} = a_{k+1}/\|a_{k+1}\|$. Clearly, $\|e_{k+1}\| = 1$. Also for $1 \leq p \leq k$, we have

$$< e_{k+1}, e_p > =< x_{k+1}, e_p > - \sum_{j=1}^{k} < x_{k+1}, e_j >< e_j, e_p >$$

$$= < x_{k+1}, e_p > - \sum_{j=1}^{k} < x_{k+1}, e_j > \delta_{jp} = < x_{k+1}, e_p > - < x_{k+1}, e_p > = 0.$$

Thus $e_1, e_2, \cdots, e_k, e_{k+1}$ are orthonormal. Now the following calculation completes the proof

$$G(e_1, e_2, \cdots, e_k, e_{k+1}) = G[G(e_1, e_2, \cdots, e_k), e_{k+1}]$$

$$= G\left[G(e_1, e_2, \cdots, e_k), x_{k+1} - \sum_{k=1}^{k} < x_{k+1}, e_j > e_j\right]$$

$$= G[G(e_1, e_2, \cdots, e_k), x_{k+1}] = G[G(x_1, x_2, \cdots, x_k), x_{k+1}]$$

$= G(x_1, x_2, \cdots, x_k, x_{k+1}).$ □

13-2.8. **Theorem on Minimum Distance** Let M be a complete convex subset of H. Then for every $x \in H$, there is a unique $y \in M$ such that $\|x - y\|$ is the distance from x to M. See §5-2.6.

13-2.9. **Theorem** Let M be a complete vector subspace of H and let $x \in H$, $y \in M$. Then $x - y$ is orthogonal to M iff $\|x - y\|$ is the distance from x to M.

Proof. ($\Leftarrow$) Suppose $\|x - y\|$ is the distance d from x to M. Let $z = x - y$. Then for all $0 \neq a \in M$, we have

$$d^2 \leq \left\| x - \left(y + \frac{< z, a >}{\|a\|^2} a \right) \right\|^2 = \left\| z - \frac{< z, a >}{\|a\|^2} a \right\|^2$$
$$= \|z\|^2 - | < z, a > |^2 = d^2 - | < z, a > |^2,$$

i.e. $< z, a > = 0$. Therefore $x - y = z$ is orthogonal to M.
($\Rightarrow$) Suppose $x - y$ is orthogonal to the complete convex set M. There is $z \in M$ satisfying $\|x - z\| = d$. By part (a), we have $(x - z) \perp M$. Now observe

$$\|x - z\|^2 = \|(x - y) + (y - z)\|^2 = \|x - y\|^2 + \|y - z\|^2,$$

and $$\|x - y\|^2 = \|(x - z) + (z - y)\|^2 = \|x - z\|^2 + \|z - y\|^2.$$

Adding them together, we have $\|y - z\|^2 = 0$, i.e. $y = z$. Therefore the distance from x to M is $\|x - y\| = \|y - z\| = d$. □

13-2.10. **Theorem** Let M, N be complete subsets of H. If $M \perp N$, then the set $M + N$ is also complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in $M + N$. Then there is $a_n \in M$ and $b_n \in N$ such that $x_n = a_n + b_n$. For any $\varepsilon > 0$, there is an integer p such that for all $m, n \geq p$, we have $\|x_m - x_n\| \leq \varepsilon$. Because $a_m - a_n$ is orthogonal to $b_m - b_n$, we have $\varepsilon^2 \geq \|x - m - x_n\|^2 = \|(a_m - a_n) + (b_m - b_n)\|^2 = \|a_m - a_n\|^2 + \|b_m - b_n\|^2$. Thus both $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences. Since M, N are complete, let $a = \lim a_n \in M$ and $b = \lim b_n \in N$. Then $\lim x_n = \lim a_n + \lim b_n = a + b \in M + N$. Therefore $M + N$ is also complete. □

13-3 Orthonormal Bases of Hilbert Spaces

13-3.1. An orthonormal set B in an inner product space H is said to be *maximal* if for every orthonormal set S containing B, we have $S = B$. A maximal orthonormal set is also called an *orthonormal basis*. Let B be an orthonormal basis of H. For each $x \in H$, the numbers $\{< x, b >: b \in B\}$ are called the *Fourier coefficients* and the sum $\sum_{b\in B} < x, b > b$ the *Fourier series.*

13-3.2. **Theorem** Every orthonormal set S in an inner product space H can be extended to an orthonormal basis.

Proof. Let $\mathbb{P}$ be the family of all orthonormal subsets of H. Then $\mathbb{P}$ becomes a partially order set under inclusion. For every chain C in $\mathbb{P}$, the union of all sets in C is an upper bound of C in $\mathbb{P}$. Hence by Zorn's Lemma, every orthonormal set is contained in a maximal orthonormal set, i.e. an orthonormal basis. □

13-3.3. **Pythagora's Theorem for Infinite Series** Let $\{x_n\}$ be an orthogonal sequence in a Hilbert space H.

(a) If $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, then the series $\sum_{n=1}^{\infty} x_n$ converges in H. Furthermore the sum $\sum_{n=1}^{\infty} x_n$ is independent of the order of enumeration.

(b) If the series $\sum_{n=1}^{\infty} x_n$ converges to $x \in H$, then we have $\|x\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2$.

Proof. Since $\{x_n\}$ is orthogonal, we obtain

$$\|x_{m+1} + x_{m+2} + \cdots + x_{m+p}\|^2 = \|x_{m+1}\|^2 + \|x_{m+2}\|^2 + \cdots + \|x_{m+3}\|^2$$
$$\leq \sum_{n=m+1}^{\infty} \|x_n\|^2 \to 0, \text{ as } m \to \infty$$

Hence the partial sums $\sum_{j=1}^{n} x_j$ is a Cauchy sequence in the Hilbert space H. Therefore the series $\sum_{j=1}^{\infty} x_j$ converges. The second assertion is obvious since we can rearrange the order of $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Part (b) follows from simple calculation:

$$\|x\|^2 = \lim_{n\to\infty} \|x_1 + x_2 + \cdots + x_n\|^2$$
$$= \lim_{n\to\infty} (\|x_1\|^2 + \|x_2\|^2 + \cdots + x_n\|^2) = \sum_{n=1}^{\infty} \|x_n\|^2. \quad \square$$

13-3.4. **Theorem** Let B be an orthonormal set in a Hilbert space H. Then the following statements are equivalent.

(a) B is an orthonormal basis, i.e. a maximal orthonormal set.

(b) For each $x \in H$, if $x \perp B$ then $x = 0$.

(c) $x = \sum_{b\in B} < x, b > b, \forall\, x \in H$.

(d) $< x, y > = \sum_{b\in B} < x, b >< b, y >, \forall\, x, y \in H$.

(e) $\|x\|^2 = \sum_{b\in B} | < x, b > |^2, \forall\, x \in H$; Parseval's identity.

(f) The vector subspace spanned by B is dense in H.

Proof. we shall prove ($a \Leftrightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow b$) and also ($c \Rightarrow f \Rightarrow b$).

($a \Rightarrow b$) Suppose to the contrary that for some $x \neq 0 \in H$, $x \perp B$. Let $y = x/\|x\|$ and $M = B \cup \{y\}$. Then M is an orthonormal set containing B but not equal to B. This shows that B is not maximal.

($b \Rightarrow a$) Suppose to the contrary that B is not maximal. There is an orthonormal set M containing B but not equal to B. Then there is $x \in M \setminus B$. Therefore we have $x \neq 0$ and $x \perp B$.

($b \Rightarrow c$) Let $J = \{b \in B :< x, b >\neq 0\}$. Then the set J is countable and the series $\sum_{b\in B} < x, b > b$ converges to some limit $y \in H$. For each $a \in B \setminus J$, we have

$$< x - y, a > = < x, a > - \sum_{b\in J} < x, b >< b, a > = 0 - \sum_{b\in J} < x, b > 0 = 0,$$

where $< x, a > = 0$ because $a \notin J$ and $< b, a > = 0$ because $a \perp b$. On the other hand, for $a \in J$, we also have

$$< x - y, a > = < x, a > - \sum_{b\in J} < x, b >< b, a >$$

$$= < x, a > - \sum_{b\in J} < x, b > \delta_{ab} = < x, a > - < x, a > = 0.$$

Therefore $(x - y) \perp B$, i.e. $x - y = 0$. Consequently, we have

$$x = y = \sum_{b\in J} < x, b > b = \sum_{b\in B} < x, b > b.$$

($c \Rightarrow d$) Since the set $\{b \in B :< x, b >\neq 0\} \cup \{b \in B :< y, b >\neq 0\}$ is countable. Let $b_1, b_2, b_3, \cdots$ be an enumeration of this set. Then we have $x = \sum_{n=1}^{\infty} < x, b_n > b_n$ and $y = \sum_{n=1}^{\infty} < y, b_n > b_n$. The continuity of the inner product gives

$$< x, y > = \lim_{n\to\infty} \left\langle \sum_{j=1}^{n} < x, b_j > b_j, \sum_{k=1}^{n} < y, b_k > b_k \right\rangle$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} < x, b_j >< y, b_k >^{-} < b_j, b_k > \quad \text{; complex conjugate}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} < x, b_j >< b_k, y > \delta_{jk} = \lim_{n\to\infty} \sum_{j=1}^{n} < x, b_j >< b_j, y >$$

$$= \sum_{j=1}^{\infty} < x, b_j >< b_j, y > = \sum_{b\in B} < x, b >< b, y >$$

($d \Rightarrow e$) Letting $x = y$, it follows immediately.

($e \Rightarrow b$) Suppose $x \perp B$. Then $< x, b > = 0$ for all $b \in B$. Therefore we have $x = \sum_{b\in B} < x, b > b = 0$.

($c \Rightarrow f$) Since every vector in H is the limit of a sequence in the vector subspace spanned by B. Therefore B is dense in H.

($f \Rightarrow b$) Suppose $x \perp B$. Define $f(y) = < x, y >$ for all $y \in H$. Let M be the vector subspace spanned by B. Then for every $y \in M$, there are $b_j \in B$ and $\beta_j \in \mathbb{C}$ such that $y = \sum_{j=1}^{n} \beta_j b_j$. Hence $f(y) = < x, y > = \sum_{j=1}^{n} \beta_j^{-} < x, b_j > = 0$. Therefore $f(M) = \{0\}$. Now $f(H) = f(\overline{M}) \subset \overline{f(M)} = \overline{\{0\}} = \{0\}$. In particular, $f(x) = 0$, i.e. $< x, x > = 0$, or $x = 0$. This completes the whole proof. □

13-3.5. **Exercise** Let $e_n = (\delta_{1n}, \delta_{2n}, \cdots)$ be a sequence given by $\delta_{jn} = 1$ for $j = n$ and $\delta_{jn} = 0$ otherwise. Show that the set $\{e_n : n \geq 1\}$ forms an orthonormal basis of ℓ_2.

13-3.6. **Exercise** Let $e_n = (\cdots, \delta_{(-2)n}, \delta_{(-1)n}, \delta_{0n}, \delta_{1n}, \delta_{2n}, \cdots)$ be a bisequence given by $\delta_{jn} = 1$ for $j = n$ and $\delta_{jn} = 0$ otherwise. Show that $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis of b_2 where $\mathbb{Z}$ denotes the set of all integers.

13-3.7. **Exercise** Identify $P_2[-\pi, \pi]$ as continuous functions on the unit circle. Use Stone-Weierstrass Theorem to prove that every function in $P_2[-\pi, \pi]$ can be uniformly approximated by a trigonometric polynomial of the form $\sum_{k=-n}^{n} \alpha_k e^{inx}$ where $\alpha_k \in \mathbb{C}$. Show that the set $\{e^{inx} : n \in \mathbb{Z}\}$ forms an orthogonal basis of $P_2[-\pi, \pi]$. A study of Fourier Analysis is beyond the scope of this book.

13-4 Orthogonal Complements

13-4.1. Let H denote a Hilbert space. The *orthogonal complement* of a subset M of H is defined by $M^\perp = \{x \in H : x \perp M\}$.

13-4.2. **Theorem** Let M, N be non-empty subsets of a Hilbert space H.
(a) $M^\perp$ is a closed vector subspace of H.
(b) $M \subset M^{\perp\perp}$.
(c) If $M \subset N$, then $N^\perp \subset M^\perp$.
(d) $M^{\perp\perp\perp} = M^\perp$.
(e) $M \cap M^\perp \subset \{0\}$.

Proof. (a) For each $a \in M$, let $f_a(x) = < x, a >, \forall\, x \in H$. By Cauchy-Schwartz inequality, each f_a is a continuous linear form on H. Therefore the set $M^\perp = \cap\{\ker(f_a) : a \in M\}$ is a closed vector subspace of H.
(b,c) Both follow immediately from definition.
(d) It is direct consequence of (b) and (c).
(e) For each $x \in M \cap M^\perp$, we have $x \in M$ and $x \in M^\perp$. Hence $< x, x > = 0$, i.e. $x = 0$. □

13-4.3. **Theorem** Let M be a closed vector subspace of a Hilbert space H.
(a) $H = M \oplus M^\perp$.
(b) $M^{\perp\perp} = M$.

Proof. Since $M \cap M^\perp = \{0\}$, we need only to show $H = M + M^\perp$. Take any $x \in H$. Since M is a closed vector subspace, it is a complete convex set. There is a unique $y \in M$ such that $\|x - y\|$ is the distance from x to M. Hence $x - y$ is orthogonal to M, i.e. $(x - y) \in M^\perp$. Therefore $x = y + (x - y) \in M + M^\perp$. Consequently, H is the algebraic direct sum of M and $M^\perp$. By the way, as a result of Closed Graph Theorem, it is also the topological direct sum.

(b) It suffices to show $M^{\perp\perp} \subset M$. Take any $x \in M^{\perp\perp}$. There is $a \in M$ and $b \in M^\perp$ such that $x = a + b$. Since $M^{\perp\perp}$ is a subspace, we have

$$b = x - a \in M^{\perp\perp} + M = M^{\perp\perp}.$$

Therefore $b \in M^\perp \cap M^{\perp\perp}$, i.e. $b = 0$. Thus $x = a \in M$. Consequently, $M^{\perp\perp} \subset M$. This completes the proof. □

13-4.4. **Theorem** Let H be the algebraic direct sum of two vector subspaces M, N. If $M \perp N$, then $M = N^\perp$ and $N = M^\perp$. In particular, both M, N are closed.

Proof. Since $M \perp N$, we have $M \subset N^\perp$. Take any $x \in N^\perp$. Since $H = M + N$, write $x = a + b$ where $a \in M$ and $b \in N$. Then $b = x - a \in N^\perp - M = N^\perp$, i.e. $b \in N \cap N^\perp$, or $b = 0$. Therefore $x = a \in M$. Consequently, $N^\perp \subset M$. □

13-4.5. **Riesz Representation Theorem** Let f be a continuous linear form on a Hilbert space H. Then there is a unique vector $a \in H$ such that $f(x) = < x, a >$ for all $x \in H$. Furthermore, we have $\|f\| = \|a\|$.

Proof. If $f = 0$, then $a = 0$ satisfies all requirement. Assume $f \neq 0$. Take any $b \in H$ such that $f(b) \neq 0$. Since f is continuous, $M = \ker(f)$ is a closed vector subspace of H. Write $b = u + v$ where $u \in M$ and $v \in M^\perp$. Hence we obtain $f(v) = f(b) - f(u) = f(b) \neq 0$ and in particular $v \neq 0$. Define $a = \dfrac{f(v)^-}{\|v\|^2} v$. Now choose any $x \in H$. Since $f\left[x - \dfrac{f(x)}{f(v)} v\right] = 0$, we have $x - \dfrac{f(x)}{f(v)} v \in \ker(f) = M$. Since $v \in M^\perp$, we have $\left\langle x - \dfrac{f(x)}{f(v)} v, v \right\rangle = 0$, i.e. $< x, v > - \dfrac{f(x)}{f(v)} < v, v > = 0$, or $f(x) = \dfrac{f(v) < x, v >}{\|v\|^2} = < x, a >$. This proves the existence. For uniqueness, let $a, b \in H$ satisfy $f(x) = < x, a >$ and $f(x) = < x, b >, \forall\, x \in H$. Then $< x, a - b > = 0, \forall\, x \in H$. Take $x = a - b$. Then $\|a - b\|^2 = 0$, i.e. $a = b$. Finally, since $|f(x)| = |< x, a >| \leq \|x\|\,\|a\|$, we have $\|f\| \leq \|a\|$. If $a = 0$, then $\|f\| = \|a\| = 0$. Assume $a \neq 0$. Then $\|a\| = \left\langle \dfrac{a}{\|a\|}, a \right\rangle = \left| f\left(\dfrac{a}{\|a\|}\right) \right| \leq \|f\|$. Therefore we also have $\|f\| = \|a\|$ as required. □

13-4.6. **Exercise** Prove that every weakly Cauchy sequence in H is weakly convergent. See §7-8.4.

13-5 Adjoints

13-5.1. Let G, H be Hilbert spaces. A function $\varphi : G \times H \to \mathbb{C}$ is called a *sesquilinear form* on $G \times H$ if for all $a, b \in G; x, y \in H$ and all $\lambda \in \mathbb{C}$ the following conditions hold:
(a) $\varphi(a+b, x) = \varphi(a, x) + \varphi(b, x)$, additive;
(b) $\varphi(\lambda a, x) = \lambda\varphi(a, x)$, homogeneous;
(c) $\varphi(a, x+y) = \varphi(a, x) + \varphi(a, y)$, additive;
(d) $\varphi(a, \lambda x) = \lambda^{-}\varphi(a, x)$, conjugate homogeneous.
The *norm* of a sesquilinear form φ on $G \times H$ is defined by

$$\|\varphi\| = \sup\{|\varphi(a, x)| : \|a\| \le 1, \|x\| \le 1\}.$$

When $G = H$, we simply say that φ is a sesquilinear form on H rather than a sesquilinear form on $H \times H$.

13-5.2. **Theorem** Let φ be a sesquilinear form on $G \times H$. Then the following statements are equivalent.
(a) φ is continuous on $G \times H$.
(b) φ is continuous at $(0,0) \in G \times H$.
(c) $\|\varphi\| < \infty$.
In this case, we have $|\varphi(a, x)| \le \|\varphi\|\ \|a\|\ \|x\|, \forall\ a \in G,\ x \in H$.

Proof. $(a \Rightarrow b)$ It is obvious.
$(b \Rightarrow c)$ For $\varepsilon = 1$, there is $\delta > 0$ such that for all $\|b\| \le \delta$ in G and all $\|y\| \le \delta$ in H, we have $|\varphi(b, y)| \le \varepsilon = 1$. Take any $\|a\| \le 1$ in G and $\|x\| \le 1$ in H. If either $a = 0$ or $x = 0$, then $|\varphi(a, x)| = 0 \le 1/\delta^2$. Assume that $a \ne 0$ and $x \ne 0$. Then $\left|\varphi\left(\frac{\delta a}{\|a\|}, \frac{\delta x}{\|x\|}\right)\right| \le 1$, i.e. $|\varphi(a, x)| \le \frac{1}{\delta^2}\|a\|\|x\| \le \frac{1}{\delta^2}$. Therefore $\|\varphi\| \le \frac{1}{\delta^2} < \infty$.
$(c \Rightarrow a)$ Take $a \in G$ and $x \in H$. For $a = 0$ or $x = 0$, $|\varphi(a, x)| = 0 \le \|\varphi\|\ \|a\|\ \|x\|$. Assume that $a \ne 0$ and $x \ne 0$. Then we have $\left|\varphi\left(\frac{a}{\|a\|}, \frac{x}{\|x\|}\right)\right| \le \|\varphi\| < \infty$, i.e. $|\varphi(a, x)| \le \|\varphi\|\|a\|\ \|x\|$. Now fix $b \in G$ and $y \in H$. Consider

$$\begin{aligned}|\varphi(a, x) - \varphi(b, y)| &\le |\varphi(a-b, x-y)| + |\varphi(a-b, y)| + |\varphi(b, x-y)| \\ &\le \|\varphi\|(\|a-b\|\ \|x-y\| + \|a-b\|\ \|y\| + \|b\|\|x-y\|)\end{aligned}$$

which is small whenever $\|a-b\|$ and $\|x-y\|$ are both small. Therefore φ is continuous at $(b,y) \in G \times H$. Since b, y are arbitrary, φ is continuous on $G \times H$. This completes the proof. □

13-5.3. **Theorem** Let $B : G \to H$ be a continuous linear map. Let $\varphi : G \times H \to \mathbb{C}$ be given by $\varphi(a,x) = < Ba, x >$ for all $a \in G$ and all $x \in H$. Then φ is a continuous sesquilinear form on $G \times H$. Furthermore, we have $\|\varphi\| = \|B\|$. In this case, φ is called the *sesquilinear form associated with* the continuous linear map B.

Proof. It is routine to verify that φ is a sesquilinear form. Now for each $\|a\| \le 1$ in G and $\|x\| \le 1$ in H, we have

$$|\varphi(a,x)| = |< Ba, x >| \le \|Ba\| \, \|x\| \le \|B\| \, \|a\| \, \|x\| \le \|B\|.$$

Hence φ is continuous and $\|\varphi\| \le \|B\|$. On the other hand, take any $\|a\| \le 1$ in G. If $Ba = 0$, then clearly $\|Ba\| \le \|\varphi\|$. Assume $Ba \neq 0$. Consider

$$\|Ba\|^2 = < Ba, Ba > = |\varphi(a, Ba)| \le \|\varphi\| \, \|a\| \, \|Ba\|,$$

i.e. $\|Ba\| \le \|\varphi\| \, \|a\| \le \|\varphi\|$. In all cases, we have $\|Ba\| \le \|\varphi\|$ for all $\|a\| \le 1$ in G. Therefore, $\|B\| \le \|\varphi\|$. □

13-5.4. **Theorem** Let φ be a continuous sesquilinear form on $G \times H$. Then there is a unique continuous linear map $B : G \to H$ such that

$$\varphi(a,x) = < Ba, x >, \forall\, a \in G, \forall\, x \in H.$$

Proof. For each $a \in G$, let $f_a(x) = \varphi(a,x)^-$ for all $x \in H$. Since φ is a continuous sesquilinear form, f_a is a continuous linear form on H. There is a unique vector $Ba \in H$ such that $f_a(x) = < x, Ba >, \forall\, x \in H$, i.e. $\varphi(a,x) = < Ba, x >$. We have defined a map $B : G \to H$. To show its linearity, take any $a, b \in G$ and $\alpha, \beta \in \mathbb{C}$. Then for each $x \in H$, we have

$$< B(\alpha a + \beta b), x > = \varphi(\alpha a + \beta b, x) = \alpha\varphi(a,x) + \beta\varphi(b,x)$$

$$= \alpha < Ba, x > + \beta < Bb, x > = < \alpha Ba + \beta Bb, x > .$$

Hence $B(\alpha a + \beta b) = \alpha Ba + \beta Bb$. Therefore B is linear. Now for all $a \in G$, we get $\|Ba\|^2 = < Ba, Ba > = |\varphi(a, Ba)| \le \|\varphi\| \, \|a\| \, \|Ba\|$, i.e. $\|Ba\| \le \|\varphi\| \, \|a\|$. Therefore B is a continuous linear map. Suppose $A, B : G \to H$ are linear maps satisfying $\varphi(a,x) = < Aa, x > = < Ba, x >$ for all $a \in G$ and $x \in H$. Then $< Aa - Ba, x > = 0, \forall\, x \in H$. Hence $Aa - Ba = 0$, or $Aa = Ba, \forall\, a \in G$, i.e. $A = B$. □

13-5.5. **Adjoint Theorem** For each continuous linear map $B : G \to H$, there is a unique continuous linear map $B^* : H \to G$ so that $< Ba, x > = < a, B^*x >$, for all $a \in G, x \in H$. In this case, B^* is called the *adjoint* of B. Furthermore, we have $\|B^*\| = \|B\|$.

Proof. Let $\varphi(x,a) = < x, Ba >, \forall\ x \in H, a \in G$. Then φ is a continuous sesquilinear form on $H \times G$. There is a continuous linear map $B^* : H \to G$ such that for all $a \in G, x \in H$, we have $\varphi(x,a) = < B^*x, a >$, i.e. $< x, Ba > = < B^*x, a >$, or $< Ba, x > = < a, B^*x >$. Clearly we have $\|\varphi\| = \|B^*\|$. Also

$$\begin{aligned}\|\varphi\| &= \sup\{|\varphi(x,a)| : \|a\| \leq 1, \|x\| \leq 1\}\\ &= \sup\{| < x, Ba > | : \|a\| \leq 1, \|x\| \leq 1\}\\ &= \sup\{\ | < Ba, x > | : \|a\| \leq 1, \|x\| \leq 1\} = \|B\|.\end{aligned}$$

Therefore $\|B^*\| = \|B\|$. The uniqueness is left as an exercise. □

13-5.6. **Lemma** For all continuous linear maps $A, B : G \to H$, we have
(a) $(A+B)^* = A^* + B^*$;
(b) $(\lambda A)^* = \lambda^- A^*$;
(c) $A^{**} = A$;
(d) $I^* = I$ where $I : H \to H$ denotes the identity map.

13-5.7. **Theorem** Let G, H, K be Hilbert spaces. Let $A : G \to H$ and $B : H \to K$ be continuous linear maps. Then we have $(BA)^* = A^*B^*$. Furthermore, if A is bijective, then A^{-1} is also a continuous linear map satisfying $(A^{-1})^* = (A^*)^{-1}$.

Proof. For all $a \in G$ and $y \in K$, we have

$$< a, (BA)^*y > = < BAa, y > = < Aa, B^*y > = < a, A^*B^*y >,$$

i.e. $(BA)^* = A^*B^*$. Next, suppose A is bijective. By Open-Map Theorem, A^{-1} is also a continuous linear map. Let I_G, I_H denote the identity maps on G, H respectively. Since $AA^{-1} = I_H$, we have $(A^{-1})^*A^* = (AA^{-1})^* = I_H^* = I_H$. Similarly, $A^*(A^{-1})^* = I_G$. Therefore $(A^*)^{-1} = (A^{-1})^*$. □

13-5.8. **Theorem** Let G, H be Hilbert spaces and $B : G \to H$ a continuous linear map.
(a) $\|B^*B\| = \|B\|^2 = \|BB^*\|$.
(b) $B^*B = 0$ iff $B = 0$.

Proof. (a) For every $\|a\| \leq 1$ in G, we have

$$\|Ba\|^2 = < Ba, Ba > = < a, B^*Ba > = | < a, B^*Ba > |$$
$$\leq \|a\| \, \|B^*Ba\| \leq \|a\| \, \|B^*B\| \, \|a\| \leq \|B^*B\|,$$

i.e. $\|Ba\| \leq \sqrt{\|B^*B\|}$. Taking supremum over $\|a\| \leq 1$, we obtain $\|B\| \leq \sqrt{\|B^*B\|}$, or $\|B\|^2 \leq \|B^*B\|$. On the other hand, because of $\|B^*B\| \leq \|B^*\| \, \|B\| = \|B\|^2$, we get $\|B^*B\| = \|B\|^2$. Replacing B by B^*, we have $\|BB^*\| = \|B^*\|^2 = \|B\|^2$.

(b) $\|B^*B\| = 0$ iff $\|B\|^2 = 0$ iff $\|B\| = 0$ iff $B = 0$. □

13-5.9. **Theorem** Let $B : G \to H$ a continuous linear map and M, N be vector subspaces of G, H respectively.
(a) If $B(M) \subset N$ then $B^*(N^\perp) \subset M^\perp$.
(b) If both M, N are closed, then the converse is also true.

Proof. Assume $B(M) \subset N$. Take any $x \in N^\perp$. Then $x \in B(M)^\perp$. For all $a \in M$, we have $< a, B^*x > = < Ba, x > = 0$. Hence $B^*x \in M^\perp$. Therefore $B^*(N^\perp) \subset M^\perp$. Conversely suppose $B^*(N^\perp) \subset M^\perp$. Since M, N are closed, we get $M^{\perp\perp} = M$ and $N^{\perp\perp} = N$. By (a), $B(M) = B(M^{\perp\perp}) \subset N^{\perp\perp} = N$. □

13-5.10. **Theorem** Let $B : G \to H$ be a continuous linear map.
(a) $\ker(B) = (\text{Im } B^*)^\perp$.
(b) $[\ker(B)]^\perp = \overline{\text{Im } B^*}$.

Proof. (a) From $B[\ker(B)] \subset \{0\}$ by definition, it follows

$$B^*(H) = B^*(\{0\}^\perp) \subset [\ker(B)]^\perp.$$

Thus, $$\ker(B) = \ker(B)^{\perp\perp} \subset [B^*(H)]^\perp = (\text{Im } B^*)^\perp.$$

Conversely, by definition we get $B^*(H) \subset \text{Im } B^*$. Hence,

$$B[(\text{Im } B^*)^\perp] \subset H^\perp = \{0\}.$$

Therefore we have, $(\text{Im } B^*)^\perp \subset \ker(B)$. Consequently, $\ker(B) = (\text{Im } B^*)^\perp$.
(b) $[\ker(B)]^\perp = (\text{Im } B^*)^{\perp\perp} = \overline{\text{Im } B^*}$. □

13-5.11. **Exercise** For ℓ_2, find the adjoints of left and right shifts. For b_2, define the left and right shifts and find their adjoints.

13-5.12. **Exercise** The hermitian of a complex matrix Q is defined as $Q^* = Q^{t-}$, the complex conjugate of the transpose. Let A be a continuous linear map from $G \to H$. Prove that for every orthonormal basis of G, H respectively, the matrix representations are related by $[A^*] = [A]^*$. See §7-5.8.

13-6 Quadratic Forms

13-6.1. Let H be a Hilbert space. A function $q : H \to \mathbb{C}$ is called a *quadratic form* if there is a sesquilinear form φ on H such that $q(x) = \varphi(x,x), \forall\, x \in H$. In this case, q is called the quadratic form *associated with* the sesquilinear form φ. The *norm* of a quadratic form q on H is defined by $\|q\| = \sup\{|q(x)| : \|x\| \le 1\}$. In this section, the relationship among operators, sesquilinear forms and quadratic forms will be established. It will be used later to characterize certain properties such as being normal, self-adjoint, etc.

13-6.2. **Lemma** $|q(x)| \le \|q\|\ \|x\|^2$, for all $x \in H$.

Proof. If $x = 0$, then $|q(x)| = 0 \le \|q\|\ \|x\|^2$. If $x \neq 0$, then we have

$$|q(x)| = \left|\varphi\left(\frac{x}{\|x\|}, \frac{x}{\|x\|}\right)\right|\ \|x\|^2 \le \|q\|\ \|x\|^2. \qquad \square$$

13-6.3. **Polarization Formula** Let p, q be quadratic forms on H associated with the sesquilinear forms ξ, φ respectively.

(a) $4\varphi(x,y) = q(x+y) - q(x-y) + iq(x+iy) - iq(x-iy), \forall\, x, y \in H$.

(b) $p = q$ iff $\xi = \varphi$.

Proof. It is a routine exercise. $\square$

13-6.4. A sesquilinear form φ on a Hilbert space H is said to be

(a) *real* or *hermitian* if $\varphi(x,x)$ is real for all $x \in H$;

(b) *positive* if $\varphi(x,x) \ge 0$, for all $x \in H$;

(c) *non-degenerate* if for each $x \in H$, $\varphi(x,x) = 0$ implies $x = 0$.

13-6.5. **Lemma** A sesquilinear form φ on H is real iff $\varphi(x,y) = \varphi(y,x)^-$, for all $x, y \in H$.

Proof. Suppose φ is real. Let $\xi(x,y) = \varphi(y,x)^-$ for all $x, y \in H$. Clearly ξ is also a sesquilinear form on H. Furthermore, $\xi(x,x) = \varphi(x,x), \forall\, x \in H$. Hence $\xi = \varphi$. Therefore $\varphi(y,x)^- = \varphi(x,y)$, $\forall\, x, y \in H$. The converse is obvious. $\square$

13-6.6. **Theorem** Let φ be a sesquilinear form on a Hilbert space H. If φ is positive, then we have $|\varphi(x,y)|^2 \le \varphi(x,x)\varphi(y,y)$ for all $x, y \in H$.

Proof. If $\varphi(x,x) > 0$ or $\varphi(y,y) > 0$, the proof is identical with the Cauchy-Schwartz inequality. Now assume $\varphi(x,x) = \varphi(y,y) = 0$. Let $\lambda = \varphi(x,y)$. Then we have

$$\begin{aligned} 0 &\le \varphi(x - \lambda y, x - \lambda y) = \varphi(x,x) - \lambda\varphi(y,x) - \lambda^-\varphi(x,y) + \lambda\lambda^-\varphi(y,y) \\ &= -\varphi(x,y)\varphi(y,x) - \varphi(x,y)^-\varphi(x,y) = -2|\varphi(x,y)|^2, \end{aligned}$$

i.e. $|\varphi(x,y)|^2 = 0 \leq \varphi(x,x)\varphi(y,y)$. □

13-6.7. **Theorem** Let q be the quadratic form associated with a sesquilinear form φ on a Hilbert space H. Then the following statements are equivalent.
(a) q is continuous on H.
(b) $\|q\| < \infty$.
(c) φ is continuous on H.
In this case, we have $\|q\| \leq \|\varphi\| \leq 2\|q\|$.

Proof. Suppose q is continuous on H. For $\varepsilon = 1$, there is $\delta > 0$ such that $\|y\| \leq \delta$ in H implies $|q(y)| \leq 1$. Hence for all $\|x\| \leq 1$, we have

$$|q(x)| = |\varphi(\delta x, \delta x)/\delta^2| = |q(\delta x)|/\delta^2 \leq 1/\delta^2.$$

Taking supremum over $\|x\| \leq 1$, we have $\|q\| \leq 1/\delta^2 < \infty$. Next, assume $\|q\| < \infty$. For $\|x\| \leq 1$ and $\|y\| \leq 1$ in H, the polarization formula gives

$$\begin{aligned}
&|\varphi(x,y)| = \tfrac{1}{4}|q(x+y) - q(x-y) + iq(x+iy) - iq(x-iy)| \\
&\leq \tfrac{1}{4}\|q\|(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2) \\
&= \tfrac{1}{4}\|q\|2(\|x\|^2 + \|y\|^2 + \|x\|^2 + \|iy\|^2), \text{ by parallelogram law} \\
&= \tfrac{1}{4}\|q\|4(\|x\|^2 + \|y\|^2) \leq \|q\|(1+1) = 2\|q\|.
\end{aligned}$$

Taking supremum over $\|x\| \leq 1$ and $\|y\| \leq 1$, we have $\|\varphi\| \leq 2\|q\|$. It follows that φ is continuous on H. Finally, suppose φ is continuous on H. Clearly q is also continuous. Moreover for all $\|x\| \leq 1$, we have $|q(x)| = |\varphi(x,x)| \leq \|\varphi\|\ \|x\|^2 \leq \|\varphi\|$. Therefore $\|q\| \leq \|\varphi\|$. □

13-6.8. **Theorem** Let q be a quadratic form associated with a sesquilinear form φ on a Hilbert space H. If φ is real, then $\|\varphi\| = \|q\|$.

Proof. We have prove $\|q\| \leq \|\varphi\|$. Take any $\|x\| \leq 1$ and $\|y\| \leq 1$ in H. Let $r \geq 0$ and $\theta \geq 0$ satisfy $\varphi(x,y) = re^{i\theta}$. Then polarization formula gives

$$\begin{aligned}
&4r = 4\varphi(e^{-i\theta}x, y) \\
&= q(e^{-i\theta}x + y) - q(e^{-i\theta}x - y) + iq(e^{-i\theta}x + iy) - iq(e^{-i\theta}x - iy).
\end{aligned}$$

Since φ is real, q is real. Hence the imaginary terms vanish. Therefore

$$\begin{aligned}
&|\varphi(x,y)| = r = \tfrac{1}{4}[q(e^{-i\theta}x + y) - q(e^{-i\theta}x - y)] \\
&\leq \tfrac{1}{4}\|q\|(\|e^{-i\theta}x + y\|^2 + \|e^{-i\theta}x - y\|^2) = \tfrac{1}{4}\|q\|2(\|x\|^2 + \|y\|^2) \leq \|q\|.
\end{aligned}$$

Taking supremum over $\|x\| \leq 1$ and $\|y\| \leq 1$, we have $\|\varphi\| = \|q\|$. □

13-6.9. **Exercise** Prove $\ker(A^*A) = \ker(A)$ for every operator on H.

13-7 Normal Operators

13-7.1. Let H be a Hilbert space. As in normed spaces, a continuous linear map from H into itself is also called an *operator*. The identity operator on H is denoted by I. An operator A on H is said to be

(a) *isometric* or an *isometry* if $A^*A = I$;

(b) *unitary* if $A^*A = AA^* = I$;

(c) *normal* if $A^*A = AA^*$;

(d) *self-adjoint* if $A^* = A$;

(e) *skew-adjoint* if $A^* = -A$.

13-7.2. **Lemma** Let A, B be operators on H. If $< Ax, x > = < Bx, x >$ for all $x \in H$, then we have $A = B$.

Proof. From given quadratic forms, we get $< Ax, y > = < Bx, y > \forall\, x, y \in H$. Hence $A = B$. □

13-7.3. **Theorem** For every operator A on H, the following statements are equivalent:

(a) A is an isometry, i.e. $A^*A = I$.

(b) $\|Ax\| = \|x\|$, for all $x \in H$.

(c) $< Ax, Ay > = < x, y >$, for all $x, y \in H$.

Proof. $(a \Rightarrow c)$ $< Ax, Ay > = < A^*Ax, y > = < Ix, y > = < x, y >$.

$(c \Rightarrow b)$ Take $x = y$.

$(b \Rightarrow a)$ $< A^*Ax, x > = < Ax, Ax > = \|Ax\|^2 = \|x\|^2 = < x, x >$. □

13-7.4. **Lemma** If A is an isometry on H, then $\mathrm{Im}(A)$ is a closed subspace of H.

Proof. It suffices to show that $\mathrm{Im}(A)$ is complete. Being an isometry, Cauchy and convergent sequences in H and $\mathrm{Im}(A)$ correspond. □

13-7.5. **Theorem** For every operator A on H, the following statements are equivalent:

(a) A is unitary, i.e. $A^*A = AA^* = I$.

(b) A^* is unitary.

(c) Both A and A^* are isometric.

(d) A is isometric and A^* is injective.

(e) A is isometric and surjective.

(f) A is bijective and $A^{-1} = A^*$.

Proof. ($a \Leftrightarrow b$) It follows immediately from $A^{**} = A$.
($a \Rightarrow c$) It follows from definitions.
($c \Rightarrow d$) Every isometry is injective.
($d \Rightarrow e$) Since A is an isometry, $\text{Im}(A)$ is closed in H. Hence

$$\text{Im}(A) = \overline{\text{Im}(A)} = [\ker(A^*)]^{\perp} = \{0\}^{\perp} = H.$$

Therefore A is surjective.
($e \Rightarrow f$) Since A is isometric, A is injective. By (e), A is bijective. Therefore A^{-1} is also an operator on H. Since A is an isometry, we have $A^*A = I$, i.e. A^* is a left inverse of A. Therefore $A^{-1} = A^*$. This proves (f).
($f \Rightarrow a$) $A^*A = A^{-1}A = I$ and $AA^* = AA^{-1} = I$. □

13-7.6. **Theorem** For every operator A on H, the following statements are equivalent:
(a) A is normal, i.e. $A^*A = AA^*$.
(b) A^* is normal.
(c) $\|A^*x\| = \|Ax\|$, for all $x \in H$.
(d) $< A^*x, A^*y > = < Ax, Ay >$, for all $x, y \in H$.

Proof. ($a \Leftrightarrow b$) It follows immediately from $A^{**} = A$.
($a \Rightarrow d$) $< A^*x, A^*y > = < AA^*x, y > = < A^*Ax, y > = < Ax, Ay >$.
($d \Rightarrow c$) Take $x = y$.
($c \Rightarrow a$) For all $x \in H$, we have

$$< A^*Ax, x > = < Ax, Ax > = \|Ax\|^2 = \|A^*x\|^2 = < A^*x, A^*x > = < AA^*x, x >.$$

Hence $A^*A = AA^*$. Therefore A is normal. □

13-7.7. **Corollary** If A is a normal operator on H, then $\|A^2\| = \|A\|^2$.

Proof. Replacing x by Ax in (c) of last theorem, we have $\|A^*(Ax)\| = \|A(Ax)\|$, i.e. $\|(A^*A)x\| = \|A^2x\|$, $\forall\ x \in H$. Taking supremum over $\|x\| \le 1$, we obtain $\|A^*A\| = \|A^2\|$. Since $\|A^*A\| = \|A\|^2$ is always true, we have $\|A^2\| = \|A\|^2$. □

13-7.8. **Theorem** Let A, B be normal operators on H. If $AB^* = B^*A$, then $A + B$, AB and BA are all normal.

Proof. Taking the adjoint of $AB^* = B^*A$, we have $BA^* = A^*B$. Now observe that

$$(A + B)^*(A + B) = A^*A + B^*A + A^*B + B^*B$$
$$= AA^* + AB^* + BA^* + BB^* = (A + B)(A + B)^*;$$

and also that

$$\begin{aligned}(AB)^*(AB) &= B^*A^*AB = B^*AA^*B = AB^*A^*B \\ &= AB^*BA^* = ABB^*A^* = (AB)(AB)^*.\end{aligned}$$

Therefore $A + B$ and AB are normal. Similarly, BA is also normal. □

13-7.9. **Exercise** Prove that scalar multiplications of normal operators are normal.

13-7.10. **Exercise** Show that the product of two isometric operators is isometric. Show that the product of two unitary operators is unitary.

13-7.11. **Exercise** Let $\{e_n : n \in J\}$ be an orthonormal basis in H. Prove that an operator A on H is unitary iff $\{Ae_n : n \in J\}$ is an orthonormal basis.

13-7.12. **Exercise** Prove that the norm of an isometric operator is one.

13-7.13. **Lemma** For every normal operator A, e^{A^*-A} is unitary.

Proof. By §8-8.15c, $\left(e^{A^*-A}\right)^* = e^{A-A^*} = e^{-(A^*-A)} = \left(e^{A^*-A}\right)^{-1}$. □

13-7.14. **Fuglede's Theorem** Let A, B be normal operators. Then for every operator T, if $AT = TB$, then $TA^* = TB^*$.

Proof. Following [Rosenblum], clearly $A^nT = TB^n$ by induction. Since $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$ converges in norm, we have $e^AT = Te^B$, or $T = e^{-A}Te^B$. Thus $e^{A^*}Te^{-B^*} = PTQ$ where $P = e^{A^*-A}$ and $Q = e^{B-B^*}$. Since P, Q are unitary, we have $\|e^{A^*}Te^{-B^*}\| \le \|P\|\ \|T\|\ \|Q\| \le \|T\|$. Next, for every $\lambda \in \mathbb{C}$, let $f(\lambda) = e^{\lambda A^*}Te^{-\lambda B^*}$. Then $f : \mathbb{C} \to L(H)$ is an entire map. Replacing A, B by $\lambda^- A, \lambda^- B$ respectively, we have $\|f(\lambda)\| \le \|T\|$ for all λ. By Liouville's Theorem, f is a constant map, that is

$$f'(\lambda) = e^{\lambda A^*}A^*Te^{-\lambda B^*} + e^{\lambda A^*}Te^{-\lambda B^*}(-B^*) = 0.$$

The result follows by setting $\lambda = 0$. □

13-8 Self-Adjoint Operators

13-8.1. **Theorem** For every operator A on a Hilbert space H, the following statements are equivalent:

(a) A is self-adjoint, i.e. $A^* = A$.

(b) $< Ax, y > = < Ay, x >^-$, $\forall\ x, y \in H$, i.e. a hermitian form.

(c) $< Ax, x >$ is real for all $x \in H$, i.e. a real-valued quadratic form.

Proof. We have proved $(b \Leftrightarrow c)$ already. Observe that

$$< A^*x, x > = < x, Ax > = < Ax, x >^-.$$

Hence $A^* = A$ iff $< Ax, x > = < A^*x, x >, \forall\ x$, i.e. $< Ax, x > = < Ax, x >^-$, or $< Ax, x >$ is real. We also proved $(a \Leftrightarrow c)$ □

13-8.2. **Theorem** Let A, B be self-adjoint operators on H.
(a) $A + B$ is self-adjoint.
(b) λA is self-adjoint for all real number λ.
(c) AB is self-adjoint iff $AB = BA$.

Proof. For (c), if AB is self-adjoint, then $AB = (AB)^* = B^*A^* = BA$. Conversely if $AB = BA$, then $(AB)^* = (BA)^* = A^*B^* = AB$, i.e. AB is self-adjoint. Both (a) and (b) are left as an exercise. □

13-8.3. **Theorem** Let A be an operator on H. Let $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$.
(a) Both B, C are self-adjoint. Also $A = B + iC$.
(b) If M, N are self-adjoint operators satisfying $A = M + iN$, then we have $M = B$ and $N = C$.
(c) A is normal iff $BC = CB$.

Proof. For (c), $A^*A = B^2 - iCB + iBC + C^2$ and $AA^* = B^2 + iCB - iBC + C^2$. Therefore $A^*A = AA^*$ iff $BC = CB$. □

13-8.4. **Exercise** Let A be a normal operator on H. Let B, C be the real and imaginary parts of A. Prove that $\|Ax\|^2 = \|Bx\|^2 + \|Cx\|^2$ for all $x \in H$.

13-8.5. **Exercise** Prove that an operator is skew-adjoint iff $A = iC$ for some self-adjoint operator C.

13-8.6. **Theorem** Let A be a self-adjoint operator on H.
(a) $\|A\| = \sup\{| < Ax, x > | : \|x\| \leq 1\}$.
(b) $\|A^n\| = \|A\|^n$ for all integer $n \geq 1$.

Proof. (a) Let φ and q be the sesquilinear and quadratic forms associated with A. Then $\|A\| = \|\varphi\| = \|q\|$.
(b) It suffices to show $\|A\|^n \leq \|A^n\|$. Take any $\|x\| \leq 1$ in H. Then for every integer $m \geq 1$, we have

$$\|A^m x\|^2 = < A^m x, A^m x > = < A^* A^m x, A^{m-1} x >$$
$$= < A^{m+1} x, A^{m-1} x > \leq \|A^{m+1} x\| \ \|A^{m-1} x\|.$$

Suppose $Ax \neq 0$. Letting $m = 1, 2, \cdots$, we have $A^k x \neq 0$ for all $k \geq 1$. Hence we get

$$\frac{\|Ax\|}{\|x\|} \le \frac{\|A^2x\|}{\|Ax\|} \le \frac{\|A^3x\|}{\|A^2x\|} \le \cdots \le \frac{\|A^nx\|}{\|A^{n-1}x\|},$$

or
$$\left(\frac{\|Ax\|}{\|x\|}\right)^n \le \frac{\|Ax\|}{\|x\|} \cdot \frac{\|A^2x\|}{\|Ax\|} \cdot \frac{\|A^3x\|}{\|A^2x\|} \cdots \frac{\|A^nx\|}{\|A^{n-1}x\|} = \frac{\|A^nx\|}{\|x\|},$$

that is $\|Ax\|^n \le \|A^nx\| \; \|x\|^{n-1} \le \|A^n\|$. If $Ax = 0$, then this is obvious. Taking supremum over $\|x\| \le 1$, we have $\|A\|^n \le \|A^n\|$. □

13-8.7. <u>**Exercise**</u> Prove that the set of all self-adjoint operators on H is closed in $L(H)$. Is it true for normal, skew-adjoint operators?

13-9 Projectors and Closed Vector Subspaces

13-9.1. Let H be a Hilbert space. An operator A on H is called a *projector* or an *orthogonal projection* if $A^* = A = A^2$. In this section, a bijection between the sets of projectors and closed vector subspaces will be established.

13-9.2. <u>**Theorem**</u> Let M be a closed vector subspace of a Hilbert space H.
(a) There is a unique projector P onto M.
(b) $M = P(H) = \{x \in H : Px = x\} = \{x \in H : \|Px\| = \|x\|\}$.

<u>*Proof*</u>. Suppose P, Q are projectors such that $M = P(H) = Q(H)$. Take any $x \in H$. Then $Px \in M = Q(H)$, i.e. $Px = Qy$ for some $y \in H$. Hence $QPx = Q^2y = Qy = Px$. Since $x \in H$ is arbitrary, we have $QP = P$. Similarly, $PQ = Q$. Now $P = P^* = (QP)^* = P^*Q^* = PQ = Q$ gives the uniqueness of the required projector. Next, we want to construct a projector from H onto M. Since M is closed, $H = M \oplus M^\perp$. For every $x \in H$, there is unique $a \in M$ and $b \in M^\perp$ such that $x = a + b$. Define $Px = a$. Then clearly, P is a linear map from H onto M satisfying $P^2 = P$ and $M = P(H) = \{x \in H : Px = x\}$. Since $a \perp b$, we obtain $\|Px\|^2 = \|a\|^2 \le \|a\|^2 + \|b\|^2 = \|x\|^2$, i.e. $\|Px\| \le \|x\|$. Hence P is continuous and therefore P is an operator on H. Now for each $x \in H$, we get

$$< P^*x, x > = < x, Px > = < a+b, a > = < a, a > = < a, a+b > = < Px, x >,$$

i.e. $P^* = P$. Hence P is a projector. If $Px = x$, then obviously $\|Px\| = \|x\|$. Conversely, suppose that $\|Px\| = \|x\|$ for some $x \in H$. Then we obtain $\|a\|^2 = \|Px\|^2 = \|x\|^2 = \|a\|^2 + \|b\|^2$, i.e. $\|b\| = 0$, or $b = 0$. Therefore $x = a \in M$. This completes the proof. □

13-9.3. <u>**Theorem**</u> Let P be a projector in H, $Q = I - P$, $M = P(H)$ and $N = Q(H)$.

(a) Q is also a projector. Furthermore, $PQ = QP = 0$.
(b) $M = \ker(Q) = N^\perp$; $N = \ker(P) = M^\perp$ and $H = M \oplus N$.
(c) Both M, N are closed vector subspace of H.
The operator Q is also denoted by $P^\perp$.

Proof. The calculations: $Q^2 = (I - P)^2 = I - P - P + P^2 = I - P = Q$ and $Q^* = (I - P)^* = I^* - P^* = I - P = Q$ show that Q is a projector. Furthermore, $PQ = P(I - P) = P - P^2 = P - P = 0$. Similarly, $QP = 0$. Next, suppose $x \in M$. Then $x = Py$ for some $y \in H$. Now $Qx = QPy = 0$. Hence $M \subset \ker(Q)$. Next, suppose $Qx = 0$. Then $x = Ix = (P + Q)x = Px$, i.e. $x \in P(H) = M$. Hence $\ker(Q) \subset M$. Therefore $M = \ker(Q)$. Consequently, M is a closed vector subspace. Finally, take any $x \in H$. Then $x = Px + Qx \in M + N$. Hence $H = M + N$. Because of $M \cap N = \ker(Q) \cap Q(H) = \{0\}$, we get $H = M \oplus N$. Now $x \in M^\perp$ iff for all $y \in H$, we have $< x, Py > = 0$, i.e. $< Px, y > = 0$, or $Px = 0$. Therefore $M^\perp = \ker(P)$. Since P, Q are symmetric, the proof is complete. □

13-9.4. **Theorem** Let P be an operator on H. Then P is a projector iff we have $\|Px\|^2 = < Px, x >$ for all $x \in H$.

Proof. Suppose P is a projector. Let $M = P(H)$ and $N = \ker(P)$. We have proved $H = M \oplus N$. For each $x \in H$, let $x = a + b$ where $a \in M$ and $b \in N$. Since $M \perp N$, we get $\|Px\|^2 = < Px, Px > = < a, a > = < a, a + b > = < Px, x >$. Conversely, suppose $\|Px\|^2 = < Px, x >$ for all $x \in H$. Then we have

$$< Px, x > = \|Px\|^2 = < Px, Px > = < P^*Px, x >.$$

Hence $P = P^*P$. Thus $P^* = (P^*P)^* = P^*P^{**} = P^*P = P$ and $P = P^*P = P^2$. Consequently, P is a projector. □

13-9.5. **Theorem** For every projector P, we have either $\|P\| = 1$ or $\|P\| = 0$.

Proof. It follows from $\|P\| = \|PP\| = \|P^*P\| = \|P\|^2$. □

13-9.6. **Theorem** Let P, Q be projectors in H. Then PQ is a projector iff $PQ = QP$. In this case, we have $PQ(H) = P(H) \cap Q(H)$.

Proof. Suppose PQ is a projector. Then $PQ = (PQ)^* = Q^*P^* = QP$. Conversely, suppose $PQ = QP$. Then we have $(PQ)^2 = PQPQ = PPQQ = PQ$ and $(PQ)^* = Q^*P^* = QP = PQ$. Therefore PQ is a projector. Finally, assume that PQ is a projector. Choose any $x \in PQ(H)$. Write $x = PQy$ for some $y \in H$. Then $x = P(Qy) \in P(H)$ and $x = Q(Py) \in Q(H)$, i.e. $x \in P(H) \cap Q(H)$.

On the other hand, take any $x \in P(H) \cap Q(H)$. Then $x = Px$ and $x = Qx$. Hence $x = PQx \in PQ(H)$. Consequently, we have $PQ(H) = P(H) \cap Q(H)$. □

13-9.7. **Theorem** Let P, Q be projectors in H, $M = P(H)$ and $N = Q(H)$. Then the following statements are equivalent.

(a) $P + Q$ is a projector.

(b) $M \perp N$.

(c) $P(N) = 0$.

(d) $PQ = 0$.

In this case, $P + Q$ is a projector onto $M + N$.

Proof. ($a \Rightarrow b$) Let $x \in M$. Then $Px = x$ and

$$\|x\|^2 + \|Qx\|^2 = \|Px\|^2 + \|Qx\|^2 = < Px, x > + < Qx, x >$$
$$= < (P + Q)x, x > \leq \ \|P + Q\| \ \|x\|^2 \leq \|x\|^2.$$

Hence $\|Qx\| = 0$, i.e. $x \in \ker(Q)$. Therefore $M \subset \ker(Q) = N^{\perp}$. Consequently, we have $M \perp N$.

($b \Rightarrow c$) Since $M \perp N$, we have $N \subset M^{\perp} = \ker(P)$, i.e. $P(N) = 0$.

($c \Rightarrow d$) Suppose $P(N) = 0$. Then $PQ(H) = P[Q(H)] = P(N) = 0$, i.e. $PQ = 0$.

($d \Rightarrow a$) Suppose $PQ = 0$. Then $QP = Q^*P^* = (PQ)^* = 0$ and thus we get $(P + Q)^2 = P^2 + PQ + QP + Q^2 = P + Q$. Clearly $P + Q$ is self-adjoint. Therefore $P + Q$ is also a projector. Finally, assume that $P + Q$ is also a projector. Take any $x \in P(H) + Q(H)$. Then $x = a + b$ with $a = Pa \in P(H)$ and $b = Qb \in Q(H)$. Hence $(P + Q)x = Pa + Pb + Qa + Qb = a + PQb + QPa + b = x$, i.e. $x = (P + Q)x \in (P + Q)(H)$. Therefore $P(H) + Q(H) \subset (P + Q)(H)$. The reversed inequality is obvious. □

13-9.8. **Theorem** Let $P_1, P_2, \cdots, P_n$ be projectors in a Hilbert space H and let $P = P_1 + P_2 + \cdots + P_n$. Then P is a projector iff $P_j P_k = 0$ for all $1 \leq j \neq k \leq n$.

Proof. Suppose P is a projector and $j \neq k$. Then for all $x \in H$, we have

$$\sum_{j=1}^{n} \|P_j x\|^2 = \sum_{j=1}^{n} < P_j x, x >= \left\langle \sum_{j=1}^{n} P_j x, x \right\rangle$$
$$= < Px, x > \leq \|P\| \ \|x\|^2 \leq \|x\|^2.$$

Replacing x by $P_k x$, we get $\sum_{j=1}^{n} \|P_j P_k x\|^2 \leq \|P_k x\|^2$, i.e. $\sum_{j \neq k} \|P_j P_k x\|^2 \leq 0$. Hence $P_j P_k x = 0$. Since $x \in H$ is arbitrary, we have $P_j P_k = 0$. The converse follows from simple calculations. □

13-9.9. **Exercise** Prove that A is a projector iff $A = A^* A$.

13-9.10. **Exercise** Prove that every compact projector is finite dimensional.

13-9.11. **Exercise** Prove that if A is an isometry then AA^* is a projector.

13-9.12. **Exercise** Let A, B be projectors. Prove that $A + B - AB$ is a projector iff $AB = BA$.

13-9.13. Let A be an operator on H and M a vector subspace of H. Recall that M is *invariant* under A if $A(M) \subset M$. Also M is said to *reduce* A if both M and $M^\perp$ are invariant under A.

13-9.14. **Theorem** Let A be an operator on H and P a projector in H onto a vector subspace M. Then the following statements are equivalent.

(a) M is an invariant subspace of A.

(b) $AP = PAP$.

(c) $M^\perp$ is an invariant subspace of A^*.

Proof. Suppose M is an invariant subspace of A. Take any $x \in H$. Then $Px \in M$. Since M is invariant under A, $APx \in M$ and hence $PAPx = APx$. Since x is arbitrary, we have $PAP = AP$. Conversely, if $PAP = AP$, then we get $A(M) = AP(H) = PAP(H) \subset M$ and hence M is invariant under A. Therefore (a) and (b) are equivalent. Next, $M^\perp$ is invariant under A^* iff $A^*P^\perp = P^\perp A^* P^\perp$ iff $A^*(I - P) = (I - P)A^*(I - P)$ iff $PA^* = PA^*P$ iff $AP = PAP$. □

13-9.15. **Theorem** Let A be an operator on H and P a projector in H onto a vector subspace M. Then the following statements are equivalent.

(a) M reduces A.

(b) $AP = PA$.

(c) M is invariant under both A and A^*.

(d) $M^\perp$ reduces A.

Proof. ($a \Rightarrow b$) Observe that $P^\perp = I - P$ is the projector onto $M^\perp$. Since $M^\perp$ is invariant under A, we have $AP^\perp = P^\perp AP^\perp$, i.e. $A(I-P) = (I-P)A(I-P)$, or $PA = PAP$. Since M is also invariant under A, we get $AP = PAP$. Therefore we obtain $AP = PA$.

($b \Rightarrow c$) Since $PAP = (PA)P = (AP)P = AP$, M is invariant under A. Similarly, $PAP = P(AP) = P(PA) = PA$. Taking adjoint, $PA^*P = A^*P$, i.e. M is invariant under A^*.

($c \Rightarrow a$) Because M is invariant under A^*, $M^\perp$ is also invariant under $A^{**} = A$. Therefore M reduces A.

($c \Leftrightarrow d$) Since M is closed, we have $M^{\perp\perp} = M$. The result follows immediately from definition. □

13-9.16. **Corollary** Let A be an operator on H and M a closed vector subspace of H.
(a) Suppose A is self-adjoint. Then M reduces A iff $A(M) \subset M$.
(b) Suppose A is unitary. Then M reduces A iff $A(M) = M$.

Proof. (b) Assume that M reduces A. Then we have $A^*(M) \subset M$, that is $M = AA^*(M) \subset A(M)$. Since M reduces A, it is invariant under A. Consequently $A(M) = M$. Conversely, assume $A(M) = M$. Then we have $A^*(M) = A^*A(M) = I(M) = M$. Hence M is invariant under A^*. Therefore M reduces A. □

13-9.17. **Exercise** Let M be the set of $x \in \ell_2$ with zero as the first coordinate $x_1 = 0$. Show that M is invariant under the right-shift but it does not reduce the right-shift.

13-9.18. **Exercise** Prove that if a closed vector subspace is invariant under a self-adjoint operator, it also reduces the operator.

13-10 Partial Order of Operators

13-10.1. Let A, B be self-adjoint operators on a Hilbert space H. Define $A \leq B$ if $< Ax, x > \leq < Bx, x >$ for all $x \in H$. On the other hand, an operator C on H is said to be *positive* if $C = D^*D$ for some operator D. Note that the order is defined in terms of quadratic forms but positive operators are in line of §13-7.1. It is trivial to show that every positive operator C satisfies $C \geq 0$ but the proof of the converse is delayed until §14-6.9.

13-10.2. **Lemma** Let A, B, C be self-adjoint operators on H.
(a) $A \leq A$.
(b) If $A \leq B$ and $B \leq C$, then $A \leq C$.
(c) If $A \leq B$ and $B \leq A$, then $A = B$.
(d) If $A \leq B$, then $A + C \leq B + C$ and $-B \leq -A$.
(e) If $A \leq B$ and $\lambda \geq 0$, then $\lambda A \leq \lambda B$.
(f) If $\alpha \leq \beta$ in $\mathbb{R}$ and if $A \geq 0$, then $\alpha A \leq \beta A$.

Proof. (c) For all $x \in H$, we have $< Ax, x > \leq < Bx, x > \leq < Ax, x >$, i.e. $< Ax, x > = < Bx, x >$ and hence $A = B$. The rest is left as exercise. □

13-10.3. **Theorem** If $A \geq 0$, then for all $x, y \in H$ we have
(a) $| < Ax, y > |^2 \leq < Ax, x > < Ay, y >$;
(b) $\|Ax\|^2 \leq \|A\| < Ax, x >$.

Proof. (a) This is the Cauchy-Schwartz inequality for sesquilinear forms.
(b) It is obvious if $Ax = 0$. Assuming $\|Ax\| \neq 0$, it follows by dividing $\|Ax\|^2$ from

$$\|Ax\|^4 = | < Ax, Ax > |^2 \leq < Ax, x > < A(Ax), Ax > \text{ ; by (a)}$$

$$\leq < Ax, x > \|A^2 x\| \; \|Ax\| \leq < Ax, x > \|A\| \; \|Ax\|^2. \qquad \square$$

13-10.4. **Theorem** (a) If $A \leq B$, then $T^*AT \leq T^*BT$ for every operator T.
(b) If $0 \leq A \leq B$, then $\|A\| \leq \|B\|$.

Proof. (a) It follows immediately from

$$< T^*ATx, x > = < ATx, Tx > \leq < BTx, Tx > = < T^*BTx, x >, \quad \forall\, x \in H.$$

(b) If $A = 0$, it is obvious. Assume $\|A\| \neq 0$. For each $\|x\| \leq 1$ in H, observe

$$\|Ax\|^2 \leq \|A\| < Ax, x > \leq \|A\| < Bx, x > \leq \|A\| \; \|B\| \; \|x\|^2 \leq \|A\| \; \|B\|.$$

Taking supremum over $\|x\| \leq 1$, we have $\|A\|^2 \leq \|A\|\|B\|$. The result follows by dividing $\|A\|$. $\square$

13-10.5. **Theorem** Let $A_n \leq A_{n+1}$ be an increasing sequence of self-adjoint operators on H. If it has an upper bound, i.e. a self-adjoint operator $B \geq A_n, \forall\, n$; then there is a self-adjoint operator A such that $A_n x \to Ax$ for each $x \in H$ and $A = \sup_n A_n$. Similar result holds for decreasing sequence.

Proof. Replacing A_n and B by $A_n - A_1$ and $B - A_1$ respectively, we may assume $A_n \geq 0$. For each $x \in H$, we have a bounded increasing sequence of real numbers $< A_n x, x > \leq < A_{n+1} x, x > \leq < Bx, x >$ and hence it converges. Take any $m > n \geq 1$. Since $0 \leq A_n \leq B$, we have $\|A_n\| \leq \|B\|$. Now observe

$$\|A_m x - A_n x\|^2 \leq \|A_m - A_n\| < (A_m - A_n)x, x >$$

$$\leq 2\|B\|(< A_m x, x > - < A_n x, x >) \to 0 \text{ as } m, n \to \infty.$$

Hence $\{A_n x\}$ is a Cauchy sequence. By Banach-Steinhaus Theorem, the limit $Ax = \lim_{n\to\infty} A_n x$ defines an operator A on H. Since for all x the number $< Ax, x > = \lim_{n\to\infty} < A_n x, x >$ is real, A is self-adjoint. For every $m > n$, we get $< A_n x, x > \leq < A_m x, x >$. Letting $m \to \infty$, we have $< A_n x, x > \leq < Ax, x >$, i.e. $A_n \leq A$. Therefore A is an upper bound of the set $\{A_n\}$. On the other hand, let U be any upper bound of $\{A_n\}$. Then

$< A_n x, x > \leq < Ux, x >$. Letting $n \to \infty$, we obtain $< Ax, x > \leq < Ux, x >$ for all $x \in H$, i.e. $A \leq U$. □

13-10.6. Let M, N be vector subspaces of H. Then $N \cap M^{\perp}$ is called the *orthogonal complement* of M in N. It is also denoted by $N \ominus M$.

13-10.7. **Theorem** Let P, Q be projectors in H. Let $M = P(H)$ and $N = Q(H)$. Then the following statements are equivalent.
(a) $M \subset N$.
(b) $QP = P$.
(c) $PQ = P$.
(d) $Q - P$ is a projector.
(e) $< (Q - P)x, x > \geq 0$, for all $x \in H$.
(f) $\|Px\| \leq \|Qx\|$, for all $x \in H$.
(g) $P \leq Q$
In this case, $Q - P$ is a projector onto $N \ominus M$.

Proof. ($a \Rightarrow b$) Take any $x \in H$. Then $Px \in M \subset N$. Hence $QPx = Px$. Since x is arbitrary, $QP = P$.
($b \Rightarrow c$) $P = P^* = (QP)^* = P^*Q^* = PQ$.
($c \Rightarrow d$) Assume $PQ = P$. Then $P = P^* = (PQ)^* = Q^*P^* = QP$ and hence $(Q - P)^2 = Q^2 - PQ - QP + P^2 = Q - P - P + P = Q - P$. Clearly, $Q - P$ is self-adjoint. Therefore $Q - P$ is a projector.
($d \Rightarrow e$) Since $Q-P$ is a projector, $< (Q-P)x, x > = \|(Q-P)x\|^2 \geq 0$, $\forall\, x \in H$.
($e \Rightarrow f$) For all $x \in H$, $< (Q - P)x, x > \geq 0$ gives $< Qx, x > \geq < Px, x >$, i.e. $\|Qx\|^2 \geq \|Px\|^2$, or $\|Qx\| \geq \|Px\|$.
($f \Rightarrow a$) For any $x \in M$, we have $\|x\| = \|Px\| \leq \|Qx\| \leq \|Q\|\ \|x\| \leq \|x\|$, i.e. $\|Qx\| = \|x\|$, or $x \in N$. Hence $M \subset N$.
($e \Leftrightarrow g$) It follows from definition.
Finally, suppose $Q-P$ is a projector. Then $QP^{\perp} = Q(I-P) = Q-QP = Q-P$. Hence $QP^{\perp}$ is also a projector. Therefore we have

$$(Q - P)(H) = QP^{\perp}(H) = Q(H) \cap P^{\perp}(H)$$
$$= Q(H) \cap [P(H)]^{\perp} = N \cap M^{\perp} = N \ominus M.$$ □

13-10.8. **Exercise** Is the left-shift on ℓ_2 an isometric, unitary, normal, self-adjoint, skew-adjoint, positive operator? Work on the same question for right-shift on ℓ_2. Answer the same question for left and right shifts for b_2.

13-10.9. **Exercise** Prove that if A is a positive operator on H, then

(a) $< Ax, x >$ is positive for all $x \in H$;
(b) A is self-adjoint.

13-10.10. **Exercise** Let A, B be positive operators. Prove that if $A + B = 0$ then $A = B = 0$.

13-10.11. Let A_n, B be operators on H.
(a) $A_n \to B$ *uniformly* or in norm if

$$\lim_{n\to\infty} \sup_{\|x\|\le 1} \|A_n x - Bx\| = \lim_{n\to\infty} \|A_n - B\| = 0.$$

(b) $A_n \to B$ *strongly* or $B =$ s-lim A_n if $\lim_{n\to\infty} \|A_n x - Bx\| = 0$ for each $x \in H$.
(c) $A_n \to B$ *weakly* or $B =$ w-lim A_n if $\lim_{n\to\infty} < A_n x - Bx, y >= 0$, $\forall\, x, y \in H$.
Write $B = \lim A_n =$ u-lim A_n for uniform convergence or convergence in norm. It is an exercise to show that uniform convergence implies strong convergence which in turn implies weak convergence but the converses are false. Similar concept of Cauchy sequences is assumed. Note that weak convergence of vectors was defined in §7-7,8.

13-10.12. **Lemma** If $\{A_n\}$ is weakly Cauchy, then $\sup_n \|A_n\| < \infty$.

Proof. For all $x, y \in H$; let $B_n^x(y) =< y, A_n x >$. Since $\{A_n\}$ is weakly Cauchy, $B_n^x(y) \to B^x(y)$ in $\mathbb{C}$. Because B_n^x are continuous linear maps, Banach-Steinhaus Theorem ensures $M^x = \sup_n \|B_n^x\| < \infty$. For $D_{ny}(x) =< A_n x, y >$ where $\|y\| \le 1$; we have $|D_{ny}(x)| = | < y, A_n x > | = |B_n^x(y)| \le M^x$. Uniform boundedness theorem gives $M = \sup_{ny} \|D_{ny}\| < \infty$, that is $| < A_n x, y > | \le M$ for all $\|x\| \le 1$, $\|y\| \le 1$ and $n \ge 1$. Therefore for all n we have

$$\|A_n\| = \sup\{| < A_n x, y > | : \|x\| \le 1,\ \|y\| \le 1\} \le M.$$ □

13-10.13. **Theorem** Strongly Cauchy sequences of operators are strongly convergent. Weakly Cauchy sequences of operators are weakly convergent.

Proof. The first statement is obvious because pointwise limits of continuous operators are continuous. Next, let $\{A_n\}$ be a weakly Cauchy sequence of operators. Then $M = \sup_n \|A_n\| < \infty$. For all $x, y \in H$; let $\varphi(x, y)$ be the limit of the Cauchy sequence $\{< A_n x, y >\}$. Obviously φ is sesquilinear. Also for $\|x\| \le 1$, $\|y\| \le 1$; we get $|\varphi(x, y)| = \lim | < A_n x, y > | \le M$. Thus φ is continuous. There is an operator A such that $< Ax, y >= \varphi(x, y)$ $= \lim < A_n x, y >$ for all $x, y \in H$. Consequently, $A_n \to A$ weakly. □

13-10.14. **Example** In ℓ_2, let $Q_n x = (0, \cdots, 0, x_{n+1}, x_{n+2}, x_{n+3}, \cdots)$. Then Q_n are projectors with $\|Q_n\| = 1$. Hence $Q_n \to 0$ uniformly is false. However for each $x \in \ell_2$, $\|Q_n x\|^2 = \sum_{k>n} |x_k|^2 \to 0$ as $n \to \infty$, that is $Q_n \to 0$ strongly. There is counter example of weakly convergent sequence of operators which is not strongly convergent.

13-10.15. **Example** Let $Sx = (x_2, x_3, \cdots)$ and $A_n = S^n$. Then $A_n \to 0$ strongly for $\|A_n x\|^2 = \sum_{j>n} |x_j|^2 \to 0$. On the other hand, $S^* x = (0, x_1, x_2, \cdots)$ and $\|A_n^* e_1\| = 1$. Hence $A_n^* \to 0$ strongly is false.

13-10.16. **Exercise** Prove that if $A_n \to 0$ weakly, then $A_n^* \to 0$ weakly.

13-10.17. **Theorem** If $A_n \to A$ and $B_n \to B$ strongly, then $A_n B_n \to AB$ strongly. This is false for weak convergence.

Proof. For all $x \in H$, we have $A_n x \to Ax$ and $B_n x \to Bx$. By uniform boundedness theorem, $M = \sup_n \|A_n\| < \infty$. The result follows from

$$\begin{aligned}\|A_n B_n x - ABx\| &\le \|A_n\| \, \|B_n x - Bx\| + \|A_n Bx - ABx\| \\ &\le M\|B_n x - Bx\| + \|A_n(Bx) - A(Bx)\| \to 0. \qquad \square\end{aligned}$$

13-10.18. **Theorem** Let P_n be projectors onto the closed subspace M_n.
(a) If $\{P_n\}$ is increasing, then $P = \text{s-lim } P_n = \sup_n P_n$ exists and is a projector onto the closed subspace $\overline{\cup_n M_n}$.
(b) If $\{P_n\}$ is decreasing, then $P = \text{s-lim } P_n = \inf_n P_n$ exists and is a projector onto $\cap_n M_n$.

Proof. We have prove the existence of $P = \text{s-lim } P_n$. Letting $n \to \infty$ in $P_n^2 = P_n = P_n^*$, P is a projector. Let Q be the projector onto the closed subspace $N = \overline{\cup_n M_n}$. Since $M_n \subset N$, we have $P_n \le Q$. As the supremum, we obtain $P_n \le P \le Q$. From $M_n \le P(H) \le N$, it follows $N = \overline{\cup_n M_n} \subset P(H) \le N$, that is $P(H) = N$. Part (b) is left as an exercise. $\square$

13-10.19. **Exercise** Let P, Q be projectors onto M, N respectively. Prove that R =s-lim $(PQP)^n$ =s-lim $(PQ)^n$ is the projector onto $M \cap N$.

13-11 Eigenvalues

13-11.1. Let H be a Hilbert space. Suppose A is an operator on H. Recalled that a complex number λ is called an ***eigenvalue*** of A if there is a non-zero vector $x \in H$ such that $Ax = \lambda x$. A vector $x \in H$ is called an ***eigenvector*** of A if there is $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$. In this case, x is called an eigenvector of λ. The set of all eigenvectors of λ is called the ***eigenspace*** of λ. This section provides the motivation for corresponding properties of spectrum of an operator studied in next chapter.

13-11.2. **Theorem** The eigenspace of every number is a closed subspace. The non-zero eigenvectors of distinct eigenvalues are linearly independent.

13-11.3. **Theorem** Let A be a normal operator on H.
(a) If $Ax = \lambda x$ for $\lambda \in \mathbb{C}$ and $x \in H$, then $A^*x = \lambda^- x$.
(b) The eigenvectors of distinct eigenvalues of A are orthogonal.

Proof. (a) Since A is normal, so is $A - \lambda I$. The result follows from:

$$\|A^*x - \lambda^- x\| = \|(A - \lambda I)^* x\| = \|(A - \lambda I)x\| = 0.$$

(b) Let $\lambda \neq \mu$ be eigenvalues of A and x, y eigenvectors of λ, μ respectively. Then $\lambda < x, y > = < \lambda x, y > = < Ax, y > = < x, A^*y > = < x, \mu^- y > = \mu < x, y >$, i.e. $(\lambda - \mu) < x, y > = 0$, or $< x, y > = 0$. Therefore $x \perp y$. □

13-11.4. **Theorem** Let A be an operator on H.
(a) If A is self-adjoint, then all eigenvalues are real.
(b) If A is skew-adjoint, then all eigenvalues are purely imaginary.
(c) If A is isometric, then the absolute value each eigenvalue is one. Note that every unitary operator is isometric.
(d) If A is positive, then all eigenvalues are positive (≥ 0).

Proof. Let λ be an eigenvalue of A and x a non-zero eigenvector of λ.
(a) Observe that

$$\lambda \|x\|^2 = < \lambda x, x > = < Ax, x >$$
$$= < x, A^*x > = < x, Ax > = < x, \lambda x > = \lambda^- \|x\|^2,$$

i.e. $\lambda = \lambda^-$. Therefore λ is real.
(b) As above $\lambda \|x\|^2 = < x, A^*x > = < x, -Ax > = < x, -\lambda x > = -\lambda^- \|x\|^2$, i.e. $\lambda = -\lambda^-$. Therefore λ is purely imaginary.
(c) Observe that $\|x\|^2 = \|Ax\|^2 = \|\lambda x\|^2 = |\lambda|^2 \|x\|^2$, i.e. $|\lambda| = 1$.
(d) Since $0 \leq < Ax, x > = < \lambda x, x > = \lambda \|x\|^2$, we have $\lambda \geq 0$. □

13-11.5. **Lemma** Let A, B be operators on H. If $AB = BA$, then every eigenspace of B is invariant under A.

Proof. Let $M = \ker(B - \lambda I)$ be an eigenspace of B where $\lambda \in \mathbb{C}$. For every $x \in M$, $(B - \lambda I)Ax = A(B - \lambda I)x = A0 = 0$, i.e. $Ax \in \ker(B - \lambda I) = M$. Therefore M is invariant under A. □

13-11.6. **Theorem** If A is a normal operator on H, then every eigenspace reduces A.

Proof. Since A commute with both A and A^*, every eigenspace of A is invariant under both A and A^*. □

13-11.7. **Exercise** Let A, B be operators on H. Suppose that a vector orthogonal every eigenspace of B must be zero. Prove that if each eigenspace of B is invariant under A then we have $AB = BA$.

13-11.8. **Exercise** Let A be an operator and x a non-zero vector. Prove that if $| < Ax, x > | = \|Ax\| \; \|x\|$ then x is an eigenvector of A.

13-99. **References and Further Readings** : Fan-78,79,96, Ando, Young, Berberian-61, Rudin-80, Dineen-89, Arazy, Burbrea, Hwang and Isidro.

Chapter 14

Spectral Properties of Hilbert Spaces

14-1 Spectrum of an Operator

14-1.1. The properties of an operator reflected by its eigenvalue has been proved by algebraic method in last chapter. In this chapter, we shall show that similar result holds for spectrum. The following lemma is a crucial tool.

14-1.2. **Lemma** Let A be an operator on a Hilbert space H. If both A and A^* are bounded below, then A is invertible.

Proof. Since A^* is bounded below, A^* is injective, i.e. $\ker(A^*) = \{0\}$. Hence $(\text{Im } A)^- = [\ker(A^*)]^\perp = \{0\}^\perp = H$. Therefore $\text{Im}(A)$ is dense in H. Since A is also bounded below, A is invertible. □

14-1.3. **Theorem** For every operator A on H, $I + A^*A$ is invertible.

Proof. The operator is bounded below because for every $x \in H$,

$$\|(I + A^*A)x\|^2 = < x + A^*Ax, x + A^*Ax >$$

$$= \|x\|^2 + < x, A^*Ax > + < A^*Ax, x > + \|A^*Ax\|^2 \geq \|x\|^2.$$

Since $A^* = A$ is also bounded below, A is invertible. □

14-1.4. **Lemma** $\sigma(A^*) = [\sigma(A)]^-$, the complex conjugate.

Proof. The right hand side is interpreted as $\{\lambda^- : \lambda \in \sigma(A)\}$. It follows from the fact that $A - \lambda I$ is invertible iff $A^* - \lambda^- I$ is. □

14-1.5. **Theorem** If A is self-adjoint, then its spectrum consists of real numbers only.

Proof. Suppose $\lambda = \alpha + i\beta$ where α, β are real and $\beta \neq 0$. It suffices to show $\lambda \notin \sigma(A)$, i.e. $A - \lambda I$ is invertible. Now for each $x \in H$, we have

$$\|(A - \alpha I \pm \beta i I)x\|^2 = < (A - \alpha I \pm \beta i I)x, (A - \alpha I \pm \beta i I)x >$$

$$= \|(A - \alpha I)x\|^2 + < \pm\beta i x, (A - \alpha I)x > + < (A - \alpha I)x, \pm\beta i x > + \|\beta i x\|^2$$

$$= \|(A - \alpha I)x\|^2 \pm \beta i < x, (A - \alpha I)x > \mp \beta i < (A - \alpha I)x, x > + \beta^2 \|x\|^2$$

$$= \|(A - \alpha I)x\|^2 \pm \beta i < (A - \alpha I)x, x > \mp \beta i < (A - \alpha I)x, x > + \beta^2 \|x\|^2;$$
$$\geq \beta^2 \|x\|^2.$$

Since $\beta \neq 0$, $A - \lambda I = A - \alpha I - \beta i I$ and $(A - \lambda I)^* = A - \alpha I + \beta i I$ are bounded below. Consequently $A - \lambda I$ is invertible, i.e. $\lambda \notin \sigma(A)$. □

14-1.6. **Theorem** If A is positive, then its spectrum contains positive (≥ 0) numbers only.

Proof. Since A is self-adjoint, $\sigma(A)$ contains real numbers only. Let $\lambda < 0$. Take any $x \in H$. By $A \geq 0$, we obtain $< Ax, x > \geq 0$. Since

$$\|(A - \lambda I)x\|^2 = \|Ax\|^2 - 2\lambda < Ax, x > + \lambda^2 \|x\|^2 \geq \lambda^2 \|x\|^2,$$

$A_\lambda I$ is bounded below. Because $A - \lambda I$ is self-adjoint, it is invertible. Therefore $\lambda \notin \sigma(A)$. As a result, $\sigma(A)$ contains positive ($\geq$) numbers only.

14-1.7. **Theorem** If A is unitary, then its spectrum is a compact subset of the unit circle of the complex plane.

Proof. Since A is an isometry, $\|A\| = 1$. Therefore for every $\lambda \in \sigma(A)$, we obtain $|\lambda| \leq \|A\| = 1$. Take any $|\lambda| < 1$. Let $B = A - \lambda I$. Then we have

$$\|A - B\| = \|\lambda I\| = |\lambda| < 1 = \frac{1}{\|A^*\|} = \frac{1}{\|A^{-1}\|}.$$

Since A is invertible, so is B. Hence $\lambda \notin \sigma(A)$. This completes the proof. □

14-1.8. **Exercise** Let φ be a continuous sesquilinear form on H. Prove that if there is a constant $\lambda > 0$ such that $\varphi(x, x) \geq \lambda \|x\|^2, \forall\ x \in H$ then every continuous linear form f on H has a unique $a \in H$ such that $f(x) = \varphi(x, a)$ for all $x \in H$.

14-2 Approximate Spectrum

14-2.1. Let A be an operator on a Hilbert space H. A complex number λ is called an *approximate eigenvalue* of A if there is $\|x_n\| = 1$ in H such that $\|Ax_n - \lambda x_n\| \to 0$. The set of all approximate eigenvalues is called the *approximate point spectrum* of A and denoted by $\alpha(A)$. It will be used to calculate the spectral radius of a self-adjoint operator. For convenience, the set of all eigenvalues is called the *point spectrum* of A and denoted by $\pi(A)$.

14-2.2. **Lemma** A complex number λ is not an approximate eigenvalue iff $A - \lambda I$ is bounded below.

14-2.3. **Theorem** $\pi(A) \subset \alpha(A) \subset \sigma(A)$.

Proof. Let λ be an eigenvalue of A. There is a non-zero eigenvector y of λ. Let $x_n = y/\|y\|$. Then $\|x_n\| = 1$ and $\|Ax_n - \lambda x_n\| = 0$. Hence λ is an approximate eigenvalue. Therefore $\pi(A) \subset \alpha(A)$. Next, take any $\lambda \notin \sigma(A)$. Then $A - \lambda I$ is invertible. Hence $A - \lambda I$ is bounded below. Therefore λ is not an approximate eigenvalue, i.e. $\lambda \notin \alpha(A)$. Consequently, $\alpha(A) \subset \sigma(A)$. □

14-2.4. **Theorem** If A is normal, then $\alpha(A) = \sigma(A)$.

Proof. Let $\lambda \notin \alpha(A)$. Then $A - \lambda I$ is bounded below and hence its range is closed in H. Suppose to the contrary that $\text{Im}(A - \lambda I)$ is not dense in H. Then $\text{Im}(A - \lambda I) = \overline{\text{Im}(A - \lambda I)} \neq H$. There is $x \neq 0$ such that $x \in [\text{Im}(A - \lambda I)]^{\perp} = \ker(A - \lambda I)^*$. Since A is normal, so is $A - \lambda I$. Hence $\|(A - \lambda I)x\| = \|(A - \lambda I)^* x\| = 0$. Since $A - \lambda I$ is bounded below, $\|x\| = 0$, i.e. $x = 0$. This contradiction shows that $\text{Im}(A - \lambda I)$ is dense in H. So, $A - \lambda I$ is invertible, i.e. $\lambda \notin \sigma(A)$. This proves $\sigma(A) \subset \alpha(A)$. □

14-2.5. **Theorem** If A is a self-adjoint operator on H, then either $\|A\|$ or $-\|A\|$ is in the spectrum. Furthermore, $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, the spectral radius.

Proof. Let $\lambda = \|A\|$. There are $\|x_n\| = 1$ in H such that $\|Ax_n\| \to \lambda$. Observe that

$$\begin{aligned}
&\|A^2x_n - \lambda^2 x_n\|^2 = < A^2x_n - \lambda^2 x_n, A^2 x_n - \lambda^2 x_n > \\
&= \|A^2 x_n\|^2 - < \lambda^2 x_n, A^2 x_n > - < A^2 x_n, \lambda^2 x_n > + \lambda^4 \|x_n\|^2 \\
&\leq \|A^2\|^2 \|x_n\|^2 - 2\lambda^2 < Ax_n, Ax_n > + \lambda^4 \|x_n\|^2 \text{ ; since } A \text{ is self-adjoint} \\
&\leq \|A\|^4 - 2\lambda^2 \|Ax_n\|^2 + \lambda^4 = 2\lambda^2(\lambda^2 - \|Ax_n\|^2) \to 0.
\end{aligned}$$

Therefore λ^2 is an approximate eigenvalue of A^2, i.e. $\sigma(A^2) = [\sigma(A)]^2$. There is $\mu \in \sigma(A)$ such that $\lambda^2 = \mu^2$, i.e. $\mu = \pm\|A\|$. Therefore either $\|A\|$ or $-\|A\|$ is in the spectrum. Since $\sup\{|\lambda| : \lambda \in \sigma(A)\} \leq \|A\|$ is always true, the proof is complete. □

14-2.6. **Exercise** Prove that if A has an approximate eigenvalue λ such that $|\lambda| = \|A\|$, then $\|A\| = \sup_{\|x\| \leq 1} | < Ax, x > |$.

14-2.7. **Exercise** Prove that if λ is an approximate eigenvalue of a normal operator A then λ^- is an approximate eigenvalue of A^*.

14-2.8. **Exercise** Show that the point spectrum of the right-shift on ℓ_2 is empty but its spectrum is the closed unit ball at the origin.

14-2.9. **Exercise** Show that the spectrum of the left shift on ℓ_2 is the closed unit ball at the origin. Show that if λ is an eigenvalue, then $|\lambda| < 1$.

14-2.10. We shall work on a self-adjoint operator A on a Hilbert space H. Let $M = \sup\{< Ax, x >: \|x\| = 1\}$ and $m = \inf\{< Ax, x >: \|x\| = 1\}$.

14-2.11. **Lemma** (a) If A is self-adjoint, then $-\|A\|I \leq A \leq \|A\|I$.
(b) $M = \inf\{\lambda \in \mathbb{R} : A \leq \lambda I\}$.
(c) $m = \sup\{\lambda \in \mathbb{R} : \lambda I \leq A\}$.
(d) $mI \leq A \leq MI$.

Proof. (a) $< Ax, x > \leq | < Ax, x > | \leq \|A\| \ \|x\|^2 = < \|A\| Ix, x >, \forall\ x \in H$. Hence $A \leq \|A\|I$. Replacing A by $-A$, $-A \leq \|-A\|I = \|A\|I$, i.e. $-\|A\|I \leq A$.
(b,d) If $A \leq \lambda I$, then for all $\|x\| = 1$ we have $< Ax, x > \leq < \lambda Ix, x > = \lambda$, i.e. $M \leq \lambda$. Therefore $M \leq \inf\{\lambda \in \mathbb{R} : A \leq \lambda I\}$. On the other hand, by definition of M we get $\left\langle A\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \leq M$ for every $x \neq 0$. Thus $< Ax, x > \leq M\|x\|^2 = < M\ Ix, x >$ for all $x \in H$. Hence $A \leq MI$. Therefore $\inf\{\lambda \in \mathbb{R} : A \leq \lambda I\} \leq M$.
(c,d) If $A \geq \lambda I$, then for all $\|x\| = 1$ we have $< Ax, x > \geq < \lambda Ix, x > = \lambda$, i.e. $m \geq \lambda$. Therefore $m \geq \sup\{\lambda \in \mathbb{R} : A \geq \lambda I\}$. On the other hand, by definition of m we get $\left\langle A\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \geq m$ for every $x \neq 0$. Thus $< Ax, x > \geq m\|x\|^2 = < mIx, x >$ for all $x \in H$. Hence $A \geq mI$. Therefore $\sup\{\lambda \in \mathbb{R} : A \geq \lambda I\} \geq m$. □

14-2.12. **Theorem** Let $\sigma(A)$ be the spectrum of a self-adjoint operator A.
(a) $M = \sup \sigma(A)$.
(b) $m = \inf \sigma(A)$.
(c) $\|A\| = \max\{|m|, |M|\}$.

Proof. (a) For every $\lambda > M$ and every $x \neq 0$, we have

$$\|(\lambda I - A)x\| \ \|x\| \geq < (\lambda I - A)x, x > = \|x\|^2 \left\langle (\lambda I - A)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle$$

$$= \|x\|^2 \left(\lambda - \left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right) \geq \|x\|^2(\lambda - M),$$

or $\|(\lambda I - A)x\| \geq (\lambda - M)\|x\|$. Thus $\lambda I - A$ is bounded below. Hence $\lambda \notin \alpha(A)$, the approximate spectrum of A. Since A is normal, $\alpha(A) = \sigma(A)$. So, we obtain $\lambda \notin \sigma(A)$. Therefore $\sup \sigma(A) \leq M$. On the other hand, by definition of M

there is $\|x_n\| = 1$ such that $M - \frac{1}{n} \leq < Ax_n, x_n >$. Suppose to the contrary that $M \notin \sigma(A)$. Then $MI - A$ is invertible. Hence

$$\begin{aligned}1 &= \|x_n\|^2 = \|(MI - A)^{-1}(MI - A)x_n\|^2 \\ &\leq \|(MI - A)^{-1}\|^2\|(MI - A)x_n\|^2 \\ &\leq \|(MI - A)^{-1}\|^2\|(MI - A)\| < (MI - A)x_n, x_n > \\ &= \|(MI - A)^{-1}\|^2\|(MI - A)\|[M - < Ax_n, x_n >] \\ &\leq \|(MI - A)^{-1}\|^2\|(MI - A)\|[M - (M - \tfrac{1}{n})] \\ &= \|(MI - A)^{-1}\|^2\|(MI - A)\| \cdot \tfrac{1}{n} \to 0.\end{aligned}$$

This is a contradiction. Hence $M \in \sigma(A)$. Therefore $M = \sup \sigma(A)$.

(b) It follows from the following calculation:

$$\begin{aligned}m &= \inf\{< Ax, x >: \|x\| = 1\} = -\sup\{< -Ax, x >: \|x\| = 1\} \\ &= -\sup \sigma(-A) = -\sup[-\sigma(A)] = \inf \sigma(A).\end{aligned}$$

(c) Since $\sigma(A)$ is compact, we have $m, M \in \sigma(A)$. Hence $|m|, |M| \leq \|A\|$, or $\max\{|m|, |M|\} \leq \|A\|$. Next, for all $0 < \|x\| \leq 1$ we have

$$m \leq \left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle \leq M$$

by definition of m, M. Hence we obtain $\left|\left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle\right| \leq \max\{|m|, |M|\}$, or $| < Ax, x > | \leq \|x\|^2 \max\{|m|, |M|\} \leq \max\{|m|, |M|\}$. Since A is self-adjoint, taking supremum over $\|x\| \leq 1$ we have

$$\|A\| = \sup\{| < Ax, x > | : \|x\| \leq 1\} \leq \max\{|m|, |M|\}. \qquad \square$$

14-2.13. **Corollary** A positive operator A is invertible iff there is $\delta > 0$ such that $A \geq \delta I$.

Proof. ($\Rightarrow$) Since A is invertible, $0 \notin \sigma(A)$. Because $A \geq 0$, we have $\sigma(A) \geq 0$. Thus $m = \inf \sigma(A) > 0$ and $A \geq mI$.

($\Leftarrow$) From $\delta\|x\|^2 =< \delta Ix, x > \leq < Ax, x > \leq \|Ax\| \; \|x\|$, we have $\|Ax\| \geq \delta\|x\|$. Since $A \geq 0$, both A, A^* are bounded below. Therefore A is invertible. $\square$

14-3 Weak Convergence

14-3.1. Weak convergence in Banach spaces are applicable in Hilbert spaces. We shall prove directly that the unit ball of a Hilbert space is weakly sequentially compact even though it is reducible to reflexive Banach spaces.

14-3.2. **Theorem** Let H be a Hilbert space. If $x_n \to a$ weakly in H and if $\|x_n\| \to \|a\|$, then $x_n \to a$ strongly.

Proof. It follows easily from the following calculation

$$\|x_n - a\|^2 = < x_n - a, x_n - a >$$
$$= \|x_n\|^2 - < a, x_n > - < x_n, a > + < a, a >$$
$$\to \|a\|^2 - < a, a > - < a, a > + < a, a > = 0.$$
□

14-3.3. **Theorem** Every weakly Cauchy sequence $\{x_n\}$ in H is weakly convergent. In other words, every Hilbert space is weakly sequentially complete.

Proof. Instead of quoting the corresponding result in reflexive Banach space, we give a direct proof here. Let $z \in H$ be given. Since $\{< x_n, z >\}$ is Cauchy in $\mathbb{C}$, it converges and so does its complex conjugate $\{< z, x_n >\}$. Define $f(z) = \lim < z, x_n >, \forall\ z \in H$. Then f is a linear form in z on H. Since $\{x_n\}$ is weakly Cauchy, it is bounded. There is $\lambda > 0$ such that $\|x_n\| \leq \lambda$ for all n. Because $|f(z)| = \lim | < z, x_n > | \leq \|z\|\lambda$, f is a continuous linear form on H. There is $a \in H$ such that $f(z) = < z, a >, \forall\ z \in H$. Therefore $< x_n, z > \to < a, z >$ for all z. Consequently $x_n \to a$ weakly. □

14-3.4. **Lemma** For all in integers $m, n \geq 1$, let a_{mn} be complex numbers such that the sequence $\{a_{mn} : n \geq 1\}$ is a bounded sequence for each given m. Then there is a sequence $1 \leq n(1) < n(2) < n(3) < \cdots$ of integers such that for each m, the sequence $\{a_{mn(j)} : j \geq 1\}$ converges.

Proof. It is an exercise to apply the standard *diagonal process*. □

14-3.5. **Theorem** Every bounded sequence $\{x_n\}$ in H contains a weakly convergent subsequence.

Proof. Let $\{x_n\}$ be a bounded sequence in H. Let M be the closed vector subspace spanned by $\{x_n : n \geq 1\}$. If M is finite dimensional, then $\{x_n\}$ has a strongly convergent subsequence which is also weakly convergent. Hence we need only to consider the case when M is infinite dimensional. Applying the orthonormalization process to $\{x_n\}$ and deleting some x_k if it is linearly dependent on the preceding terms, we can construct an orthonormal basis $\{e_m : m \geq 1\}$ for M. Since $\{x_n\}$ is bounded, the sequence $\{< e_m, x_n >\}$ is bounded for each given m. There are integers $1 \leq n(1) < n(2) < \cdots$ such that $\{< e_m, x_{n(j)} >: j \geq 1\}$ converges for each m. Now take any $u \in M$, we claim that $\{< u, x_{n(j)} >: j \geq 1\}$ is Cauchy. In fact, suppose $\varepsilon > 0$ is given. There is a finite linear combination $y = \sum_{k=1}^{p} \alpha_k e_k$ such that $\|u - y\| \leq \varepsilon$. Then the sequence given by $< y, x_{n(j)} > = \sum_{k=1}^{p} \alpha_k < e_k, x_{n(j)} >$ converges as $j \to \infty$. Hence $\{< y, x_{n(j)} >: j \geq 1\}$ is Cauchy. There is an integer j_0 such that for all $i, j \geq j_0$, we have $| < y, x_{n(i)} > - < y, x_{n(j)} > | \leq \varepsilon$. Since $\{x_n\}$ is bounded, there is $\lambda > 0$ satisfying $\|x_n\| \leq \lambda, \forall\, n$. Now for all $i, j \geq j_0$, we have

$$| < u, x_{n(i)} > - < u, x_{n(j)} > | = | < u, x_{n(i)} - x_{n(j)} > |$$

$$\leq | < u - y, x_{n(i)} - x_{n(j)} > | + | < y, x_{n(i)} - x_{n(j)} > |$$

$$\leq \|u - y\| \, \|x_{n(i)} - x_{n(j)}\| + | < y, x_{n(i)} - x_{n(j)} > | \leq \varepsilon(\lambda + \lambda) + \varepsilon = (2\lambda + 1)\varepsilon.$$

Therefore $\{< u, x_{n(j)} >: j \geq 1\}$ is Cauchy and hence converges in $\mathbb{C}$. Now take any $z \in H$. Write $z = u + v$ where $u \in M$ and $v \in M^{\perp}$. Since $x_{n(j)} \in M$, the sequence $< z, x_{n(j)} > = < u, x_{n(j)} > + < v, x_{n(j)} > = < u, x_{n(j)} >$ converges as $j \to \infty$. Therefore its complex conjugate $\{< x_{n(j)}, z >: j \geq 1\}$ also converges. Consequently, $\{x_{n(j)}\}$ is weakly convergent. □

14-3.6. **Corollary** Every sequence in the closed unit ball of H has a subsequence which is weakly convergent to some point of the closed unit ball. In other words, the closed unit ball of a Hilbert space is *weakly sequentially compact*.

Proof. A sequence $\{x_n\}$ in the closed unit ball is bounded. It has a subsequence $\{y_n\}$ weakly convergent to some $a \in H$. Hence for each $z \in H$, we have $< z, y_n > \to < z, a >$. It follows from the Banach-Steinhaus Theorem that $\|a\| \leq \liminf \|y_n\| \leq 1$, i.e. a lies in the closed unit ball. □

14-4 Diagonal Operators

14-4.1. Let $\{u_n\}$ and $\{e_n\}$ be orthonormal sequences in Hilbert spaces H, G respectively. Let $\{\lambda_n : n \geq 1\}$ be a *bounded* sequence of complex numbers. We allow λ_n to be zero and the sequence may be finite or infinite. For each n, let $u_n \otimes e_n(x) = < x, u_n > e_n$, for every $x \in H$. Clearly, each $u_n \otimes e_n$ is a one dimensional continuous linear map from H into G. In this section, we shall study the following series: $A = \sum_{n \geq 1} \lambda_n u_n \otimes e_n$. Diagonal operators will be introduced as a special case.

14-4.2. **Lemma** A is a continuous linear operator on H. Furthermore we have $A u_n = \lambda_n e_n$, $\lambda_n = < A u_n, e_n >$ and $\|A\| = \sup |\lambda_n|$.

Proof. Let $t = \sup |\lambda_n|$. By Bessel's Inequality, we have

$$\sum_{n\geq 1} \|\lambda_n < x, u_n > e_n\|^2 \leq t^2 \sum_{n\geq 1} | < x, u_n > |^2 \leq t^2 \|x\|^2.$$

Hence the series $Ax = \sum_{n\geq 1} \lambda_n < x, u_n > e_n$ converges in norm and its sum is independent of the order of summation. Clearly, A is linear. Because

$$\|Ax\|^2 = \sum_{n\geq 1} \|\lambda_n < x, u_n > e_n\|^2 \leq t^2\|x\|^2,$$

A is continuous and $\|A\| \leq t$. On the other hand, for each $j \geq 1$, we obtain

$$Au_j = \sum_{n\geq 1} \lambda_n < u_j, u_n > e_n = \lambda_j e_j.$$

Clearly $\lambda_j = < \lambda_j e_j, e_j > = < Au_j, e_j >$. Furthermore we have

$$\|A\| \geq \|Au_j\| = |\lambda_j| \; \|u_j\| = |\lambda_j|.$$

Since j is arbitrary, $\|A\| \geq t$. This completes the proof. □

14-4.3. **Corollary** Let $B = \sum_{n\geq 1} \nu_n u_n \otimes e_n$. If $A = B$, then $\lambda_n = \nu_n$, $\forall\ n$.

Proof. $\lambda_n = < Au_n, e_n > = < Bu_n, e_n > = \nu_n$. □

14-4.4. **Theorem** $A^* = \sum_{n\geq 1} \lambda_n^- e_n \otimes u_n$.

Proof. Since $\{\lambda_n^-\}$ is bounded, $B = \sum_{n\geq 1} \lambda_n^- e_n \otimes u_n$ is a continuous linear map from G into H. For all $x \in H$ and $y \in G$, observe that

$$< x, A^*y > = < Ax, y > = \left\langle \sum_{n\geq 1} \lambda_n < x, u_n > e_n, y \right\rangle$$

$$= \sum_{n\geq 1} \lambda_n < x, u_n >< e_n, y >= \left\langle x, \sum_{n\geq 1} \lambda_n^- < y, e_n > u_n \right\rangle = < x, By >.$$

Therefore $A^* = B$. □

14-4.5. **Theorem** $BA = \sum_{n\geq 1} \lambda_n \nu_n u_n \otimes w_n$ for $B = \sum_{n\geq 1} \nu_n e_n \otimes w_n$ where $\{w_n\}$ be an orthonormal sequence in a Hilbert space K and $\{\nu_n\}$ a bounded sequence of complex numbers.

Proof. $BAx = \sum_{n\geq 1} \nu_n < Ax, e_n > w_n$

$$= \sum_{n\geq 1} \nu_n \left\langle \sum_{k\geq 1} \lambda_k < x, u_k > e_k, e_n \right\rangle w_n$$

$$= \sum_{n\geq 1} \nu_n \sum_{k\geq 1} \lambda_k < x, u_k >< e_k, e_n > w_n$$

$$= \sum_{n\geq 1} \nu_n \sum_{k\geq 1} \lambda_k < x, u_k > \delta_{nk} w_n = \sum_{n\geq 1} \nu_n \lambda_n < x, u_n > w_n.$$ □

14-4.6. For the rest of this section, we shall study a special case when $H = G$ and $u_n = e_n$ for all n. A *diagonal* operator has the form $A = \sum_{n\geq 1} \lambda_n e_n \otimes e_n$ where $\{e_n\}$ is an orthonormal sequence in H and $\{\lambda_n\}$ a bounded sequence of complex numbers. The sequence can be finite or infinite. An operator is said to be *diagonable* if it can be represented as a diagonal operator. We shall study a diagonal operator given above.

14-4.7. **Theorem** Every diagonal operator is normal.

Proof. It follow from the following simple computation
$A^*Ax = \sum_{n\geq 1} \lambda_n^- \lambda_n < x, e_n > e_n = \sum_{n\geq 1} \lambda_n \lambda_n^- < x, e_n > e_n = AA^*x.$ □

14-4.8. **Lemma** Every non-zero eigenvalue of A is some λ_n.

Proof. Let $\|x\| = 1$ be an eigenvector corresponding to a non-zero eigenvalue α. Then $Ax = \alpha x$, i.e. $\sum_{j\geq 1} \lambda_j < x, e_j > e_j = \alpha x$. Since $\alpha x \neq 0$, there is some n satisfying $< x, e_n > \neq 0$. Hence we have

$$\alpha < x, e_n > = < \alpha x, e_n > = < Ax, e_n > = < \sum_j \lambda_j < x, e_j > e_j, e_n >$$
$$= \sum_j \lambda_j < x, e_j >< e_j, e_n > = \lambda_n < x, e_n >.$$

Since $< x, e_n > \neq 0$, we have $\alpha = \lambda_n$. □

14-4.9. **Theorem** A diagonable operator is self-adjoint (respectively skew-adjoint) iff all eigenvalues are real (respectively purely imaginary).

Proof. A is self-adjoint iff $A^* = A$ iff $\lambda_n^- = \lambda_n$ for each n iff all λ_n are real. Similarly we can the case for skew-adjoint operators. □

14-4.10. **Theorem** A diagonable operator is positive iff all eigenvalues are positive (≥ 0).

Proof. Assume that all $\lambda_n \geq 0$. Then $\{\sqrt{\lambda_n}\}$ is also a bounded sequence. Define an operator by $B = \sum_{n\geq 1} \sqrt{\lambda_n} e_n \otimes e_n$. Direct calculation gives $A = B^*B$. Hence A is positive. The converse was proved in §13-11.4d. □

14-4.11. **Theorem** For every diagonable operator A, the following statements are equivalent:

(a) A is a projector.

(b) A is an idempotent.

(c) All eigenvalues are either zero or one.

Proof. For $(b \Rightarrow c)$ If $A^2 = A$, then $\lambda_n^2 = \lambda_n$, i.e. $\lambda_n = 0$ or 1. The case: $(b \Rightarrow c)$ follows by definition and $(c \Rightarrow a)$ by direct calculation. □

14-5 Compact Operators

14-5.1. In this section, we shall prove that compact normal operators are diagonable and that every compact operator can be approximated in norm by finite dimensional operators. Let H, G be Hilbert spaces.

14-5.2. **Theorem** A continuous linear map $A : H \to G$ is compact iff it takes every weakly convergent sequence into a strongly convergent sequence.

Proof. ($\Leftarrow$) Let $\{x_n\}$ be any bounded sequence in H. It has a weakly convergent subsequence $\{y_n\}$. Hence $\{Ay_n\}$ converges strongly. Therefore A is a compact operator.

($\Rightarrow$) Let A be a compact operator on H and $\{x_n\}$ a sequence weakly convergent to some $x \in H$. Suppose to the contrary that $\|Ax_n - Ax\| \not\to 0$. Then there is $\varepsilon > 0$ and a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\|Ay_n - Ax\| > \varepsilon$ for all n. Since $\{x_n\}$ is weakly convergent, it is bounded. By compactness of A, there is $b \in G$ and a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\|Az_n - b\| \to 0$ as $n \to \infty$. Since $x_n \to x$ weakly, we have $z_n \to x$ weakly and consequently $Az_n \to Ax$ weakly. Hence $b = Ax$. Now the contradiction of $\|Az_n - Ax\| > \varepsilon$ and $\|Az_n - b\| \to 0$ completes the proof. □

14-5.3. **Lemma** If $x_n \to a$ weakly and $y_n \to b$ strongly both in H, then we have $< x_n, y_n > \to < a, b >$.

Proof. Since $\{x_n\}$ is weakly convergent, it is bounded. There is $\lambda > 0$ such that $\|x_n\| \le \lambda$ for all n. Now

$$\begin{aligned} &|< x_n, y_n > - < a, b >| \le |< x_n, y_n - b >| + |< x_n, b > - < a, b >| \\ &\le \|x_n\|\ \|y_n - b\| + |< x_n - a, b >| \le \lambda\|y_n - b\| + |< x_n - a, b >| \to 0. \end{aligned}$$ □

14-5.4. **Theorem** If $A : H \to G$ is a compact operator, then so is $A^* : G \to H$.

Proof. Let $x_n \to b$ weakly in G. Since AA^* is compact, $AA^*x_n \to AA^*b$ strongly. Hence $< AA^*x_n, x_n > \to < AA^*b, b >$, i.e. $\|A^*x_n\|^2 \to \|A^*b\|^2$, or $\|A^*x_n\| \to \|A^*b\|$. Since $A^*x_n \to A^*b$ weakly, $A^*x_n \to A^*b$ strongly. Consequently, A^* is compact. □

14-5.5. **Theorem** Let $\{u_n\}$ and $\{e_n\}$ be orthonormal sequences in H, G respectively. Let $\{\lambda_n : n \ge 1\}$ be a bounded sequence of complex numbers. Then the operator $A = \sum_{n\ge1} \lambda_n u_n \otimes e_n$ is compact iff $\lambda_n \to 0$ as $n \to \infty$.

Proof. ($\Rightarrow$) Assume that A is compact. Suppose to the contrary that $\lambda_n \not\to 0$. Then for some $\varepsilon > 0$ there is a subsequence $|\lambda_{n(j)}| \ge \varepsilon, \forall\, j$. Since $u_{n(j)} \to 0$ weakly, we have $Au_{n(j)} \to A0 = 0$ in norm. Now a contradiction is obtained from $\|Au_{n(j)}\| = \|\lambda_{n(j)}e_{n(j)}\| = |\lambda_{n(j)}| \ge \varepsilon$.

($\Leftarrow$) Assume $\lambda_n \to 0$. Then for every $\varepsilon > 0$ there is an integer p such that for all $n \ge p$ we have $|\lambda_n| \le \varepsilon$. Then $Q = \sum_{j=1}^{p} \lambda_j u_j \otimes e_j$ is a finite dimensional linear map from H into G. Now for all $\|x\| \le 1$,

$$\|(A-Q)x\|^2 = \left\|\sum_{j=p+1}^{\infty} \lambda_j < x, u_j > e_j\right\|^2 \le \sum_{j=p+1}^{\infty} |\lambda_j|^2 | < x, u_j > |^2 \|e_j\|^2$$

$$\le \varepsilon^2 \sum_{j=p+1}^{\infty} | < x, u_j > | \le \varepsilon^2 \|x\|^2 \le \varepsilon^2 \quad \text{; by Bessel's inequality.}$$

Taking supremum over $\|x\| \le 1$, we have $\|A - Q\| \le \varepsilon$. Since Q is finite dimensional, A is compact. □

14-5.6. **Theorem** Let A be a compact normal operator on a Hilbert space $H \ne \{0\}$. Then there is an eigenvalue λ of A such that $|\lambda| = \|A\|$.

Proof. If $A = 0$, then $\lambda = 0$ is a required eigenvalue. Assume $A \ne 0$. Due to $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, there is a sequence $\{x_n\}$ in H such that $\|x_n\| = 1$ and $\lim \|Ax_n\| = \|A\|$. Since A is compact, replacing by subsequence we may assume $Ax_n \to y$ strongly for some $y \in H$. Let $B = A^*A$ and $\beta = \|B\|$. Then we have $\beta = \|B\| = \|A^*A\| = \|A\|^2 \ne 0$. Observe that

$$\|Bx_n - \beta x_n\|^2 = \|Bx_n\|^2 - 2 < Bx_n, x_n > +\beta^2 \|x_n\|^2$$

$$\le \|B\|^2\|x_n\|^2 - 2 < A^*Ax_n, x_n > +\beta^2 = \beta^2 - 2\|Ax_n\|^2 + \beta^2 \to 0 \text{ as } n \to \infty.$$

Hence $Bx_n - \beta x_n \to 0$. Let $z = A^*y$. Because of $Ax_n \to y$, we obtain $Bx_n = A^*(Ax_n) \to A^*y = z$. Therefore $\beta x_n = (\beta x_n - Bx_n) + Bx_n \to 0 + z = z$. Thus $\beta = \lim \|\beta x_n\| = \|z\|$, i.e. $z \ne 0$. Also $Bz = B(\lim \beta x_n) = \beta \lim Bx_n = \beta z$. Therefore β is an eigenvalue of B. Let $N = \ker(\beta I - B)$. Because $z \ne 0$ is in N, we get $N \ne \{0\}$. Since $B = A^*A$ is compact and $\beta \ne 0$, we have $\dim N < \infty$. Now the normality of A gives $AB = BA$ and thus $A(N) \subset N$. Considering the characteristic polynomial of $A|N$, there is $u \in N$ such that $\|u\| = 1$ and $Au = \lambda u$ for some $\lambda \in \mathbb{C}$. Since $u \in N$, we have $Bu = \beta u$. On the other hand,

$$Bu = A^*Au = A^*(\lambda u) = \lambda A^*u = \lambda\lambda^- u.$$

Therefore $\beta u = \lambda\lambda^- u$, i.e. $|\lambda|^2 = \beta = \|B\| = \|A\|^2$, or $|\lambda| = \|A\|$. □

14-5.7. **Theorem** Let $\{H_n\}$ be a sequence, finite or infinite, of orthogonal closed subspaces of a Hilbert space H. Then the set $\sum_{n\ge1} H_n$ of all convergent series $x = \sum_{n\ge1} x_n$ for $x_n \in H_n$ is the closed vector subspace spanned by $\bigcup_{n\ge1} H_n$. Furthermore, if Q, P_k are the projectors onto $\sum_{n\ge1} H_n$ and H_k respectively, then for each $x \in H$ we have a convergent series $Qx = \sum_{n\ge1} P_n x$.

Proof. Allowing $H_n = \{0\}$, we may work with the infinite sequence. Take any $x = \sum_{n=1}^{\infty} x_n$ and $y = \sum_{n=1}^{\infty} y_n$ where $x_n, y_n \in H_n$. Since

$$\sqrt{\sum_{n=1}^{\infty} \|x_n + y_n\|^2} \le \sqrt{\sum_{n=1}^{\infty} \|x_n\|^2} + \sqrt{\sum_{n=1}^{\infty} \|y_n\|^2} = \|x\| + \|y\| < \infty,$$

the series $\sum_{n=1}^{\infty}(x_n + y_n)$ converges and hence $x + y \in \sum_{n=1}^{\infty} H_n$. Similarly $\lambda x \in \sum_{n=1}^{\infty} H_n$. Thus $\sum_{n=1}^{\infty} H_n$ is a vector subspace of H. Next we claim that $\sum_{n=1}^{\infty} H_n$ is closed. In fact, take any closure point y of $\sum_{n=1}^{\infty} H_n$. Write $y = \lim x^j$ where $x^j = \sum_{n=1}^{\infty} x_n^j$ and $x_n^j \in H_n$. Since $\{x^j\}$ is a Cauchy sequence, for every $\varepsilon > 0$ there is an integer p such that for all $j, k \geq p$ we have $\|x^j - x^k\| \leq \varepsilon$. By $\sum_{n=1}^{\infty} \|x_n^j - x_n^k\|^2 = \sum_{n=1}^{\infty} \|x^j - x^k\|^2 \leq \varepsilon^2$, for each n the sequence $\{x_n^j : j \geq 1\}$ is Cauchy. Since H_n is closed in the complete space H, we can write $\lim_{j\to\infty} x_n^j = x_n \in H_n$. Letting $k \to \infty$ in the inequality $\sum_{n=1}^{m} \|x_n^j - x_n^k\|^2 \leq \varepsilon^2$, we have $\sum_{n=1}^{m} \|x_n^j - x_n\|^2 \leq \varepsilon^2$. Since m is arbitrary, we get $\sum_{n=1}^{\infty} \|x_n^j - x_n\|^2 \leq \varepsilon^2$. Hence the series $\sum_{n=1}^{\infty}(x_n^j - x_n)$ is convergent, and so does $x = \sum_{n=1}^{\infty} x_n$. Moreover, $\|x^j - x\|^2 = \sum_{n=1}^{\infty} \|x_n^j - x_n\|^2 \leq \varepsilon^2, \forall\, j \geq p$. In other words, $y = \lim x^j = x \in \sum_{n=1}^{\infty} H_n$. Therefore $\sum_{n=1}^{\infty} H_n$ is a closed vector subspace of H. Clearly, it is the smallest one containing $\bigcup_n H_n$. Finally take any $x \in H$. Since $Qx \in \sum_{n=1}^{\infty} H_n$, write $Qx = \sum_{n=1}^{\infty} x_n$ where $x_n \in H_n$. For each j, $P_j x = P_j Q x = \sum_{n=1}^{\infty} P_j x_n = x_j$. Therefore $Qx = \sum_{n=1}^{\infty} P_n x$. □

14-5.8. Let A be a *compact normal* operator on a H. Let $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ be an enumeration of all *non-zero* eigenvalues of A repeated according to its multiplicity. It may be a null, or finite or infinite sequence. If it is null, then A must be the zero operator. Hence we assume A has at least one non-zero eigenvalue in the following context. For each $n \geq 1$, let $H_n = \ker(A - \lambda_n I)$ denote the eigenspace of λ_n and let $H_0 = \ker(A)$. For every $n \geq 0$, let P_n be the projector onto the closed vector subspace H_n. Note that for each $n \geq 1$, H_n is finite dimensional but H_0 may be infinite dimensional. The following theorem shows the existence of a spectral resolution of the identity.

14-5.9. **Theorem** For each $x \in H$, the series $x = \sum_{n\geq 0} P_n x$ converges and we also have $P_j P_k = 0$ for all $j \neq k$. It is called the *spectral expansion* of x with respect to the enumeration $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ of eigenvalues of A.

Proof. Let $M = \sum_{n\geq 1} H_n$. Clearly for all $n \geq 1$, we have $A(H_n) \subset H_n$ and hence $A(M) \subset M$, or $A^*(M^\perp) \subset M^\perp$. Next we claim that every eigenvalue λ of the restriction $A^*|M^\perp$ must be zero. Suppose to the contrary that $\lambda \neq 0$ is an eigenvalue of $A^*|M^\perp$. Then there is $x \neq 0$ in $M^\perp$ satisfying $A^*x = \lambda x$, or $Ax = \lambda^- x$. Since $\{\lambda_n : n \geq 1\}$ is an enumeration of *all* non-zero eigenvalues of A, we have $\lambda^- = \lambda_n$ for some n. Then $x \in H_n \subset M$. This is a contradiction to $M \cap M^\perp = \{0\}$. Therefore $\lambda = 0$. Since A is compact normal, so is $A^*|M^\perp$. Thus $\|A^*|M^\perp\|$ is maximum of all eigenvalues. It follows that $A^*|M^\perp$ is a zero operator. Hence $M^{\perp} \subset \ker(A^*) = \ker(A)$ by normality of A. Now take any $x \in H$. Since $H = M^{\perp} \oplus M$, we can write $x = x_0 + \sum_{n\geq 1} x_n$ where $x_n \in H_n$. Therefore $x = \sum_{n\geq 0} P_n x$. Because $H_j \perp H_k$ for all $j \neq k$, we have $P_j P_k = 0$. □

14-5.10. **Lemma** If the enumeration $|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_k| > 0$ of all non-zero eigenvalues of a compact normal operator A on H is *finite*, then $A = \sum_{n=1}^{k} \lambda_n P_n$. Consequently, A is finite dimensional.

Proof. Take any $x \in H$. Then there is $x_n \in H_n$ for each $n \geq 0$ such that $x = \sum_{n=0}^{k} x_n$. Now $Ax = \sum_{n=0}^{k} Ax_n = \sum_{n=1}^{k} \lambda_n x_n = \sum_{n=1}^{k} \lambda_n P_n x$. Therefore $A = \sum_{n=1}^{k} \lambda_n P_n$. Since all H_n are finite dimensional, so is A. □

14-5.11. **Spectral Theorem** If $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ is an enumeration of all non-zero eigenvalues of a compact normal operator A, then $A = \sum_{n=1}^{\infty} \lambda_n P_n$ in the sense that $\|A - \sum_{n=1}^{k} \lambda_n P_n\| = |\lambda_{k+1}| \to 0$ as $k \to \infty$.

Proof. Assume that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ is an infinite sequence. Take any $x \in H$. Let $x = \sum_{n=0}^{\infty} P_n x$ be the spectral expansion of x. By continuity of A, we have $Ax = \sum_{n=0}^{\infty} Ax_n = 0 + \sum_{n=1}^{\infty} \lambda_n x_n = \sum_{n=1}^{\infty} \lambda_n P_n x$. Therefore

$$\left\|\left(A - \sum_{n=1}^{k} \lambda_n P_n\right) x\right\|^2 = \left\|\sum_{n=k+1}^{\infty} \lambda_n P_n x\right\|^2 = \sum_{n=k+1}^{\infty} \|\lambda_n P_n x\|^2$$
$$= \sum_{n=k+1}^{\infty} |\lambda_n|^2 \|P_n x\|^2 \leq |\lambda_{k+1}|^2 \sum_{n=k+1}^{\infty} \|P_n x\|^2$$
$$\leq |\lambda_{k+1}|^2 \sum_{n=0}^{\infty} \|P_n x\|^2 = |\lambda_{k+1}|^2 \|x\|^2.$$

Consequently, $$\left\|A - \sum_{n=1}^{k} \lambda_n P_n\right\| \leq |\lambda_{k+1}|.$$

On the other hand, let $\|x\| = 1$ be in H_{k+1}. Then

$$|\lambda_{k+1}| = \|\lambda_{k+1} x\| = \left\|\left(A - \sum_{n=1}^{k} \lambda_n P_n\right) x\right\| \leq \left\|A - \sum_{n=1}^{k} \lambda_n P_n\right\|.$$

Therefore $$\left\|A - \sum_{n=1}^{k} \lambda_n P_n\right\| = |\lambda_{k+1}|.$$

The fact $\lambda_k \to 0$ as $k \to \infty$ has been proved in Normed Spaces. We leave the case when $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ is a finite sequence as an exercise. □

14-5.12. **Diagonal Representation of Compact Normal Operators** Let A be a compact normal operator on H. Then there is an orthonormal sequence $\{e_n : n \geq 1\}$ of eigenvectors with corresponding eigenvalues λ_n such that for each $x \in H$, we have $x = x_0 + \sum_{n\geq 1} < x, e_n > e_n$ where $x_0 \perp e_n$ for all $n \geq 1$. It is called the *coordinate expansion* of x with respect to $\{e_n\}$. Furthermore we have the following diagonal representation: $A = \sum_{n\geq 1} \lambda_n e_n \otimes e_n$.

Proof. For each $n \geq 1$ let Q_n be an orthonormal basis of H_n. Let $Q = \bigcup_n Q_n$. Since H_n is finite dimensional for each $n \geq 1$, every Q_n is a finite set. Since A is normal, Q is a countable orthonormal set. Now take any $x \in H$. Then $P_n x \in H_n$. For each $n \geq 1$, because Q_n is an orthonormal basis of H_n, we obtain $P_n x = \sum_{e \in Q_n} < x, e > e$. Now the spectral representation becomes the coordinate representation: $x = x_0 + \sum_{n \geq 1} \sum_{e \in Q_n} < x, e > e$. By continuity of A, we get

$$\begin{aligned} Ax &= Ax_0 + \sum_{n \geq 1} \sum_{e \in Q_n} < x, e > Ae \\ &= 0 + \sum_{n \geq 1} \sum_{e \in Q_n} < x, e > \lambda_n e = \sum_{n \geq 1} \sum_{e \in Q_n} \lambda_n < x, e > e. \end{aligned}$$ □

14-5.13. Let H, G be Hilbert spaces and $A : H \to G$ a compact linear map. Clearly A^*A is a compact positive operator on H. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ be an enumeration of all non-zero eigenvalues of A^*A repeated according to its multiplicity. Then $s_n(A) = \sqrt{\lambda_n}$ is called the n-th *singular number* of A.

14-5.14. **Finite Dimensional Approximation Theorem** Let $A : H \to G$ be a compact linear map on and s_n the n-th singular number of A. Then there are orthonormal sequences $\{u_n\}$ and $\{e_n\}$ in H, G respectively such that the following conditions hold.
(a) If A is of finite rank k, then $A = \sum_{n=1}^{k} s_n u_n \otimes e_n$.
(b) If A is infinite dimensional, then $\|A - \sum_{n=1}^{k} s_n u_n \otimes e_n\| \to 0$ as $k \to \infty$.
In symbol, we may write $A = \sum_{n \geq 1} s_n u_n \otimes e_n$. Consequently, every compact operator on a Hilbert space can be approximated in norm by finite dimensional operators. Note that in general this is false in Banach spaces.

Proof. Write $A^*A = \sum_{n \geq 1} \lambda_n u_n \otimes u_n$ where $\{u_n\}$ is an orthonormal sequence of eigenvectors of A^*A corresponding to eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots > 0$ respectively. By definition, $s_n = \sqrt{\lambda_n} > 0$. Take any $x \in H$. Let $x = x_0 + \sum_{n \geq 1} < x, u_n > u_n$ be the coordinate expansion corresponding to $\{u_n\}$ where $x_0 \perp u_n$ for all $n \geq 1$. Then $A^*Ax_0 = \sum_{n \geq 1} \lambda_n < x_0, u_n > u_n = 0$, that is $\|Ax_0\|^2 = | < A^*Ax_0, x_0 > | = 0$, or $Ax_0 = 0$. Hence we obtain $Ax = \sum_{n \geq 1} < x, u_n > Au_n$. Let $e_n = Au_n / s_n$. Since

$$\begin{aligned} < e_m, e_n > &= \frac{< Au_m, Au_n >}{s_m s_n} = \frac{< A^*Au_m, u_n >}{s_m s_n} \\ &= \frac{\sum_j \lambda_j < u_m, u_j >< u_j, u_n >}{s_m s_n} = \frac{s_n^2 < u_m, u_n >}{s_m s_n} = \delta_{mn}, \end{aligned}$$

$\{e_n\}$ is an orthonormal sequence. Furthermore, we have

$$Ax = \sum_{n\geq 1} < x, u_n > Au_n = \sum_{n\geq 1} s_n < x, u_n > e_n.$$

If $rank(A) = k < \infty$, then $A = \sum_{n=1}^{k} s_n u_n \otimes e_n$ because $\{e_n\}$ is a linearly independent set. Suppose A is infinite dimensional. Take any $\|x\| \leq 1$. Observe

$$\left\| \left(A - \sum_{n=1}^{k} s_n u_n \otimes e_n \right) x \right\|^2 \leq \sum_{n=k+1}^{\infty} |s_n < x, u_n > e_n|^2$$
$$= \sum_{n=k+1}^{\infty} |s_n|^2 \; | < x, u_n > |^2 = \sum_{n=k+1}^{\infty} \lambda_n | < x, u_n > |^2$$
$$\leq \lambda_{k+1} \|x\|^2 \leq \lambda_{k+1} \to 0$$

as $k \to \infty$. This completes the proof. □

14-5.15. **Corollary** $s_n(A^*) = s_n(A)$ for every compact linear map $A : H \to G$.

Proof. Let $A = \sum_{n\geq 1} s_n(A) u_n \otimes e_n$ be a spectral representation. Then we have $A^* = \sum_{n\geq 1} s_n(A) e_n \otimes u_n$. Therefore $s_n(A^*) = s_n(A)$ by §14-4.3. □

14-5.16. **Exercise** Let A be a compact normal operator on a nontrivial Hilbert space H. Let B, C be the real and imaginary parts of A. Prove that A has an eigenvalue λ satisfying $\max(\|B\|, \|C\|) \leq \lambda$.

14-6 Functional Calculus of Self-Adjoint Operators

14-6.1. Holomorphic maps of an operator was defined in terms of Cauchy integral formula. For a self-adjoint operator A, we shall define $f(A)$ when f is a continuous function on the spectrum of A. As a result, we can take square roots of positive operators. We proved that if an operator A is positive, i.e. $A = B^*B$ for some B, then $A \geq 0$, i.e. $< Ax, x > \geq 0$. In this section, the converse will be proved.

14-6.2. **Theorem** For every normal operator A, $\|A\| = \sup\{ \, |\lambda| : \lambda \in \sigma(A)\}$.

Proof. Since $A^*A = AA^*$, it is easy to show $(A^n)^*(A^n) = (A^n)(A^n)^*$. Hence A^n is also normal. Repeated application of $\|A^2\| = \|A\|^2$, we have $\left\| A^{2^k} \right\| = \|A\|^{2^k}$.

Therefore we have $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{k\to\infty} \left\| A^{2^k} \right\|^{1/2^k} = \|A\|$. □

14-6.3. **Exercise** Prove that a normal quasinilpotent operator must be zero.

14-6.4. **Theorem** For every complex polynomial

$$f(t) = \alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_1 t + \alpha_0,$$

let
$$f^{\sim}(t) = \alpha_n^- t^n + \alpha_{n-1}^- t^{n-1} + \cdots + \alpha_1^- t + \alpha_0^-$$
which is f^- when t is real. Then for every normal operator A, we have

(a) $[f(A)]^* = f^{\sim}(A^*)$;

(b) $f(A)$ is normal;

(c) $\|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\}$.

Proof. Both (a) and (b) follow by routine calculations:
$$\begin{aligned}[f(A)]^* &= [\alpha_n A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I]^* \\ &= \alpha_n^- (A^n)^* + \alpha_{n-1}^- (A^{n-1})^* + \cdots + \alpha_1^- A^* + \alpha_0^- I^* \\ &= \alpha_n^- (A^*)^n + \alpha_{n-1}^- (A^*)^{n-1} + \cdots + \alpha_1^- A^* + \alpha_0^- I = f^{\sim}(A^*)\end{aligned}$$
and
$$\begin{aligned}[f(A)]^*[f(A)] &= [f^{\sim}(A^*)][f(A)] = \left[\sum_{j=0}^n \alpha_j^- (A^*)^j\right]\left[\sum_{k=0}^n \alpha_k A^k\right] \\ &= \sum_{j=0}^n \sum_{k=0}^n \alpha_j^- \alpha_k (A^*)^j A^k = \left[\sum_{k=0}^n \alpha_k A^k\right]\left[\sum_{j=0}^n \alpha_j^- (A^*)^j\right] \\ &= [f(A)][f^{\sim}(A^*)] = [f(A)][f(A)]^*.\end{aligned}$$
For (c), the Spectral Polynomial Theorem gives
$$\|f(A)\| = \sup\{|\mu| : \mu \in \sigma[f(A)]\} = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\}. \qquad \square$$

14-6.5. **Theorem** Let A be a *self-adjoint* operator on H. Let $C_\infty[\sigma(A)]$ be the sup-norm algebra of continuous complex functions on $\sigma(A)$. Then there is a unique linear map $f \to f(A)$ form $C_\infty[\sigma(A)]$ into the algebra $L(H)$ of operators on H such that the following conditions hold:

(a) Isometry: $\|f(A)\| = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\}$.

(b) Algebra Isomorphism: $(f \cdot g)(A) = f(A)g(A)$.

(c) Polynomials: If $f(x) = 1$ then $f(A) = I$. If $f(x) = x$, then $f(A) = A$.

Proof. Let P be the set of all complex polynomials on $\sigma(A)$. Since A is self-adjoint, $\sigma(A)$ consists of real numbers only and hence P is self-conjugate. By Stone-Weierstrass Theorem, P is a dense subspace of $C_\infty[\sigma(A)]$. We have proved that $f \to f(A)$ is a linear isometry from P into $L(H)$. Since $L(H)$ is complete, there is a linear extension over $C_\infty[\sigma(A)]$ into $L(H)$. The uniqueness is left as an exercise. □

14-6.6. **Corollary** Let A be a *self-adjoint* operator on H and let f_n, g be functions continuous on $\sigma(A)$. If $f_n \to g$ uniformly on $\sigma(A)$, then $f_n(A) \to g(A)$ in norm.

Proof. $\|(f_n - g)(A)\| = \|f_n - g\|_\infty \to 0$. □

14-6.7. **Corollary** Let A be a *self-adjoint* operator on H, f a continuous function on $\sigma(A)$ and B any operator.
(a) Commutativity: If $AB = BA$, then $f(A)B = Bf(A)$.
(b) Eigenvalues: If $Ax = \lambda x$, then $f(A)x = f(\lambda)x$.

Proof. Let p_n be polynomials on $\sigma(A)$ such that $\|f-p_n\|_\infty \to 0$. By $AB = BA$, we have $p_n(A)B = Bp_n(A)$. Letting $n \to \infty$, we obtain $f(A)B = Bf(A)$. Since $p_n(A)x = p_n(\lambda)x$, we have $f(A)x = f(\lambda)x$. □

14-6.8. **Corollary** Let A be a self-adjoint operator on H. Then for every continuous function f on $\sigma(A)$, we have
(a) $[f(A)]^* = f^-(A)$;
(b) $f(A)$ is normal;
(c) if f is real-valued, then $f(A)$ is self-adjoint.

Proof. Since A is self-adjoint, $\sigma(A)$ is a subset of $\mathbb{R}$. Part (a) is true for the polynomials by direct verification. By passing limits, it is also valid for continuous functions on $\sigma(A)$. Part (b) follows from

$$[f(A)]^* f(A) = f^-(A) f(A) = (f^- \cdot f)(A)$$
$$= (f \cdot f^-)(A) = f(A) f^-(A) = [f(A)][f(A)]^*.$$

For (c), $[f(A)]^* = f^-(A) = f(A)$. □

14-6.9. **Theorem** Let A be a self-adjoint operator on H. Then the following statements are equivalent.
(a) $\sigma(A) \geq 0$.
(b) $A = B^2$ for some positive operator B.
(c) $A = B^*B$ for some operator B.
(d) $A \geq 0$.

Proof. ($a \Rightarrow b$) Let $f(\lambda) = \sqrt[4]{\lambda}$ for every $\lambda \geq 0$. Then f is continuous on $\sigma(A) \geq 0$. Since f is real-valued, $f(A)$ is self-adjoint. Hence we obtain $B = [f(A)]^2 = f(A)^* f(A)$ is positive. By $f(\lambda)^4 = \lambda$ for all $\lambda \geq 0$, we have $A = [f(A)]^4 = B^2$.
($b \Rightarrow c$) Since B is positive, $B^* = B$.
($c \Rightarrow d$) It was an exercise §13-10.9a.
($d \Rightarrow a$) Let $h(\lambda) = 0$ if $\lambda \geq 0$ and $h(\lambda) = \sqrt{-\lambda}$ if $\lambda < 0$. Then h is continuous on $\mathbb{R}$. Let $B = h(A)$. Since h is real-valued, B is self-adjoint. Hence $B^2 = B^*B$ is positive. By ($c \Rightarrow d$), we have $B^2 \geq 0$. Since $h(\lambda)\lambda h(\lambda) = -[h(\lambda)]^4$ for all λ, we get $BAB = -B^4$. By $0 \leq BAB = -B^4 \leq 0$, we obtain $0 = B^4 = [h(A)]^4$.

Because $f \to f(A)$ is injective, $[h(\lambda)]^4 = 0$, i.e. $h(\lambda) = 0$ for all $\lambda \in \sigma(A)$. Therefore $\sigma(A) \geq 0$. □

14-6.10. **Square Root Theorem** Let A be a positive operator. Then there is a unique positive operator denoted by $\sqrt{A}$ such that $A = (\sqrt{A})^2$. Furthermore if A is invertible, so is $\sqrt{A}$.

Proof. The existence was proved by ($c \Rightarrow b$) of last theorem. We give another more intuitive proof here. Let $f(\lambda) = \sqrt{\lambda}$ for all $\lambda \geq 0$. Then f is continuous on $\sigma(A)$ and $[f(\lambda)]^2 = \lambda$. Hence $f(A)$ is an operator satisfying $[f(A)]^2 = A$. Now suppose B is any positive operator satisfying $B^2 = A$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ uniformly on $\sigma(A)$. Define $g_n(\lambda) = p_n(\lambda^2)$. Then $g_n(\lambda) \to \lambda$ uniformly on $\sigma(B)$. Hence

$$B = \lim g_n(B) = \lim p_n(B^2) = \lim p_n(A) = f(A).$$

This proves the uniqueness. Finally if A is invertible, then

$$0 \notin \sigma(A) = \{\lambda^2 : \lambda \in \sigma(\sqrt{A})\},$$

i.e. $0 \notin \sigma(\sqrt{A})$. Therefore $\sqrt{A}$ is invertible. □

14-6.11. **Theorem** Let A be a self-adjoint operator and f a continuous function on $\sigma(A)$. Then we have $\sigma[f(A)] = f[\sigma(A)]$.

Proof. Let $\lambda \in \sigma(A)$. Suppose to the contrary that $f(\lambda) \notin \sigma[f(A)]$. Then $f(\lambda)I - f(A)$ is invertible. There is a polynomial g such that

$$\sup_{\mu\in\sigma(A)} |(g - f)(\mu)| \leq \tfrac{1}{3} \left\| [f(\lambda)I - f(A)]^{-1} \right\|^{-1}.$$

Hence we have

$$\begin{aligned}
&\|[f(\lambda)I - f(A)] - [g(\lambda)I - g(A)]\| \leq \|f(A) - g(A)\| + |f(\lambda) - g(\lambda)| \\
&\leq \sup_{\mu\in\sigma(A)} \|(f - g)(\lambda)\| + |f(\lambda) - g(\lambda)| = \frac{2}{3} \left\|[f(\lambda)I - f(A)]^{-1}\right\|^{-1} \\
&< \left\|[f(\lambda)I - f(A)]^{-1}\right\|^{-1}.
\end{aligned}$$

Hence $g(\lambda)I - g(A)$ is invertible. Hence $g(\lambda) \notin \sigma[g(A)]$. This contradicts the Spectral Polynomial Theorem. Therefore $f[\sigma(A)] \subset \sigma[f(A)]$. Conversely, suppose to the contrary that there is $\mu \in \sigma[f(A)]$ but $\mu \notin f[\sigma(A)]$. For each $\lambda \in \sigma(A)$, define $g(\lambda) = 1/[\mu - f(\lambda)]$. Then g is continuous on $\sigma(A)$. Hence $g(A)$ is an operator on H. Since $g(\lambda)[\mu - f(\lambda)] = 1$, $\forall\ \lambda \in \sigma(A)$, we have $g(A)[\mu I - f(A)] = I$ and $[\mu I - f(A)]g(A) = I$. Therefore $\mu I - f(A)$ is invertible, i.e. $\mu \notin \sigma[f(A)]$. This contradiction establishes the proof. □

14-6.12. **Theorem** Let A be a self-adjoint operator. Let f be a continuous function on $\sigma(A)$.

(a) $f(A)$ is self-adjoint iff f is real-valued.

(b) $f(A)$ is unitary iff $|f| = 1$.

(c) $f(A)$ is invertible iff $f(\lambda) \neq 0, \forall\ \lambda \in \sigma(A)$.

(d) $f(A) \geq 0$ iff $f[\sigma(A)] \geq 0$.

Proof. (a) $f(A)$ is self-adjoint iff $\|f(A)^* - f(A)\| = 0$ iff $\|(f^- - f)(A)\| = 0$ iff $f^- - f = 0$ on $\sigma(A)$ iff f is real-valued.

(b) $f(A)$ is unitary iff $[f(A)]^* f(A) = I = f(A)[f(A)]^*$ iff $f^- \cdot f = 1 = f \cdot f^-$ iff $|f| = 1$.

(c) $f(A)$ is invertible iff $0 \notin \sigma(A)$ iff $0 \notin f[\sigma(A)]$ iff $f(\lambda) \neq 0, \forall\ \lambda \in \sigma(A)$.

(d) Suppose $f[\sigma(A)] \geq 0$. Then $g = \sqrt{f}$ is a continuous real-valued function on $\sigma(A)$. Hence $g(A)$ is self-adjoint and $f(A) = [g(A)]^2$ is positive. Conversely, suppose $f(A) \geq 0$. Since $f(A)$ is self-adjoint, f is real-valued by (a). Let $v = \max\{-f, 0\}$ denote the negative part of f. By $f(A) \geq 0$, we obtain $0 \leq [v(A)]^* f(A) v(A)$. Because $v^3 \geq 0$ and by what was just proved, $v^3(A) \geq 0$. Since $v^- f v = v f v = -v^3$, we have $0 \leq [v(A)]^* f(A) v(A) = -v^3(A) \leq 0$, i.e. $v^3(A) = 0$. By injectivity, $v^3 = 0$, i.e. $v = 0$ on $\sigma(A)$. Therefore we conclude that $f \geq 0$ on $\sigma(A)$. □

14-6.13. **Exercise** Prove that the product of two commuting positive operators A, B is a positive operator. Furthermore, prove that $\sqrt{AB} = \sqrt{A}\sqrt{B}$. Show that the matrices: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are positive but their product is not.

14-6.14. **Exercise** Prove that if A is self-adjoint and if A^2 is a projector then A is a projector.

14-6.15. **Exercise** Prove that $\|\sqrt{A^*A}\| = \sqrt{\|A^*A\|}$ for every operator A.

14-6.16. **Exercise** Prove that the inverse of an invertible positive operator is positive.

14-6.17. Let $\mathcal{F}$ be a family of operators on H. An operator *commutes* with $\mathcal{F}$ if it commutes with every operator in $\mathcal{F}$. The family $\mathcal{F}'$ of operators commuting with $\mathcal{F}$ is called the *commutant* of $\mathcal{F}$. Clearly $\mathcal{F}'$ is a subalgebra of $L(H)$. The family $\mathcal{F}$ is *self-adjoint* if $A^* \in \mathcal{F}$ whenever $A \in \mathcal{F}$.

14-6.18. **Theorem** Every operator A commuting with a self-adjoint family $\mathcal{F}$ of operators on H is a linear combination of unitary operators in $\mathcal{F}'$.

Proof. Firstly assume that $A \in \mathcal{K}$ is self-adjoint with $\|A\| \leq 1$. Since $f(t) = \sqrt{1-t^2} \geq 0$ is continuous on $\sigma(A)$, the operator $B = f(A) \geq 0$ is well-defined and commutes with $\mathcal{F}$. Hence $U = A + iB \in \mathcal{F}'$. From $UU^* = U^*U = A^2 + B^2 = I$, both U, U^* are unitary. Now $A = \frac{1}{2}(U + U^*)$ is a linear combination of unitary operators in $\mathcal{F}'$. Next in general, let $A \in \mathcal{F}'$ be given. Then for every $Q \in \mathcal{F}$, we have $Q^* \in \mathcal{F}$ and hence $A^*Q = (Q^*A)^* = (AQ^*)^* = AQ^*$, that is $A^* \in \mathcal{F}'$. Therefore both $B = \frac{1}{2}(A + iA^*)$ and $C = \frac{1}{2i}(A - iA^*)$ are self-adjoint operators in $\mathcal{F}'$. By the first case, $B/(1 + \|B\|)$ and hence B itself are linear combinations of unitary operators in $\mathcal{F}'$. Similarly C and consequently $A = B + iC$ are also linear combinations of unitary operators in $\mathcal{F}'$. □

14-7 Polar Decomposition

14-7.1. Every complex number can be written in polar form $re^{i\theta}$ where r is positive (≥ 0) and $e^{i\theta}$ is a one-by-one unitary matrix. In this section, an analogue for operators will be given.

14-7.2. Let A be an operator on a Hilbert space H. Then A is called a *partial isometry* if A^*A is a projector. The *initial space* of A is defined as $init(A) = (\ker A)^\perp$.

14-7.3. **Theorem** Let A be an operator on H. Then the following statements are equivalent.

(a) A is a partial isometry, i.e. A^*A is a projector.

(b) For every x in the initial space of A, we have $\|Ax\| = \|x\|$.

(c) $A = AA^*A$.

(d) A^* is a partial isometry.

Proof. ($a \Rightarrow b$) Let $P = A^*A$ and $M = P(H)$. Then P is a projector onto M. Now $M = \ker(P)^\perp = \ker(A^*A)^\perp = \ker(A)^\perp$ is the initial space of A. Take any $x \in M$. We have $Px = x$. Hence

$$\|Ax\|^2 = < A^*Ax, x > = < Px, x > = < x, x > = \|x\|^2,$$

i.e. $\|Ax\| = \|x\|$.

($b \Rightarrow a$) Let $M = init(A) = \ker(A)^\perp$ and P the projector onto M. Then for any $x \in M$, we have $< A^*Ax, x > = \|Ax\|^2 = \|x\|^2 = < Px, x >$. On the other hand, for any $x \in M^{\perp}$, we get $x \in \ker(A)^{\perp\perp} = \ker(A)$, i.e. $Ax = 0$. Hence $< A^*Ax, x > = 0 = < Px, x >$. By linearity, we obtain

$$< A^*Ax, x > = 0 = < Px, x >$$

for all $x \in H$. Hence $A^*A = P$ is a projector. Therefore A is a partial isometry.
$(a \Rightarrow c)$ Let A^*A be a projector onto a vector subspace M of H. Then for any $x \in M$, $A^*Ax = x$ and hence $AA^*Ax = Ax$. On the other hand, for any $x \in M^{\perp}$, $A^*Ax = 0$, or $Ax = 0$ and hence $AA^*Ax = 0 = Ax$. By linearity, we get $AA^*Ax = Ax, \forall\, x \in H$. Therefore $AA^*A = A$.
$(c \Rightarrow a)$ $(A^*A)^2 = A^*(AA^*A) = A^*A$ and $(A^*A)^* = A^*A^{**} = A^*A$. Therefore A^*A is a projector.
$(c \Rightarrow d)$ From $A^* = (AA^*A)^* = A^*A^{**}A^*$, A^* is an isometry by $(c \Rightarrow a)$.
$(d \Rightarrow c)$ Apply $(c \Rightarrow d)$ to A^*. □

14-7.4. **Corollary** If A is a partial isometry, then we have

$$init(A) = \operatorname{Im}(A^*A) = \operatorname{Im}(A^*) \text{ and } \operatorname{Im}(A) = \operatorname{Im}(AA^*) = init(A^*).$$

Proof. Since A^*A is a projector, we have

$$init(A) = \ker(A)^{\perp} = \ker(A^*A)^{\perp} = \operatorname{Im}(A^*A).$$

Next,
$$\operatorname{Im}(AA^*) = AA^*(H) \subset A(H) = \operatorname{Im}(A),$$
and
$$A(H) = AA^*A(H) = AA^*(H) = \operatorname{Im}(AA^*)$$
gives $\operatorname{Im}(A) = \operatorname{Im}(AA^*)$. By symmetry, we get

$$\operatorname{Im}(A^*) = \operatorname{Im}[A^*(A^*)^*] = \operatorname{Im}(A^*A) = init(A)$$
and
$$init(A^*) = \operatorname{Im}[(A^*)^*A^*] = \operatorname{Im}(AA^*) = \operatorname{Im}(A).$$ □

14-7.5. **Polar Decomposition Theorem** Let A be an operator on H. Then there is a unique positive operator P and a unique partial isometry U such that $A = UP$ and $\ker(U) = \ker(P)$. Furthermore, we have
(a) U^*U is a projector onto the closure of $\operatorname{Im}(P)$;
(b) $\ker(A) = \ker(P)$;
(c) $P = U^*A$.

Proof. The operator $P = \sqrt{A^*A}$ is positive. For all $x \in H$, we have

$$\|Px\|^2 = < Px, Px > = < P^*Px, x > = < P^2x, x > = < A^*Ax, x > = \|Ax\|^2.$$

Hence if $Px = Py$, then $P(x - y) = 0$, or $A(x - y) = 0$, i.e. $Ax = Ay$. Now the map $V : P(H) \to H$ given by $VPx = Ax$ is well-defined. Clearly, V

is linear. Since $\|V(Px)\| = \|Px\|$, V is also continuous. There is a unique continuous linear extension over the closure of $P(H)$. This extension is still denoted by V. Define $Ux = Vx$ if $x \in P(H)^-$ and $Ux = 0$ if $x \in P(H)^\perp$. Then by linearity, U is defined on H. Since for all non-zero $x \in P(H)^-$, we get $\|Ux\| = \|Vx\| = \|x\| \neq 0$, U is a partial isometry and $A = UP$ by the construction of U. Next, $\ker(U) = P(H)^\perp = \ker(P^*) = \ker(P)$. This proves the existence. For uniqueness, suppose that P, Q are positive operators and U, V are partial isometries such that $A = UP = VQ$, $\ker(U) = \ker(P)$ and $\ker(V) = \ker(Q)$. Then $P^2 = A^*A = Q^*V^*VQ$. Now V^*V is a projector onto the subspace $init(V) = \ker(V)^\perp = \ker(Q)^\perp = \mathrm{Im}(Q^*)^- = \mathrm{Im}(Q)^-$. Hence $V^*VQ = Q$. Thus $P^2 = Q^*V^*VQ = Q^*Q = Q^2$. By uniqueness of square roots, $P = Q$. Hence $UP = VP$. Observe $\mathrm{Im}(P)^\perp = \ker(P^*) = \ker(P) = \ker(U)$. For $x \in \mathrm{Im}(P)$, we have $Ux = Vx$. On the other hand, for $x \in \mathrm{Im}(P)^\perp$, we have $Ux = 0$ and similarly $Vx = 0$. By linearity, $Ux = Vx$ for all $x \in H$. Therefore $U = V$. This proves the uniqueness. Finally, we have $P = V^*VP = V^*A$ and $\ker(P) = \ker(\sqrt{A^*A}) = \ker(A^*A) = \ker(A)$. □

14-7.6. **Theorem** For every operator A, there is a unique positive operator Q and a unique partial isometry V such that $A = QV$ and $\ker(Q) = \mathrm{Im}(V)^\perp$. Furthermore we have

(a) VV^* is a projector onto $Q(H)^-$.

(b) $\ker(A^*) = \ker(Q)$.

(c) $Q = AV^* = \sqrt{AA^*}$.

Proof. Replace A by A^* in last theorem and then take hermitian. □

14-7.7. **Exercise** Show that the square root $|A| = \sqrt{A^*A}$ obeys $|\lambda A| = |\lambda|\,|A|$ but not the triangular inequality.

14-7.8. **Exercise** Prove that if an operator A is invertible, then its absolute value $|A| = \sqrt{A^*A}$ is also invertible. Prove that if A is normal, then the converse is also true.

14-99. **References and Further Readings** : Steen, Konig-75, Gohberg-00, Berezin and Rice.

Chapter 15

Tensor Products

15-1 Algebraic Tensor Products of Vector Spaces

15-1.1. The main idea of this chapter is to convert a multilinear map on a product space into a linear map on the tensor product of its factor spaces and to study the relationship between these two maps. Multilinear maps have been introduced in §10-1 when we worked with higher derivatives. Let $E_1, E_2, \cdots, E_p$ and F be vector spaces. The vector space of all multilinear maps from $\Pi_{k=1}^p E_k$ into F is denoted by $M(E_1, E_2, \cdots, E_p; F)$. When F is the scalar field $\mathbb{K}$, it is simply denoted by $M(E_1, E_2, \cdots, E_p)$.

15-1.2. A vector space G is called a *tensor product* or more precisely an algebraic tensor product of $E_1, E_2, \cdots, E_p$ under the *tensor map* g from the product space $\Pi_{k=1}^p E_k$ into G if the following conditions hold
(a) g is a multilinear map;
(b) for every multilinear map h from $\Pi_{k=1}^p E_k$ into a vector space H, there is a *unique* linear map φ from G into H such that $h = \varphi g$. We say that h is factorized through G and φ is called the linear map *associated* with h.

15-1.3. **Existence Theorem** Let $E_1, E_2, \cdots, E_p$ be vector spaces. Let g be the linear form on the vector space M of all multilinear forms on $\Pi_{k=1}^p E_k$ given by $[g(x)](u) = u(x)$ for each $x = (x_1, x_2, \cdots, x_p) \in \Pi_{k=1}^p E_k$. Then the vector subspace G of the algebraic dual M^* of M spanned by $g(\Pi_{k=1}^p E_k)$ is a tensor product under the tensor map g.

Proof. It is easy to verify that $[g(x)](u) = u(x)$ is linear in u and hence $g(x) \in M^*$. Observe that for all $\alpha, \beta \in \mathbb{K}$ and all $x_k \in E_k$,

$$\begin{aligned}
&[g(\alpha x_1 + \beta y_1, x_2, \cdots, x_p)](u) = u(\alpha x_1 + \beta y_1, x_2, \cdots, x_p) \\
&= \alpha u(x_1, x_2, \cdots, x_p) + \beta u(y_1, x_2, \cdots, x_p) \\
&= [\alpha g(x_1, x_2, \cdots, x_p)(u) + \beta g(y_1, x_2, \cdots, x_p)](u).
\end{aligned}$$

Hence $g(x_1, x_2, \cdots, x_p)$ is linear in x_1 and similarly it is linear in every x_k. Thus $g : \Pi_{k=1}^p E_k \to M^*$ is a multilinear map. To show that G is a tensor product under g, let h be a multilinear map from $\Pi_{k=1}^p E_k$ into a vector space H. Take any $z \in G$. There are vectors $x^1, x^2, \cdots, x^m \in \Pi_{k=1}^p E_k$ and scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $z = \sum_{j=1}^m \alpha_j g(x^j)$. Define $\varphi(z) = \sum_{j=1}^m \alpha_j h(x^j)$. To show that φ is well-defined, let $z = \sum_{i=1}^n \beta_i g(y^i)$ be another representation where $y^i \in \Pi_{k=1}^p E_k$ and $\beta_i \in \mathbb{K}$. Then for each $u \in M$, we have $\sum_{j=1}^m \alpha_j h(x^j)(u) = \sum_{i=1}^n \beta_i h(y^i)(u)$, i.e. $\sum_{j=1}^m \alpha_j u(x^j) = \sum_{i=1}^n \beta_i u(y^i)$. Now let v be any linear form on H. Since h is multilinear, the composite vh is a multilinear form on $\Pi_{k=1}^p E_k$. Letting $u = vh$ we have $\sum_{j=1}^m \alpha_j vh(x^j) = \sum_{i=1}^n \beta_i vh(y^i)$, or $v\left[\sum_{j=1}^m \alpha_j h(x^j)\right] = v\left[\sum_{i=1}^n \beta_i h(y^i)\right]$. Since H^* separates points of H, we have $\sum_{j=1}^m \alpha_j h(x^j) = \sum_{i=1}^n \beta_i h(y^i)$. Therefore φ is well-defined on G into H. It is obvious that φ is linear. When $z = g(x)$, we have $\varphi g(x) = h(x)$, i.e. $\varphi g = h$. It remains to prove the uniqueness of φ. Suppose that ψ is another linear map from G into H such that $\psi g = h$. For any $x \in \Pi_{k=1}^p E_k$, we obtain $\psi[g(x)] = h(x) = \varphi[g(x)]$. Hence $\psi = \varphi$ on $g(\Pi_{k=1}^p E_k)$ which spans G. Therefore $\psi = \varphi$ on G. This completes the proof. □

15-1.4. **Uniqueness Theorem** Let G, H be tensor products of vector spaces $E_1, E_2, \cdots, E_p$ under the tensor maps g, h respectively. Then there exists a unique isomorphism $\varphi : G \to H$ such that $h = \varphi g$. We shall identify G, H under φ.

Proof. Factorizing h, g through G, H, there are linear maps $\varphi : G \to H$ and $\psi : H \to G$ such that $h = \varphi g$ and $g = \psi h$ respectively. Consequently $\psi\varphi$ is a linear map from G into itself such that $g = (\psi\varphi)g$. Let I, J be the identity maps on G, H respectively. Then I is also a linear map from G into itself such that $g = Ig$. By uniqueness of factorizing, we have $\psi\varphi = I$. Similarly, $\varphi\psi = J$. Therefore φ is a bijection from G onto H. Consequently, φ is an isomorphism. Its uniqueness is part of the definition of tensor product. □

15-1.5. The tensor product of $E_1, E_2, \cdots, E_p$ is denoted by $E_1 \otimes E_2 \otimes \cdots \otimes E_p$, or by $\bigotimes_{k=1}^p E_k$. Elements of $\bigotimes_{k=1}^p E_k$ are called *tensors*. The tensor map $\otimes$ from $\Pi_{k=1}^p E_k$ into the tensor product $\bigotimes_{k=1}^p E_k$ will be denoted by

$$x = (x_1, x_2, \cdots, x_p) \to \otimes x = x_1 \otimes x_2 \otimes \cdots \otimes x_p = \otimes_{k=1}^p x_k.$$

A tensor is said to be *decomposable* if it is of the form $\otimes x = x_1 \otimes x_2 \otimes \cdots \otimes x_p$ for some $x = (x_1, x_2, \cdots, x_p) \in \Pi_{k=1}^p E_k$. Since the tensor map is multilinear, the following results follow immediately.

15-1.6. **Corollary** (a) $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ is linear in each variable x_k.
(b) Every tensor is a sum of decomposable tensors.

15-1.7. **Exercise** Let g, h be linear maps from the tensor product $\bigotimes_{k=1}^{p} E_k$ into a vector space H. If $g(\otimes x) = h(\otimes x)$ for all decomposable tensors $\otimes x$, then $g = h$ on the tensor product $\bigotimes_{k=1}^{p} E_k$.

15-1.8. **Exercise** For every linear map $f : \bigotimes_{k=1}^{p} E_k \to F$, let $T(f)(x) = f(\otimes x)$ for every $x \in \Pi_{k=1}^{p} E_k$. Prove that T is an isomorphism from the vector space $L(\bigotimes_{k=1}^{p} E_k, F)$ onto the vector space $M(E_1, E_2, \cdots, E_p; F)$ of multilinear maps.

15-1.9. **Exercise** Prove that the algebraic dual space of $\bigotimes_{k=1}^{p} E_k$ is isomorphic to the vector space $M(E_1, E_2, \cdots, E_p)$ of multilinear forms on $\Pi_{k=1}^{p} E_k$. This justifies the way which we constructed tensor product $\bigotimes_{k=1}^{p} E_k$ as a vector subspace of the algebraic dual space of $M(E_1, E_2, \cdots, E_p)$.

15-1.10. **Exercise** Let E be a vector space and let $g : \mathbb{K} \times E \to E$ be given by $g(\lambda, x) = \lambda x$. Prove that E is the tensor product of $\mathbb{K}$ and E under the tensor map g. In symbols, we have $\mathbb{K} \otimes E = E$ and $\lambda \otimes x = \lambda x$.

15-1.11. **Exercise** State and prove a statement to identify $E \otimes (F \otimes G)$ with $(E \otimes F) \otimes G$ for vector spaces E, F, G. Hence justify the notation

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z = x \otimes y \otimes z.$$

15-2 Tensor Products of Linear Maps

15-2.1. For $k = 1, 2, \cdots, p$, let E_k, F_k be vector spaces and $f_k : E_k \to F_k$ a linear map. Suppose $L^*(E_k, F_k)$ denote the vector space of all linear maps from E_k into F_k. According to last section, the tensor product $\otimes_{k=1}^{p} f_k$ is a vector in vector space $\otimes_{k=1}^{p} L^*(E_k, F_k)$. On the other hand, a separate definition for tensor product of linear maps is used. These two definitions are consistent under the identification of §15-5.10,11. The reader may modify the proofs from normed spaces to algebraic case.

15-2.2. **Theorem** There is a unique linear map $\otimes_{k=1}^{p} f_k$ from $\bigotimes_{k=1}^{p} E_k$ into $\bigotimes_{k=1}^{p} F_k$ such that $(\otimes_{k=1}^{p} f_k)(\otimes_{k=1}^{p} x_k) = \otimes_{k=1}^{p} f_k(x_k)$ for all $x_k \in E_k$. It is called the *tensor product* of linear maps f_k.

Proof. Since the map $g : \Pi_{k=1}^{p} E_k \to \bigotimes_{k=1}^{p} F_k$ by $g(x_1, x_2, \cdots, x_p) = \otimes_{k=1}^{p} f_k(x_k)$ is multilinear, there is a linear map $\otimes_{k=1}^{p} f_k : \bigotimes_{k=1}^{p} E_k \to \bigotimes_{k=1}^{p} F_k$ such that

$(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k) = g(x_1, x_2, \cdots, x_p)$, i.e. $(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k) = \otimes_{k=1}^p f_k(x_k)$. Finally suppose $h : \bigotimes_{k=1}^p E_k \to \bigotimes_{k=1}^p F_k$ is any linear map such that $h(\otimes_{k=1}^p x_k) = \otimes_{k=1}^p f_k(x_k)$. Then we have $h(\otimes_{k=1}^p x_k) = (\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k)$. Since the decomposable tensors span $\bigotimes_{k=1}^p E_k$, it follows that $h = \otimes_{k=1}^p f_k$. □

15-2.3. **Corollary** Let f_k be a linear form on E_k for each $k = 1, 2, \cdots, p$. Then there is a unique linear form $\otimes_{k=1}^p f_k$ on $\bigotimes_{k=1}^p E_k$ such that for all $x_k \in E_k$ we have $(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k) = \Pi_{k=1}^p f_k(x_k)$.

15-2.4. **Corollary** If $\otimes_{k=1}^p x_k = 0$, then $x_j = 0$ for some j.

Proof. Suppose that $x_j \neq 0$ for every $1 \leq j \leq p$. There is a linear form f_k on E_k such that $f_k(x_k) = 1$. Then $\otimes_{k=1}^p f_k$ is a linear form on $\bigotimes_{k=1}^p E_k$ satisfying $(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k) = \Pi_{k=1}^p f_k(x_k) = 1$. Therefore $\otimes_{k=1}^p x_k \neq 0$. □

15-2.5. **Corollary** If $\otimes_{k=1}^p x_k = \otimes_{k=1}^p y_k \neq 0$, then there are scalars α_k such that $x_k = \alpha_k y_k$ for each k and also $\Pi_{k=1}^p \alpha_k = 1$.

Proof. Since $\otimes_{k=1}^p x_k \neq 0$, we have $x_k \neq 0$ for each k. Choose a linear form f_k on E_k such that $f_k(x_k) = 1$. Define $\alpha_j = \Pi_{k\neq j} f_k(y_k)$. Now for every linear form h on E_k, applying $f_1 \otimes \cdots \otimes f_{j-1} \otimes h \otimes f_{j+1} \otimes \cdots \otimes f_k$ to $\otimes_{k=1}^p x_k = \otimes_{k=1}^p y_k$, we have $h(x_j)\Pi_{k\neq j} f_k(x_k) = h(y_j)\Pi_{k\neq j} f_k(y_k)$ i.e. $h(x_j) = \alpha_j h(y_j) = h(\alpha_j y_j)$. Since h is arbitrary, we have $x_j = \alpha_j y_j$. Replacing h by f_j, we get $1 = \Pi_{k=1}^p f_k(x_k)$. Therefore the proof is completed by

$$\Pi_{k=1}^p \alpha_k = \Pi_{k=1}^p \Pi_{k\neq j} f_k(y_k) = \left(\Pi_{k=1}^p f_k(y_k)\right)^{k-1} = 1.$$ □

15-2.6. **Theorem** For each k, let E_k, F_k, G_k be vector spaces. Let $f_k : E_k \to F_k$ and $g_k : F_k \to G_k$ be linear maps. Then the composite maps satisfy the following relation $(\otimes_{k=1}^p g_k)(\otimes_{k=1}^p f_k) = \otimes_{k=1}^p (g_k f_k)$.

Proof. It suffices to verify the equality for decomposable tensors. Let $x_k \in E_k$. Then the proof is completed by the following calculation:

$$(\otimes_{k=1}^p g_k)(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p x_k) = (\otimes_{k=1}^p g_k)[\otimes_{k=1}^p f_k(x_k)]$$
$$= \otimes_{k=1}^p g_k[f_k(x_k)] = \otimes_{k=1}^p (g_k f_k)(x_k) = (\otimes_{k=1}^p g_k f_k)(\otimes_{k=1}^p x_k).$$

15-2.7. **Exercise** If I_k is the identity map on E_k, then $\otimes_{k=1}^p I_k$ is the identity map on $\bigotimes_{k=1}^p E_k$.

15-2.8. **Exercise** Prove that if every $f_k : E_k \to F_k$ is an isomorphism, then $\otimes_{k=1}^p f_k : \bigotimes_{k=1}^p E_k \to \bigotimes_{k=1}^p F_k$ is also an isomorphism. Furthermore deduce that $\left(\otimes_{k=1}^p f_k\right)^{-1} = \otimes_{k=1}^p f_k^{-1}$.

15-2.9. The above results may be abbreviated by the following notations. For all linear maps $f_k : E_k \to F_k$, define a map $f = (f_1, f_2, \cdots, f_p)$ from $\Pi_{k=1}^p E_k$ into $\Pi_{k=1}^p F_k$ by $f(x) = (f_1(x_1), f_2(x_2), \cdots, f_p(x_p))$ for $x = (x_1, x_2, \cdots, x_p) \in \Pi_{k=1}^p E_k$. Then we have some formulas such as $(\otimes f)(\otimes x) = \otimes f(x)$, $\otimes(gf) = (\otimes g)(\otimes f)$ and $(\otimes f)^{-1} = \otimes(f^{-1})$.

15-2.10. **Example** Not every tensor is decomposable.

Proof. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$. We claim that the tensor $z = e_1 \otimes e_2 + e_3 \otimes e_4$ is not decomposable. Suppose to the contrary that $z = x \otimes y$ for some $x = \sum_{i=1}^4 \alpha_i e_i$ and $y = \sum_{j=1}^4 \beta_j e_j$ in $\mathbb{R}^4$. Let f_1, f_2, f_3, f_4 be the dual basis given by $f_i(e_j) = \delta ij$. Applying $f_p \otimes f_q$ to $\sum_{i=1}^4 \alpha_i e_i = e_1 \otimes e_2 + e_3 \otimes e_4$, we have $\alpha_p \beta_q = f_p(e_1)g_q(e_2) + f_p(e_3)g_q(e_4)$. Letting $(p,q) = (1,2), (3,4), (1,4)$ respectively, we obtain $\alpha_1\beta_2 = 1$, $\alpha_3\beta_4 = 1$ and $\alpha_1\beta_4 = 0$ which is a contradiction. □

15-3 Independent Sets in Tensor Products

15-3.1. **Theorem** Let E, F be vector spaces and let x_j, y_j be vectors in E, F respectively. If $y_1, y_2, \cdots, y_m$ are linearly independent and if $\sum_{j=1}^m x_j \otimes y_j = 0$, then all $x_j = 0$.

Proof. Since $y_1, y_2, \cdots, y_m$ are linearly independent, there are linear forms g_i on F such that $g_i(y_j) = \delta_{ij}$. Now for every linear form f on E, we have

$$0 = (f \otimes g_i)\left(\sum_{j=1}^m x_j \otimes y_j\right) = \sum_{j=1}^m f(x_j)g_i(y_j) = \sum_{j=1}^m f(x_j)\delta_{ij} = f(x_i).$$

Because f is arbitrary, we have $x_i = 0$ for each $i = 1, 2, \cdots, m$. □

15-3.2. **Theorem** If $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, y_2, \cdots, y_n\}$ are linearly independent in E, F respectively, then the set $\{x_i \otimes y_j : 1 \le i \le m, 1 \le j \le n\}$ is independent in $E \otimes F$.

Proof. Let α_{ij} be scalars satisfying $\sum_{i=1}^m \sum_{j=1}^n \alpha_{ij} x_i \otimes y_j = 0$. Then we have $\sum_{j=1}^n \left(\sum_{i=1}^m \alpha_{ij} x_i\right) \otimes y_j = 0$. Since $y_1, y_2, \cdots, y_n$ are linearly independent, we obtain $\sum_{i=1}^m \alpha_{ij} x_i = 0$ for each j. Now because $x_1, x_2, \cdots, x_m$ are linearly independent, we get $\alpha_{ij} = 0$ for all i, j. Therefore the set $\{x_i \otimes y_j\}$ is independent in $E \otimes F$. □

15-3.3. Let z be a tensor in $E \otimes F$. If $z = 0$, then the *rank* of z is defined to be zero. If $z \neq 0$, then its *rank* is defined as the smallest integer $m \ge 1$ such that $z = \sum_{j=1}^m x_j \otimes y_j$ where $x_j \in E$ and $y_j \in F$.

15-3.4. **Theorem** Let $z = \sum_{j=1}^{m} x_j \otimes y_j$ be a tensor in $E \otimes F$ where $x_j \in E$ and $y_j \in F$. If $m \geq 1$ is the rank of z then both sets $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, y_2, \cdots, y_m\}$ are linearly independent.

Proof. Suppose $m = 1$. Then $z = x_1 \otimes y_1$. Since $rank(z) \geq 1$, we have $z \neq 0$, i.e. $x_1 \neq 0$ and $y_1 \neq 0$. Therefore both sets $\{x_1\}$ and $\{y_1\}$ are independent. Next assume $m \geq 2$. Suppose to the contrary that $y_1, y_2, \cdots, y_m$ are linearly dependent. There is $j \geq 2$ such that y_j is a linear combination of $y_1, y_2, \cdots, y_{j-1}$. By rearranging the order, we may assume $y_m = \sum_{j=1}^{m-1} \alpha_j y_j$ where α_j are scalars. Then we have

$$z = \sum_{j=1}^{m} x_j \otimes y_j = \sum_{j=1}^{m-1} x_j \otimes y_j + x_m \otimes \left(\sum_{j=1}^{m-1} \alpha_j y_j \right) = \sum_{j=1}^{m-1} (x_j + \alpha_j x_m) \otimes y_j.$$

This contradicts to the rank m of z. Therefore $y_1, y_2, \cdots, y_m$ are linearly independent. Similarly, $x_1, x_2, \cdots, x_m$ are also linearly independent. □

15-3.5. The following results in this section are also true for normed spaces and continuous linear forms.

15-3.6. **Lemma** Let z be a tensor in $E \otimes F$. If $(f \otimes g)(z) = 0$ for all linear forms f, g on E, F respectively, then $z = 0$.

Proof. Suppose to the contrary that $z \neq 0$. Then we can write $z = \sum_{j=1}^{m} x_j \otimes y_j$ where $m \geq 1$ is the rank of z and $x_j \in E$, $y_j \in F$. Then both $\{x_1, x_2, \cdots, x_m\}$ and $\{y_1, y_2, \cdots, y_m\}$ are independent. There are linear forms f, g on E, F respectively such that $f(x_k) = g(y_k) = 1$ for all k. Now the following contradiction establishes the proof:

$$0 = (f \otimes g)(z) = (f \otimes g) \left(\sum_{j=1}^{m} x_j \otimes y_j \right) = \sum_{j=1}^{m} f(x_j) g(y_j) = m. \qquad \square$$

15-3.7. **Theorem** Let E, F be vector spaces. Let a_i, x_j be vectors in E and b_i, y_j in F. Then the following statements are equivalent.

(a) $\sum_{i=1}^{m} a_i \otimes b_i = \sum_{j=1}^{n} x_j \otimes y_j$.

(b) $\sum_{i=1}^{m} f(a_i) b_i = \sum_{j=1}^{n} f(x_j) y_j$ for all linear forms f on E.

(c) $\sum_{i=1}^{m} g(b_i) a_i = \sum_{j=1}^{n} g(y_j) x_j$ for all linear forms g on F.

(d) $\sum_{i=1}^{m} f(a_i) g(b_i) = \sum_{j=1}^{n} f(x_j) g(y_j)$ for all linear forms f, g on E, F respectively.

Proof. $(a \Rightarrow d)$ Applying $f \otimes g$ to both sides of (a), we have

$$\sum_{i=1}^{m} (f \otimes g)(a_i \otimes b_i) = \sum_{j=1}^{n} (f \otimes g)(x_j \otimes y_j)$$

which gives (d) right away.

$(d \Rightarrow a)$ Let $z = \sum_{i=1}^m a_i \otimes b_i - \sum_{j=1}^n x_j \otimes y_j$. Then for all linear forms f, g on E, F respectively, we have

$$(f \otimes g)(z) = \sum_{i=1}^m (f \otimes g)(a_i \otimes b_i) - \sum_{j=1}^n (f \otimes g)(x_j \otimes y_j)$$
$$= \sum_{i=1}^m f(a_i)g(b_i) - \sum_{j=1}^n f(x_j)g(y_j) = 0.$$

Therefore $z = 0$ which establishes (d).

$(d \Rightarrow b)$ Since g is linear, we get $g\left[\sum_{i=1}^m f(a_i)b_i\right] = g\left[\sum_{j=1}^n f(x_j)y_j\right]$. Because g is arbitrary, we obtain (b).

$(b \Rightarrow d)$ follows immediately by applying g to both sides. Similarly, (c) is equivalent to (d). □

15-3.8. **Corollary** Let $E_1, E_2, \cdots, E_p$ be vector spaces and for each k let $x_k^i, y_k^j \in E_k$. Then the following statements are equivalent.

(a) $\sum_{i=1}^m x_1^i \otimes x_2^i \otimes \cdots \otimes x_k^i = \sum_{j=1}^n y_1^j \otimes y_2^j \otimes \cdots \otimes y_k^j$.

(b) $\sum_{i=1}^m f(x_1^i) x_2^i \otimes \cdots \otimes x_k^i = \sum_{j=1}^n f(y_1^j) y_2^j \otimes \cdots \otimes y_k^j$ for all linear forms f on E_1.

(c) $\sum_{i=1}^m \Pi_{k=1}^p f_k(x_k^i) = \sum_{j=1}^n \Pi_{k=1}^p f_k(y_k^j)$ for all linear forms f_k on E_k.

15-3.9. **Theorem** For each $1 \le k \le p$, let E_k, F_k be vector spaces and let $A_k, B_k : E_k \to F_k$ be a linear map. If $\otimes_{k=1}^p A_k = \otimes_{k=1}^p B_k \neq 0$, then there are scalars λ_j such that $B_k = \lambda_k A_k$ for each k and that $\Pi_{k=1}^p \lambda_k = 1$.

Proof. For $k \ge 2$, since $B_k \neq 0$ there exist $x_k \in E_k$ and linear form f_k on F_k such that $f_k(B_k x_k) = 1$. Now take any $x_1 \in E_1$ and any linear form f_1 on F_1, we have $(\otimes_{k=1}^p f_k)(\otimes_{k=1}^p A_k)(\otimes_{k=1}^p x_k) = (\otimes_{k=1}^p f_k)(\otimes_{k=1}^p B_k)(\otimes_{k=1}^p x_k)$, i.e. $f_1(A_1 x_1)\left(\Pi_{k=2}^p f_k(A_k x_k)\right) = f_1(B_1 x_1)\left(\Pi_{k=2}^p f_k(B_k x_k)\right)$, or $f_1(\lambda_1 A_1 x_1) = f_1(B_1 x_1)$ where $\lambda_1 = \Pi_{k=2}^p f_k(A_k x_k)$. Since f_1 and x_1 are arbitrary, we have $B_1 = \lambda_1 A_1$. Similarly, there are scalars λ_k such that $B_k = \lambda_k A_k$ for all $k \ge 2$. Finally $\otimes_{k=1}^p A_k = \otimes_{k=1}^p B_k = (\Pi_{k=1}^p \lambda_k) \otimes_{k=1}^p A_k \neq 0$ gives $\Pi_{k=1}^p \lambda_k = 1$. □

15-4 Matrix Representations

15-4.1. Identify a column vector $a = (a_1, a_2, \cdots, a_n)^t$ as a function on the index set $S = \{1, 2, \cdots, n\}$ and similarly $b = (b_1, b_2, \cdots, b_m)^t$ on $T = \{1, 2, \cdots, m\}$. Then $\mathbb{K}^n, \mathbb{K}^m$ are identified as the vector spaces $\mathbb{F}(S), \mathbb{F}(T)$ of all functions on S, T respectively. Next lemma will show that $a \otimes b$ is identified with the function c on the product set $S \times T$ defined by $c(i, j) = a_i b_j$. With reversed lexical order, we shall generalize matrix representations from linear to multilinear maps on finite dimensional vector spaces. The notation §7-5.8 of matrix representations will be used.

15-4.2. **Lemma** Let $\mathbb{F}(S), \mathbb{F}(T)$ be vector spaces of certain functions on non-empty sets S, T respectively. For every $f \in \mathbb{F}(S)$ and $g \in \mathbb{F}(T)$, let $\pi(f,g) : S \times T \to \mathbb{K}$ be defined by $\pi(f,g)(s,t) = f(s)g(t)$. Then the vector space $\mathbb{P}$ spanned by $\{\pi(f,g) : f \in \mathbb{F}(S), E, g \in \mathbb{F}(T)\}$ is a tensor product of $\mathbb{F}(S), \mathbb{F}(T)$. As a result, it justifies the usual notation : $(f \otimes g)(s,t) = f(s)g(t)$. Note that when S is metric space, we naturally restrict $\mathbb{F}(S)$ to the set of all continuous functions only.

Proof. Clearly π is bilinear. Hence there is a linear map $\varphi : \mathbb{F}(S) \otimes \mathbb{F}(T) \to \mathbb{P}$ such that for all $f \in \mathbb{F}(S)$ and $g \in \mathbb{F}(T)$ we have $\pi(f,g) = \varphi(f \otimes g)$. Since $\mathbb{P}$ is spanned by the range of π, the map φ is surjective. Suppose to the contrary that the linear map φ is not injective. Then there is $z \neq 0$ in $\mathbb{F}(S) \otimes \mathbb{F}(T)$ but $\varphi(z) = 0$ in $\mathbb{P}$. Write $z = \sum_{j=1}^{m} f_j \otimes g_j$ where $\{f_j\}$ and $\{g_j\}$ are linearly independent in $\mathbb{F}(S), \mathbb{F}(T)$ respectively. Then $\varphi(z) = \sum_{j=1}^{m} \pi(f_j, g_j) = 0$, i.e. for each $(s,t) \in S \times T$, $\sum_{j=1}^{m} f_j(s) g_j(t) = 0$. Hence $\sum_{j=1}^{m} f_j(s) g_j = 0$. By independence of $\{g_j\}$, we have $f_j(s) = 0$ for all $s \in S$, i.e. $f_j = 0$ and thus $z = \sum_{j=1}^{m} f_j \otimes g_j = 0$ which is a contradiction. Therefore φ is an isomorphism from $\mathbb{F}(S) \otimes \mathbb{F}(T)$ onto $\mathbb{P}$. □

15-4.3. Both index sets $S = \{1, 2, \cdots, n\}$ and $T = \{1, 2, \cdots, m\}$ have their own natural orders $1 < 2 < 3 < \cdots$. Their product set $S \times T$ is endowed with the reversed lexical order listed explicitly below:

$$
\begin{array}{ccccccccc}
 & (1,1) & < & (2,1) & < & & < & (n,1) \\
< & (1,2) & < & (2,2) & < & & < & (n,2) \\
< & \cdots & < & \cdots & < & & < & \cdots \\
< & (1,m) & < & (2,m) & < & \cdots & < & (n,m).
\end{array}
$$

With the notation of §15-4.1, the function $c = a \otimes b$ is identified as the column vector : $(a_1 b_1, a_2 b_1, \cdots, a_n b_1, a_1 b_2, a_2 b_2, \cdots, a_n b_2, \cdots, a_1 b_m, a_2 b_m, \cdots, a_n b_m)^t$. This motivates the following definition. We choose reversed lexical order so that §15-4.9 holds and that the notation of tensor products can be applied to higher derivatives even though it is beyond our scope.

15-4.4. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of arbitrary sizes. Then their *tensor product* is defined as the matrix given by

$$
A \otimes B = \begin{bmatrix} Ab_{11} & Ab_{12} & \cdots \\ Ab_{21} & Ab_{22} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.
$$

The following is an illustrative example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 3a & 4a & 2b & 3b & 4b \\ 5a & 6a & 7a & 5b & 6b & 7b \\ \\ 2c & 3c & 4c & 2d & 3d & 4d \\ 5c & 6c & 7c & 5d & 6d & 7d \end{bmatrix}.$$

Row vectors and column vectors are considered as matrices and hence their tensor products are also defined.

15-4.5. **Theorem** Let $\mathcal{E} = [e_1, e_2, \cdots, e_n]$ and $\mathcal{F} = [f_1, f_2, \cdots, f_m]$ be ordered bases of finite dimensional vector spaces E, F respectively. Then set

$$\mathcal{E} \otimes \mathcal{F} = \{e_i \otimes f_j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

forms a basis of $E \otimes F$. Since the reversed lexical order on $\mathcal{E} \otimes \mathcal{F}$ is always assumed, we have $[x \otimes y] = [x] \otimes [y]$ for all $x \in E$ and $y \in F$.

Proof. We have proved that $\mathcal{E} \otimes \mathcal{F}$ is independent in $E \otimes F$. Next, the expression $x \otimes y = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j e_i \otimes f_j$ shows that $\mathcal{E} \otimes \mathcal{F}$ also spans $E \otimes F$ and gives the required coordinate vector. □

15-4.6. **Theorem** In addition to the conditions of last theorem, let G, H be finite dimensional vector spaces with bases $\mathcal{G} = [g_1, g_2, \cdots, g_q]$ and $\mathcal{H} = [h_1, h_2, \cdots, h_p]$ respectively. Let $A : E \to F$ and $B : G \to H$ be linear maps. Then we have $[A \otimes B] = [A] \otimes [B]$ if both $\mathcal{E} \otimes \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{H}$ are endowed with reversed lexical orders.

Proof. It follows immediately from $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$. □

15-4.7. **Exercise** Let $\dim(E) = \dim(H) = 2$ and $\dim(F) = \dim(G) = 3$. Define $A : E \to F$ by $A(x) = (m + 4n)f_1 + (2m + 5n)f_2 + (3m + 6n)f_3$ for $x = me_1 + ne_2$ in E and also $B : G \to H$ by $B(y) = (7\alpha + 8\beta + 9\gamma)h_1 + (10\alpha + 20\beta + 30\gamma)h_2$ for $y = \alpha g_1 + \beta g_2 + \gamma g_3$ in F where $m, n, \alpha, \beta, \gamma \in \mathbb{K}$. Then $E \otimes G$ has the following ordered basis : $e_1 \otimes g_1, e_2 \otimes g_1, e_1 \otimes g_2, e_2 \otimes g_2, e_1 \otimes g_3, e_2 \otimes g_3$ and $F \otimes H$ has the ordered basis : $f_1 \otimes h_1, f_2 \otimes h_1, f_3 \otimes h_1, f_1 \otimes h_2, f_2 \otimes h_2, f_3 \otimes h_2$. Show that the coordinate vectors are

$$[x] = \begin{bmatrix} m \\ n \end{bmatrix}, [y] = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \text{ and } [x \otimes y] = \begin{bmatrix} m\alpha \\ n\alpha \\ m\beta \\ n\beta \\ m\gamma \\ n\gamma \end{bmatrix} = [x] \otimes [y].$$

Verify that the matrix representations are

$$[A] = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, [B] = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 20 & 30 \end{bmatrix}$$

$$\text{and also } [A \otimes B] = \begin{bmatrix} 7 & 28 & 8 & 32 & 9 & 36 \\ 14 & 35 & 16 & 40 & 18 & 45 \\ 21 & 42 & 24 & 48 & 27 & 54 \\ \\ 10 & 50 & 20 & 80 & 30 & 120 \\ 20 & 50 & 40 & 100 & 60 & 150 \\ 30 & 60 & 60 & 120 & 90 & 180 \end{bmatrix} = [A] \otimes [B].$$

Finally calculate $[Ax] \otimes [By]$ and $[A \otimes B][x \otimes y]$ separately and check if they are equal.

15-4.8. We shall generalize matrix representations from linear to bilinear and consequently multilinear maps. Let $A : E \times G \to F$ be a bilinear map. Using the same notation as before, the expression $A(e_i, g_j) = \sum_{k=1}^{m} a_{ij}^k f_k$ gives a column vector $C_{ij} = (a_{ij}^1, a_{ij}^2, a_{ij}^3, \cdots, a_{ij}^m)^t$ in $\mathbb{K}^m$. The *matrix representation* of the bilinear map with respect to the ordered bases $\mathcal{E}, \mathcal{G}, \mathcal{F}$ is defined as the $m \times nq$-matrix given by $[A] = [C_{11}, C_{21}, \cdots, C_{n1}, C_{12}, C_{22}, \cdots, C_{n2}, C_{1q}, C_{2q}, \cdots, C_{nq}]$. Note that the subscripts are given the reversed lexical order.

15-4.9. **Theorem** $[A(x, y)] = [A][x][y]$.

Proof. Just like in linear case, it follows from the interpretation of

$$A(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{q} A(e_i, g_j) x_i y_j = \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{q} a_{ij}^k x_i y_j. \qquad \square$$

15-4.10. **Exercise** Let $M(E, G; F)$ be the vector space of all bilinear maps from $E \times G \to F$. Prove that the map $A \to [A]$ is an isomorphism from $M(E, G; F)$ onto the vector space $mat(m, nq)$ of all $m \times nq$-matrices.

15-4.11. **Theorem** Let $A : E \times G \to F$ be a bilinear map and $B : E \otimes G \to F$ be the linear map associated to A, i.e. $A(x, y) = B(x \otimes y)$ for all $(x, y) \in E \times G$. Then we have $[A] = [B]$.

Proof. Clearly both $[A], [B]$ are of the same size $m \times nq$. Suppose $[A] = [a_{ij}^k]$ and $[B] = [b_{ij}^k]$. Observe that $\sum_{k=1}^{m} a_{ij}^k f_k = A(e_i, g_j) = B(e_i \otimes g_j) = \sum_{k=1}^{m} b_{ij}^k f_k$. Therefore $a_{ij}^k = b_{ij}^k$ for all i, j, k, i.e. $[A] = [B]$. $\square$

15-4.12. **Corollary** For every $m \times nq$-matrix A, n-column vector x and q-column vector y, we have $(Ax)y = A(x \otimes y)$. As a result, tensor product can be used as an alternative of §10-5.6 to handle the higher derivatives in finite dimensional vector spaces.

Proof. Let $E = \mathbb{K}^n$, $F = \mathbb{K}^m$ and $G = \mathbb{K}^q$. Then $(x, y) \to Axy$ is a bilinear map from $E \times G \to F$ and its matrix representation with respect to the standard bases is A itself. Let $B : E \otimes G \to F$ be the associated linear map of A. Then the proof is completed by the following computation

$$\begin{aligned} Axy &= [A][x][y] = [Axy] = [B(x \otimes y)] \\ &= [B][x \otimes y] = [A]([x] \otimes [y]) = A(x \otimes y). \end{aligned}$$ □

15-4.13. **Example** Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 20 & 30 & 40 & 50 & 60 \end{bmatrix}$. Then we have

$$\left(A\begin{bmatrix}6\\7\end{bmatrix}\right)\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix} 20 & 46 & 72 \\ 200 & 460 & 720 \end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix} 20a+46b+72c \\ 200a+460b+720c \end{bmatrix}$$

and also

$$A\left(\begin{bmatrix}6\\7\end{bmatrix}\otimes\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 10 & 20 & 30 & 40 & 50 & 60 \end{bmatrix}\begin{bmatrix}6a\\7a\\6b\\7b\\6c\\7c\end{bmatrix} = \begin{bmatrix} 20a+46b+72c \\ 200a+460b+720c \end{bmatrix}.$$

15-5 Projective Norms on Tensor Products

15-5.1. Let E, F, G be normed spaces and $\varphi : E \times F \to E \otimes F$ the tensor map. Then every continuous bilinear map $f : E \times F \to G$ factors through a linear map $g : E \otimes F \to G$, i.e. $f = g\varphi$. The identification

$$T : M(E, F; G) \to L(E \otimes F, G)$$

defined by $T(f) = g$ is an algebraic isomorphism. In this section, a norm is defined on $E \otimes F$ so that T becomes an isometry.

15-5.2. A norm on the algebraic tensor product $E \otimes F$ is called a *tensor norm* or a cross norm if $\|x \otimes y\| = \|x\|\ \|y\|$ for all decomposable tensors $x \otimes y$. It is easy to prove that if E, F contain non-zero vectors, then $\|\varphi\| = 1$ for every tensor norm on $E \otimes F$.

15-5.3. **Exercise** Prove that if E, F are separable, then so is $E \otimes F$ under every tensor norm.

15-5.4. For every $z \in E \otimes F$, write $z = \sum_{j=1}^{m} x_j \otimes y_j$ where $x_j \in E$ and $y_j \in F$. Note that x_j, y_j may be zero vectors and hence z may be the zero tensor. Let $\|z\| = \inf \sum_{j=1}^{m} \|x_j\|\ \|y_j\|$ where the infimum is taken over all representation of z as a sum of decomposable tensors. This is called the *projective norm* on the tensor product, or the π-norm. We shall assume that our results hold for finite tensor products of normed space without any further specification.

15-5.5. **Theorem** The projective norm is the largest tensor norm on $E \otimes F$.

Proof. Clearly the projective norm is positive and the projective norm of the zero tensor is zero. Let $c = \sum_{i=1}^{m} a_i \otimes b_i$ and $z = \sum_{j=1}^{n} x_j \otimes y_j$ be tensors in $E \otimes F$. Clearly we have $\|c+z\| \leq \sum_{i=1}^{m} \|a_i\|\ \|b_i\| + \sum_{j=1}^{n} \|x_j\|\ \|y_j\|$. Taking infima over the representations of c, z, we obtain $\|c+z\| \leq \|c\| + \|z\|$. Similarly, it is easy to show $\|\lambda z\| \leq |\lambda|\ \|z\|$ for every $\lambda \in \mathbb{K}$. If $\lambda \neq 0$, then $|\lambda|\ \|z\| = |\lambda|\ \|\frac{1}{\lambda}\lambda z\| \leq |\lambda|\ |\frac{1}{\lambda}|\ \|\lambda z\| = \|\lambda z\|$, i.e. $|\lambda|\ \|z\| \leq \|\lambda z\|$. Therefore $|\lambda|\ \|z\| = \|\lambda z\|$ which can be verified directly when $\lambda = 0$. Next suppose $z \neq 0$. Write $z = \sum_{i=1}^{r} u_i \otimes v_i$ where $\{u_i\}$ and $\{v_i\}$ are linearly independent sets. There are continuous linear forms f, g on E, F respectively such that $f(u_1) = g(v_1) = 1$ and $f(u_i) = g(v_i) = 0$ for all $i \geq 2$. Hence $(f \otimes g)(z) = \sum_{i=1}^{r} f(u_i)g(v_i) = 1$. Now for any representation $z = \sum_{j=1}^{n} x_j \otimes y_j$, observe that

$$\begin{aligned} 1 &= (f \otimes g)(z) = (f \otimes g)\left(\sum_{j=1}^{n} x_j \otimes y_j\right) = \sum_{j=1}^{n} f(x_j)g(y_j) \\ &= \left|\sum_{j=1}^{n} f(x_j)g(y_j)\right| \leq \sum_{j=1}^{n} |f(x_j)|\ |g(y_j)| \leq \|f\|\|g\| \sum_{j=1}^{n} \|x_j\|\ \|y_j\|. \end{aligned}$$

Taking infimum over all representations of z, we have $1 \leq \|f\|\ \|g\|\ \|z\|$. Therefore $\|z\| \neq 0$. This proves that the projective norm is a norm on $E \otimes F$. To show that the projective norm is a tensor norm, suppose $z = u \otimes v \neq 0$ is a decomposable tensor. Then both u, v are non-zero. There are continuous linear forms f, g on E, F respectively such that $\|f\| = \|g\| = 1$, $f(u) = \|u\|$ and $g(v) = \|v\|$. Thus $(f \otimes g)(z) = f(u)g(v) = \|u\|\ \|v\|$. Now for any representation $z = \sum_{j=1}^{n} x_j \otimes y_j$, calculation as above gives

$$\|u\|\ \|v\| = (f \otimes g)(z) \leq \|f\|\ \|g\| \sum_{j=1}^{n} \|x_j\|\ \|y_j\| \leq \sum_{j=1}^{n} \|x_j\|\ \|y_j\|.$$

Taking infimum, we have $\|u\|\ \|v\| \leq \|z\|$. This together with the definition shows $\|u\|\ \|v\| = \|z\|$. Therefore projective norm is a tensor norm on $E \otimes F$. Finally, let $|?|$ be any tensor norm on $E \otimes F$. Then for every $z = \sum_{j=1}^{n} x_j \otimes y_j$, we have $|z| \leq \sum_{j=1}^{n} |x_j \otimes y_j| \leq \sum_{j=1}^{n} \|x_j\|\ \|y_j\|$. Taking infimum over all representations of z, we have $|z| \leq \|z\|$. Consequently, projective norm is the largest tensor norm. □

15-5.6. **Theorem** For every continuous bilinear map $f : E \times F \to G$, there is a unique continuous linear map $g : E \otimes F \to G$ such that $f = g\varphi$ where $\varphi : E \times F \to E \otimes F$ is the tensor map. Furthermore, let

$$T : M(E, F; G) \to L(E \otimes F, G)$$

be defined by $Tf = g$. Then T is an isometric isomorphism.

Proof. Let $z = \sum_{j=1}^n x_j \otimes y_j$ be any tensor in $E \otimes F$. Observe that

$$\|g(z)\| \leq \sum_{j=1}^n \|g(x_j \otimes y_j)\| = \sum_{j=1}^n \|f(x_j, y_j)\| \leq \sum_{j=1}^n \|f\| \, \|x_j\| \|y_j\|.$$

Taking infimum over all representations of z, we have $\|g(z)\| \leq \|f\| \, \|z\|$. Hence g is continuous under the projective norm on $E \otimes F$. Furthermore, $\|g\| \leq \|f\|$. Conversely, if $g : E \otimes F \to G$ is continuous linear, then the composite $f = g\varphi$ is continuous bilinear. Since $\|f(x, y) = \|f(x \otimes y)\| \leq \|g\| \, \|x \otimes y\| = \|g\| \, \|x\| \, \|y\|$, we obtain $\|f\| \leq \|g\|$. Therefore $\|f\| = \|g\|$. It is obvious that Tf is linear in f. This proves that T is an isometric isomorphism. □

15-5.7. **Exercise** Identify the dual space of $E \otimes F$ under projective tensor norm with the space of all continuous bilinear forms on $E \times F$.

15-5.8. **Theorem** There is a unique linear injection $\varphi : L(E, F) \to (E \otimes F')'$ such that $\varphi(A)(x \otimes g) = gAx, \forall \ (x, g) \in (E \times F')$. As a result, $L(E, F)$ is identified as a subspace of $(E \otimes F')'$ by $A(x \otimes g) = gAx$, $\forall \ (x, g) \in E \otimes F'$.

Proof. It is trivial to verify that the map $\psi(A) : E \times F \to \mathbb{K}$ given by $\psi(A)(x, g) = gAx$ is continuous bilinear and hence there is a continuous linear map $\varphi(A) : E \otimes F' \to \mathbb{K}$ such that $\varphi(A)(x \otimes g) = gAx$. Thus the map $\varphi : L(E, F) \to (E \otimes F')'$ is well-defined. It is obvious that $\varphi(A)$ is linear in $A \in L(E, F)$. To prove that φ is injective, if $\varphi(A) = 0$, i.e. $gAx = 0$ for all $(x, g) \in E \times F'$, then since F' separates points we have $Ax = 0$ for all $x \in E$, or $A = 0$. Therefore φ is a required linear injection. For uniqueness, suppose $\xi : L(E, F) \to (E \otimes F')'$ is a linear map satisfying $\xi(A)(x \otimes g) = gAx$ for all $(x, g) \in (E \times F')$. Then $\xi(A)(x \otimes g) = \psi(A)(x, g)$. Therefore we obtain $\xi(A) = \varphi(A)$, i.e. $\xi = \varphi$. □

15-5.9. **Corollary** Let $A_1, A_2, \cdots, A_m : E \to F$ be continuous linear maps. If they are linearly independent, then there are points $x_1, x_2, \cdots, x_n \in E$ and continuous linear forms $g_1, g_2, \cdots, g_n$ on F such that $\sum_{j=1}^n g_1 A_1 x_j = 1$ and $\sum_{j=1}^n g_j A_i x_j = 0$ for all $i \geq 2$.

Proof. Observe that $\varphi(A_1), \varphi(A_2), \cdots, \varphi(A_m)$ are linearly independent in $(E \otimes F')'$ by injectivity of φ in the proof of last theorem. There is $z \in E \otimes F'$ such that $\varphi(A_1)(z) = 1$ and $\varphi(A_i)(z) = 0$ for all $i \geq 2$. Write $z = \sum_{j=1}^n x_j \otimes g_j$ where $x_j \in E$ and $g_j \in F'$. Now $\varphi(A)(z) = \sum_{j=1}^n g_j A_i x_j$ completes the proof. □

15-5.10. For each $1 \leq k \leq p$, let E_k, F_k be normed spaces and $L(E_k, F_k)$ the vector space of all continuous linear maps from E_k into F_k. Define

$$f(A_1, A_2, \cdots, A_p)(x_1, x_2, \cdots, x_p) = \otimes_{k=1}^p A_k x_k$$

where $A_k \in L(E_k, F_k)$ and $x_k \in E_k$. Since

$$f(A_1, A_2, \cdots, A_p) : \Pi_{k=1}^p E_k \to \bigotimes_{k=1}^p F_k$$

is a continuous multilinear map, there is a unique continuous linear map

$$g(A_1, A_2, \cdots, A_p) : \bigotimes_{k=1}^p E_k \to \bigotimes_{k=1}^p F_k$$

such that $$g(A_1, A_2, \cdots, A_p)(\otimes_{k=1}^p x_k) = \otimes_{k=1}^p A_k x_k.$$

Because $$g : \Pi_{k=1}^p L(E_k, F_k) \to L(\bigotimes_{k=1}^p E_k, \bigotimes_{k=1}^p F_k)$$

is a continuous multilinear map, there is a unique continuous linear map

$$h : \bigotimes_{k=1}^p L(E_k, F_k) \to L(\bigotimes_{k=1}^p E_k, \bigotimes_{k=1}^p F_k)$$

such that $$h(\otimes_{k=1}^p A_k) = g(A_1, A_2, \cdots, A_p).$$

Therefore $$[h(\otimes_{k=1}^p A_k)](\otimes_{k=1}^p x_k) = \otimes_{k=1}^p A_k x_k.$$

We shall prove in next theorem that h is injective. Hence the tensor product as vectors in $L(E_k, F_k)$ agrees with the tensor product as linear maps. Therefore the tensor products of linear maps $A_1 \otimes A_2 \otimes \cdots \otimes A_p$ enjoy all properties of tensor products of vectors.

15-5.11. **Theorem** The tensor product $\bigotimes_{k=1}^p L(E_k, F_k)$ can be identified as a vector subspace of $L(\bigotimes_{k=1}^p E_k, \bigotimes_{k=1}^p F_k)$.

Proof. It suffices to show that h defined above is injective. Suppose to the contrary that there is $D \in \bigotimes_{k=1}^p L(E_k, F_k)$ such that $h(D) = 0$ but $D \neq 0$. Write $D = \sum_{i=1}^r \otimes_{k=1}^p A_{ik}$ where r is the rank of the vector tensor D. Then $\{A_{ik} : 1 \leq i \leq r\}$ is an independent subset of $L(E_k, F_k)$. There are $x_{jk} \in E_k$ and $g_{jk} \in F_k'$ such that

$$\sum_{j(k)=1}^{m(k)} g_{j(k)k} A_{ik} x_{j(k)k} = \begin{cases} 1, & \text{if } i = 1; \\ 0, & \text{if } i \geq 2. \end{cases}$$

Now $h(D) = 0$ gives

$$\begin{aligned} 0 &= \sum_{j(1)=1}^{m(1)} \sum_{j(2)=1}^{m(2)} \cdots \sum_{j(p)=1}^{m(p)} (\otimes_{k=1}^p g_{j(k)k}) h(D)(\otimes_{k=1}^p x_{j(k)k}) \\ &= \sum_{j(1)=1}^{m(1)} \sum_{j(2)=1}^{m(2)} \cdots \sum_{j(p)=1}^{m(p)} (\otimes_{k=1}^p g_{j(k)k}) \left(\sum_{i=1}^r h(\otimes_{k=1}^p A_{ik})(\otimes_{k=1}^p x_{j(k)k})\right) \\ &= \sum_{i=1}^r \sum_{j(1)=1}^{m(1)} \sum_{j(2)=1}^{m(2)} \cdots \sum_{j(p)=1}^{m(p)} \otimes_{k=1}^p \left(g_{j(k)k} A_{ik} x_{j(k)k}\right) \\ &= \sum_{i=1}^r \sum_{j(1)=1}^{m(1)} \sum_{j(2)=1}^{m(2)} \cdots \sum_{j(p)=1}^{m(p)} \prod_{k=1}^p g_{j(k)k} A_{ik} x_{j(k)k} \end{aligned}$$

$$= \sum_{i=1}^{r} \prod_{k=1}^{p} \sum_{j(k)=1}^{m(k)} g_{j(k)k} A_{ik} x_{j(k)k} = \sum_{i=1}^{r} \prod_{k=1}^{p} 1 = r$$

which is a contradiction. Therefore h is injective. □

15-5.12. **Exercise** Prove that under the projective norms, both g, h are continuous. Furthermore, show that $\|g\| = \|h\|$ is either zero or one.

15-6 Inductive Norms

15-6.1. Let E, F be normed spaces. Write $E \otimes_\pi F$ to indicate that the projective norm is used. Since every $x \in E \subset E''$ is a continuous linear form on E', $x \otimes y$ is a continuous linear form on $E' \otimes_\pi F'$ defined by $(u \otimes v)(x \otimes y) = u(x)v(y)$ where $u \in E'$, $v \in F'$ and $y \in F$. Therefore we identify $E \otimes F$ as a vector subspace of $(E' \otimes_\pi F')'$. In this section, we shall study the norm on $E \otimes F$ induced by $(E' \otimes_\pi F')'$. With this norm, we can identify $E' \otimes F$ as a subspace of $L(E, F)$.

15-6.2. **Lemma** For every $f \in (E' \otimes_\pi F')'$, we have $\|f\| = \lambda(f)$ where

$$\lambda(f) = \sup\{|f(u \otimes v)| : u \in E', \|u\| \leq 1, v \in F', \|v\| \leq 1\}.$$

Proof. Since $|f(u \otimes v)| \leq \|f\| \; \|u \otimes v\| = \|f\| \; \|u\| \; \|v\| \leq \|f\|$, we have $\lambda(f) \leq \|f\|$. Conversely, take any $w \in E' \otimes_\pi F'$ with $0 < \|w\| \leq 1$. Then for every $\varepsilon > 0$, by definition of projective norm we have $w = \sum_{j=1}^{r} u_j \otimes v_j$ and $\sum_{j=1}^{r} \|u_j\| \; \|v_j\| \leq \|w\| + \varepsilon$ for some $u_j \in E'$ and $v_j \in F'$. Since $w \neq 0$, we may assume that all u_j, v_j are non-zero. Hence we obtain

$$|f(w)| \leq \sum_{j=1}^{r} |f(u_j \otimes v_j)| \leq \sum_{j=1}^{r} \left| f\left(\frac{u_j}{\|u_j\|} \otimes \frac{v_j}{\|v_j\|} \right) \right| \; \|u_j\| \; \|v_j\|$$
$$\leq \sum_{j=1}^{r} \lambda(f) \|u_j\| \; \|v_j\| = \lambda(f) \sum_{j=1}^{r} \|u_j\| \|v_j\| \leq \lambda(f)(1 + \varepsilon).$$

Taking supremum over $\|w\| \leq 1$, we have $\|f\| \leq \lambda(f)(1 + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we get $\|f\| \leq \lambda(f)$. □

15-6.3. The *inductive norm* or ε-norm on $E \otimes F$ is defined by

$$\lambda(z) = \sup\{|(u \otimes v)(z) : u \in E', \|u\| \leq 1, v \in F', \|v\| \leq 1\}.$$

We shall write $E \otimes_\varepsilon F$ to indicate that the inductive norm is used. Recall that the projective norm on $E \otimes_\pi F$ is given by $\pi(z) = \inf \sum_{j=1}^{m} \|x_j\| \; \|y_j\|$ where the infimum is taken over all representation of z as a sum of decomposable tensors.

15-6.4. **Theorem** The inductive norm λ is a tensor norm on $E \otimes_\varepsilon F$. Furthermore, we have $|(u \otimes v)(z)| \leq \|u\| \|v\| \lambda(z)$ for all $u \in E'$, $v \in F'$ and $z \in E \otimes F$.

Proof. We have proved that λ is a norm on $(E' \otimes_\pi F')'$ and hence it is also norm on its subspace $E \otimes F$. Since

$$\lambda(x \otimes y) = \sup\{|(u \otimes v)(x \otimes y)| : \|u\| \leq 1, \|v\| \leq 1\}$$
$$= \sup\{|u(x)|\ |v(y)| : \|u\| \leq 1, \|v\| \leq 1\} = \|x\|\|y\|,$$

λ is a tensor norm on $E \otimes F$. To show the inequality, we may assume that both u, v are non-zero. By definition of λ, we have

$$\left| \left(\frac{u}{\|u\|}\right) \otimes \left(\frac{v}{\|v\|}\right)(z) \right| \leq \lambda(z)$$

from which the result follows by multiplying $\|u\|\ \|v\|$ to both sides. □

15-6.5. **Lemma** For all $z \in E \otimes F$ and all $w \in E' \otimes F'$, we have

$$|w(z)| \leq \pi(w)\lambda(z) \text{ and } |w(z)| \leq \lambda(w)\pi(z).$$

Proof. Without loss of generality, assume that both w, z are non-zero otherwise the left hand side becomes zero and the result follows. Take any representation $z = \sum_{i=1}^m x_i \otimes y_i$. We may assume that all x_i, y_i are non-zero otherwise just drop them. Observe that

$$|w(z)| \leq \sum_{i=1}^m |w(x_i \otimes y_i)|$$
$$= \sum_{i=1}^m \left| w\left(\frac{x_i}{\|x_i\|} \otimes \frac{y_i}{\|y_i\|}\right) \right|\ \|x_i\|\ \|y_i\| \leq \sum_{i=1}^m \lambda(w)\|x_i\|\ \|y_i\|.$$

Taking the infimum over all representations of z, we obtain $|w(z)| \leq \lambda(w)\pi(z)$. Similarly, we can prove the other inequality. □

15-6.6. **Exercise** Let E_p, E_q denote the same vector space E equipped with two norms p, q and E'_p, E'_q their topological dual spaces respectively. The *dual norm* on E'_p is defined by $p'(u) = \sup\{|u(x)| : x \in E, \|x\| \leq 1\}$. Prove that if $p \leq q$ on E, then $E'_q \subset E'_p$ and also $q' \leq p'$ on E'_q.

15-6.7. **Theorem** If E, F are Banach spaces, then $\pi' = \lambda$ on $E' \otimes F'$.

Proof. Since $|w(z)| \leq \lambda(w)\pi(z)$, we get $\pi'(w) = \sup\{|w(z)| : \pi(z) \leq 1\} \leq \lambda(w)$. On the other hand, suppose $w \in E' \otimes F'$, $\|f\| \leq 1$ in E'' and $\|g\| \leq 1$ in F'' are given. Write $w = \sum_{j=1}^n u_j \otimes v_j$. Since E, F are Banach spaces, for every $\varepsilon > 0$ there are $x \in E$ and $y \in F$ such that $\|x\| \leq \|f\| + \varepsilon$, $\|y\| \leq \|g\| + \varepsilon$, $u_j(x) = f(u_j)$ and $v_j(y) = g(v_j)$ for all j. Since

$$(f \otimes g)(w) = \sum_{j=1}^n f(u_j)g(v_j) = \sum_{j=1}^n u_j(x)v_j(y) = w(x \otimes y),$$

we have

$$|(f \otimes g)(w)| = |w(x \otimes y)| = \left| w\left(\frac{x}{\|x\|} \otimes \frac{y}{\|y\|}\right)\right| \; \|x\| \; \|y\|$$
$$\leq \pi'(w)(\|f\| + \varepsilon)(\|g\| + \varepsilon) \leq \pi'(w)(1 + \varepsilon)^2.$$

Taking supremum over all f, g, we obtain $\lambda(w) \leq \pi'(w)(1 + \varepsilon)^2$. Because $\varepsilon > 0$ is arbitrary, we have $\lambda(w) \leq \pi'(w)$. □

15-6.8. Let E, F be normed spaces. The map $\psi : E' \times F \to L(E, F)$ defined by $\psi(f, y)(x) = f(x)y$ for $f \in E', y \in F, x \in E$ is a continuous bilinear map. Hence there is a unique linear map $\varphi : E' \otimes F \to L(E, F)$ so that $\varphi(f \otimes y)(x) = f(x)y$.

15-6.9. **Exercise** Prove that the image of $E' \otimes F$ under φ consists of all finite dimensional continuous linear maps from E into F.

15-6.10. **Theorem** The map φ is an isometric isomorphism when $E' \otimes_\varepsilon F$ is equipped with the inductive tensor norm. In particular, φ is injective. Hence we can embed $E' \otimes_\varepsilon F$ as a vector subspace of the normed space $L(E, F)$.

Proof. Let $z = \sum_{j=1}^n f_j \otimes y_j$. The following calculation completes the proof:

$$\|\varphi(z)\| = \sup\{\|\varphi(z)(x)\| : x \in E, \|x\| \leq 1\} = \sup_{\|x\|\leq 1} \left\|\sum_{j=1}^n f_j(x)y_j\right\|$$
$$= \sup_{\|x\|\leq 1} \sup\left\{\left|g\left(\sum_{j=1}^n f_j(x)y_j\right)\right| : g \in F', \|g\| \leq 1\right\}$$
$$= \sup_{\|g\|\leq 1} \sup_{\|x\|\leq 1} \left|\left(\sum_{j=1}^n g(y_j)f_j\right)(x)\right| = \sup_{\|g\|\leq 1} \left\|\sum_{j=1}^n g(y_j)f_j\right\|$$
$$= \sup_{\|g\|\leq 1} \sup_{\|h\|\leq 1} \left|h\left[\sum_{j=1}^n g(y_j)f_j\right]\right| ; h \in E''$$
$$= \sup\left\{\left|\sum_{j=1}^n g(y_j)h(f_j)\right| : \|g\| \leq 1, \|h\| \leq 1\right\} = \lambda\left(\sum_{j=1}^n f_j \otimes y_j\right) = \lambda(z). \square$$

15-6.11. The completions of $E \otimes F$ under the projective and inductive norms are denoted by $E \widehat{\otimes_\pi} F$ and $E \widehat{\otimes_\varepsilon} F$ respective. Their properties can be derived from the dense subspace $E \otimes F$. The details are left to the readers.

15-7 Tensor Product of Hilbert Spaces

15-7.1. In linear algebra, it is well-known that if a tensor product of matrices is normal, then each factor is also normal. This will be generalized to operators on Hilbert spaces. We start with introduction of natural inner products on the tensor spaces.

15-7.2. **Lemma** The completion K of a complex inner product space H is a Hilbert space.

Proof. For each $b \in H$, the map $f : H \to \mathbb{C}$ given by $f_b(a) = < a, b >$ is continuous linear in a and hence it can be extended over K, i.e. $< x, b >$ is well-defined for all $x \in K$ and $b \in H$. It is easy to show that $| < x, b > | \leq \|x\| \, \|b\|$. Therefore the map $g_x : H \to \mathbb{C}$ given by $g_x(b) = < x, b >^-$ is continuous linear in b and hence it can also be extended over K, i.e. $< x, y >$ is well-defined for all $x, y \in K$. It is easy to verify that $< x, y >$ is an inner product on K. Its norm is complete because K is the completion of H. The uniqueness follows from the fact that $H \times H$ is dense in $K \times K$ and that $< x, y >$ is jointly continuous in (x, y). □

15-7.3. **Lemma** Let H, K be complex inner Spaces. There is a unique inner product on the algebraic tensor product $H \otimes K$ such that for all $a, x \in H$ and $b, y \in K$ we have $< a \otimes b, x \otimes y > = < a, x >< b, y >$.

Proof. Define $f_{xy} : H \times K \to \mathbb{C}$ by $f_{xy}(a, b) = < a, x >< b, y >$ for every $(x, y) \in H \times K$. Clearly $f_{xy}(a, b)$ is bilinear in (a, b). Hence there is a linear map $g_{xy} : H \otimes K \to \mathbb{C}$ such that $f_{xy}(a \otimes b) = g_{xy}(a \otimes b)$ for all $(a, b) \in H \times K$. Next, for each $c \in H \otimes K$ define $p_c : H \times K \to \mathbb{C}$ by $p_c(x, y) = \overline{g_{xy}(c)}$. To show that it is linear in x, let $c = \sum_{j=1}^n a_j \otimes b_j$ and $x = \alpha u + \beta v$ where $\alpha, \beta \in \mathbb{K}$ and $u, v \in H$. Observe that

$$\begin{aligned}
&\overline{p_c(\alpha u + \beta v, y)} = g_{(\alpha u+\beta v)y}(c) \\
&= g_{(\alpha u+\beta v)y}\left(\sum_{j=1}^n a_j \otimes b_j\right) = \sum_{j=1}^n g_{(\alpha u+\beta v)y}(a_j \otimes b_j) \\
&= \sum_{j=1}^n f_{(\alpha u+\beta v)y}(a_j, b_j) = \sum_{j=1}^n < a_j, \alpha u + \beta v >< b_j, y > \\
&= \sum_{j=1}^n \overline{\alpha} < a_j, u >< b_j, y > + \sum_{j=1}^n \overline{\beta} < a_j, v >< b_j, y > \\
&= \overline{\alpha} g_{uy}(c) + \overline{\beta} g_{vy}(c) = \overline{\alpha p_c(u, y) + \beta p_c(v, y)}.
\end{aligned}$$

Therefore $p_c(x, y)$ is linear in x and similarly is also linear in y. There is a linear map $q_c : H \otimes K \to \mathbb{C}$ such that $q_c(x \otimes y) = p_c(x, y)$. For every $c, z \in H \otimes K$, write $< c, z > = q_c(z)$. Suppose $z = \sum_{i=1}^m x_i \otimes y_i$. Then it is easy to verify the identity: $< c, z > = \sum_{i=1}^m \sum_{j=1}^n < a_j, x_i >< b_j, y_i >$. From this, it follows that $< c, z >$ is the unique inner product on $H \otimes K$ required by the theorem. □

15-7.4. **Exercise** Prove that $\|x \otimes y\| = \|x\| \, \|y\|$ for all $x \in H$ and $y \in K$.

15-7.5. Let H, K be complex Hilbert spaces. Let $H \otimes_a K$ denote the inner product space of algebraic tensor product of H, K. Then the completion $H \otimes K$

of $H \otimes_a K$ is called the ***Hilbert tensor product*** of H, K. It is a Hilbert space containing all decomposable tensor and its inner product satisfies

$$< a \otimes b, x \otimes y > = < a, x >< b, y >$$

for all $a, x \in H$ and all $b, y \in K$. Let A, B be continuous linear operators on H, K respectively. Then $A \otimes B$ is a continuous linear operator on the inner product space $H \otimes_a K$ satisfying $(A \otimes B)(x \otimes y) = A(x) \otimes B(y)$ for all $(x, y) \in H \times K$. Its unique extension over the Hilbert tensor product $H \otimes K$ is also denoted by the same symbol $A \otimes B$.

15-7.6. **Lemma** Let P, Q be operators on the Hilbert tensor product $H \otimes K$ of Hilbert spaces H, K. If $< c, Pz > = < c, Qz >$ for all decomposable tensors $c, z \in H \otimes K$, then we have $P = Q$.

Proof. By linearity, $< c, Pz > = < c, Qz >$ holds for all c, z in the algebraic tensor product $H \otimes_a K$. By continuity, it holds also for all c, z in the Hilbert tensor product $H \otimes K$. Therefore $P = Q$. □

15-7.7. As a result of next theorem, it is trivial to show that the tensor products of normal operators are normal. The same is true for other classes such as unitary, self-adjoint, positive definite, etc. The interesting parts are their converses. Let A, B be operators on H, K respectively.

15-7.8. **Theorem** $(A \otimes B)^* = A^* \otimes B^*$.

Proof. For all $u, v \in H$ and $x, y \in K$ we have

$$< u \otimes v, (A \otimes B)^*(x \otimes y) > = < (A \otimes B)(u \otimes v), x \otimes y >$$
$$= < Au \otimes Bv, x \otimes y > = < Au, x >< Bv, y > = < u, A^*x >< v, B^*y >$$
$$= < u \otimes v, A^*x \otimes B^*y > = < u \otimes v, (A^* \otimes B^*)(x \otimes y) >.$$

The result follows from last lemma. □

15-7.9. **Lemma** If $A^*A = \lambda AA^*$ for some complex number λ, then A is normal.

Proof. Without loss of generality, we may assume $A \neq 0$. Choose $x \in H$ with $Ax \neq 0$. Then we have $< A^*Ax, x > = \lambda < AA^*x, x >$, i.e. $\|Ax\|^2 = \lambda \|A^*x\|^2$. Hence $\lambda > 0$. Taking norms of the given equation, we have $0 \neq \|A^*A\| = \lambda \|AA^*\|$ which gives $\lambda = 1$. Therefore A is normal. □

15-7.10. **Theorem** If $A \otimes B \neq 0$ is normal, then both A, B are normal.

Proof. Simplifying $(A \otimes B)^*(A \otimes B) = (A \otimes B)(A \otimes B)^* \neq 0$, we obtain $(A^*A) \otimes (B^*B) = (AA^*) \otimes (BB^*) \neq 0$. Hence $A^*A = \lambda AA^*$ for some number λ. Therefore A is normal. Similarly, B is also normal. □

15-7.11. **Theorem** If $A \otimes B \neq 0$ is unitary, then there exist numbers α, β and unitary operators M, N such that $A = \alpha M$, $B = \beta N$ and $\alpha\beta = 1$.

Proof. Since $A \otimes B$ is unitary, we have

$$(A^*A) \otimes (B^*B) = (AA^*) \otimes (BB^*) = I \otimes J \qquad \#1$$

where I, J are identity operators on H, K respectively. There are non-zero numbers s, t such that

$$A^*A = sI \text{ and } AA^* = tI. \qquad \#2$$

Since both left and right inverses of A exist, A is invertible and $s = t$. In particular, $A \neq 0$, i.e. $Ax \neq 0$ for some $x \in H$. Thus #2 gives $< A^*Ax, x > = s < x, x >$, or $\|Ax\|^2 = s\|x\|^2$. Hence $s > 0$. Let α be the square root of s and define $M = \frac{1}{\alpha}A$. Then it is easy to verify that M is unitary. Similarly, both β and N are defined. Substituting these into #1, we have

$$\{(\alpha M)^*(\alpha M)\} \otimes \{(\beta N)^*(\beta N)\} = I \otimes J,$$

i.e. $\alpha^2\beta^2\{(M^*M) \otimes (N^*N)\} = I \otimes J$. Since M, N are unitary, we have $\alpha^2\beta^2 I \otimes J = I \otimes J$, that is, $\alpha^2\beta^2 = 1$. Because $\alpha, \beta > 0$, we obtain $\alpha\beta = 1$. □

15-7.12. **Theorem** If $A \otimes B \neq 0$ is self-adjoint, then $A = \alpha M$ and $B = \beta N$ where M, N are some self-adjoint operators and $\alpha\beta = |\alpha| = |\beta| = 1$.

Proof. Since $A \otimes B$ is self-adjoint, we have $A^* \otimes B^* = (A \otimes B)^* = A \otimes B$. Then $A^* = \lambda A$ for some number λ. Taking hermitians, we have $A = \lambda^- A^*$, or $A = \lambda^-\lambda A$. Since $A \neq 0$, $|\lambda| = 1$. Write $\lambda = \exp(i\theta)$ where θ is real and $i^2 = -1$. Define $\alpha = \exp(-i\theta/2)$, $\beta = \exp(i\theta/2)$, $M = \frac{1}{\alpha}A$ and $N = \frac{1}{\beta}B$. Then all conditions of the theorem are satisfied. □

15-7.13. **Theorem** If $A \otimes B \neq 0$ is positive definite, then $A = \alpha M$ and $B = \beta N$ where M, N are some positive definite operators and $\alpha\beta = |\alpha| = |\beta| = 1$.

Proof. Since $A \otimes B \neq 0$ is self-adjoint, we have $A = sP$ and $B = tQ$ where P, Q are self-adjoint and s, t are complex numbers such that $st = |s| = |t| = 1$. Clearly $A \otimes B = P \otimes Q$. Since $P = \frac{1}{s}A$ is non-zero self-adjoint, there is $a \in H$ such that $< Pa, a > \neq 0$. Define $\delta = 1$ if $< Pa, a >$ is positive and $\delta = -1$ if $< Pa, a >$ is negative. Let $\alpha = \delta s$, $\beta = \delta t$, $M = \delta P$ and $N = \delta Q$. Then M, N are non-zero self-adjoint. Furthermore, $\alpha\beta = |\alpha| = |\beta| = 1$ and

$0 \leq A \otimes B = M \otimes N$. Now for every $y \in K$, we have $< (M \otimes N)(a \otimes y), a \otimes y > \geq 0$, i.e. $< Ma, a >< Ny, y > \geq 0$. Since $< Ma, a > = \delta < Pa, a >$ is strictly positive, we have $< Ny, y > \geq 0$ for all $y \in K$. Therefore N is positive definite. Since N is non-zero, there is $b \in K$ such that $< Nb, b >$ is strictly positive. By similar argument, we have $< Mx, x > \geq 0$ for all $x \in H$ and hence M is also positive definite. This completes the proof. □

15-7.14. **Theorem** If $A \otimes B \neq 0$ is a projector, then $A = \alpha M$ and $B = \beta N$ where M, N are some projectors and $\alpha\beta = 1$.

Proof. Since $A \otimes B \neq 0$ is self-adjoint, we have $A = sP$ and $B = tQ$ where P, Q are self-adjoint and s, t are complex numbers such that $st = |s| = |t| = 1$. Since $A \otimes B = P \otimes Q$ is an idempotent, we have $P^2 \otimes Q^2 = P \otimes Q \neq 0$. Hence there are numbers u, v such that $P^2 = uP$, $Q^2 = vQ$ and $uv = 1$. Since P is non-zero self-adjoint, there is a non-zero real number k in the spectrum of P. Applying the spectral polynomial theorem to $P^2 - uP = 0$, we have $k^2 - uk = 0$. Hence $u = k$ is a real number and so is $v = \frac{1}{u}$. Define $M = vP$, $N = uQ$, $\alpha = su$ and $\beta = tv$. Since v is real and P is self-adjoint, we have $M^* = M$. Also $M^2 = v^2 P^2 = v^2(uP) = vP = M$. Therefore M is a projector. Similarly, the other requirements of the theorem can be verified. □

15-7.15. **Exercise** Prove that if $A \otimes B \neq 0$ is an idempotent, then $A = \alpha M$ and $B = \beta N$ where M, N are some idempotents and $\alpha\beta = 1$.

15-7.16. **Theorem** Let A_k be an operator on a Hilbert space H_k for each $1 \leq k \leq 2r + 1$. If $\bigotimes_{k=1}^{2r+1} A_k \neq 0$ is skew-adjoint, then there are skew-adjoint operators M_k and complex numbers λ_k such that $\Pi_{k=1}^{2r+1} \lambda_k = -1$ and for each k, we have $A_k = \lambda_k M_k$, $|\lambda_k| = 1$.

Proof. Since $\bigotimes_{k=1}^{2r+1} A_k \neq 0$ is skew-adjoint, $\bigotimes_{k=1}^{2r+1} (iA_k) = (-1)^r i \bigotimes_{k=1}^{2r+1} A_k \neq 0$ is self-adjoint. There are numbers α_k and self-adjoint operators D_k such that $|\alpha_k| = 1$, $iA_k = \alpha_k D_k$ for each k and also $\Pi_{k=1}^{2r+1} \alpha_k = 1$. Define $\lambda_k = -\alpha_k$ and $M_k = iD_k$. Then it is routine to complete the proof. □

15-7.17. **Exercise** If $\{u_i\}$ and $\{v_j\}$ are orthonormal bases in H, K respectively, then $\{u_i \otimes v_j\}$ is an orthonormal basis in $H \otimes K$.

15-7.18. **Theorem** For all operators A, B on H, K respectively, we have

$$\|A \otimes B\| = \|A\| \, \|B\|.$$

Proof. Let I, J be the identity maps on H, K and $\{u_i\}$, $\{v_j\}$ be orthonormal bases in H, K respectively. We claim $\|A \otimes J\| \leq \|A\|$. Indeed, let $z = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} u_i \otimes v_j$. Observe that

$$\|(A \otimes J)z\|^2 = \left\|\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}(A \otimes J)(u_i \otimes v_j)\right\|^2$$
$$= \left\|\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_{ij} A u_i\right) \otimes v_j\right\|^2 \leq \sum_{j=1}^{n} \left\|\sum_{i=1}^{m} \alpha_{ij} A u_i\right\|^2 \|v_j\|^2$$
$$\leq \sum_{j=1}^{n} \sum_{i=1}^{m} |\alpha_{ij}|^2 \|Au_i\|^2 \leq \sum_{j=1}^{n} \sum_{i=1}^{m} |\alpha_{ij}|^2 \|A\|^2 = \|z\|^2 \|A\|^2.$$

Hence $\|A \otimes J\| \leq \|A\|$. Therefore

$$\|A \otimes B\| = \|(A \otimes J)(I \otimes B)\| \leq \|A \otimes J\| \; \|I \otimes B\| \leq \|A\| \; \|B\|.$$

On the other hand, for every $\varepsilon > 0$, there are $\|x\| = \|y\| = 1$ in H, K respectively such that $\|Ax\| \geq \|A\| - \varepsilon$ and $\|By\| \geq \|B\| - \varepsilon$. Consequently we have

$$\|A \otimes B\| \geq \|(A \otimes B)(x \otimes y)\| = \|Ax \otimes By\|$$
$$= \|Ax\| \; \|By\| \geq (\|A\| - \varepsilon)(\|B\| - \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we have $\|A \otimes B\| \geq \|A\| \; \|B\|$. □

15-99. **References and Further Readings** : Marcus, Defant-93, Freniche, Pelletier, Holub, Ichinose, John, Lewis, Ruston, Reed and Simon-79.

Chapter 16

Complex Vector Lattices

16-1 Ordered Vector Spaces

16-1.1. This chapter introduces complex vector lattices as a tool for subsequent development. The subject is elementary but sometime tricky. We shall restrict ourselves to the minimum algebraic aspects only.

16-1.2. A map $f \to f^-$ from a *complex* vector space $\mathfrak{F}$ into itself is called a *conjugation* if $(f+g)^- = f^- + g^-$, $f^{--} = f$ and $(\alpha f)^- = \alpha^- f^-$ for all $f, g \in \mathfrak{F}$, and $\alpha \in \mathbb{C}$ where α^- denotes the complex conjugate of α. For a *real* vector space, conjugation is defined as the identity map. In this case, we can unify real and complex vector spaces into one framework. A vector space with a given conjugation is called a *conjugated vector space*. For alternative approach through complexification, see for example [Schaefer-74; §11.1, p134].

16-1.3. **Example** The set $\mathbb{F}(X)$ of all complex functions on a set X is a complex conjugated vector space and the set $\mathbb{F}^r(X)$ of all real functions is a real conjugated vector space. However $\mathbb{F}(X)$ is also a real vector space but it is not a real conjugated vector space if the pointwise conjugate-operation is not the identity map. Therefore $\mathbb{F}(X)$ always means the complex vector space in this chapter.

16-1.4. Let $\mathfrak{F}$ be a conjugated vector space over $\mathbb{K}$ which can be the real or complex field. The *real part* of an element $f \in \mathfrak{F}$ is defined by $\operatorname{Re} f = \frac{1}{2}(f + f^-)$ and the *imaginary part* by $\operatorname{Im} f = \frac{1}{2i}(f - f^-)$. Note that when $\mathbb{K} = \mathbb{R}$, we have $f - f^- = 0$ and hence $\operatorname{Im} f = 0$ even scalar multiplication by $\frac{1}{2i}$ is not defined. Clearly, $f = \operatorname{Re} f + i\operatorname{Im} f$ and once again we have $i\operatorname{Im} f = 0$ for a real conjugated vector space even though the scalar multiplication by i is not defined. An element f in a conjugated vector space $\mathfrak{F}$ is said to be *real* if $f^- = f$. Clearly, the set $\mathfrak{F}^r$ of all real elements of a conjugated vector space $\mathfrak{F}$ with restriction to real scalar multiplications is a real conjugated vector space. By a complex element, we mean that it is not necessarily real but may be real or complex. Therefore it makes sense to talk about complex elements in a real conjugated vector space within the unified framework.

16-1.5. Let $\mathcal{F}$ be a conjugated vector space. A subset $\mathcal{F}^+$ of $\mathcal{F}^r$ is called a *positive cone* or simply a cone if the following conditions hold:
(a) $f+g \in \mathcal{F}^+$ and $\alpha f \in \mathcal{F}^+$, $\forall\, f,g \in \mathcal{F}^+, \forall\, \alpha \geq 0$;
(b) if both $f \in \mathcal{F}^+$ and $-f \in \mathcal{F}^+$, then $f = 0$.
A conjugated vector space with a positive cone is called an *ordered vector space*. Let $\mathcal{F}$ be an ordered vector space. Write $f \leq g$ if both f, g are real elements in $\mathcal{F}$ and $g - f \in \mathcal{F}^+$. In this case, f is said to be *majorized* by g. Clearly, for every real element f, we have $f \geq 0$ iff $f \in \mathcal{F}^+$.

16-1.6. **Exercise** A function f on a set X is *positive* if $f(x) \geq 0$ for all $x \in X$. Show that the set of all positive functions forms a cone for the real vector space $\mathbb{F}^r(X)$ and the complex vector space $\mathbb{F}(X)$.

16-1.7. **Exercise** A function $F : \mathbb{R} \to \mathbb{R}$ is *increasing* if $F(x) \leq F(y)$ whenever $x \leq y$ and *strictly* increasing if $F(x) < F(y)$ whenever $x < y$. Verify that the set of all increasing functions on $\mathbb{R}$ forms a cone for the vector space of functions on $\mathbb{R}$. This cone will play an important role in measure theory on $\mathbb{R}$.

16-1.8. **Theorem** For all real elements f, g, h, k of an ordered vector space $\mathcal{F}$, we have
(a) $f \leq f$, reflexive;
(b) if $f \leq g$ and $g \leq f$, then $f = g$, anti-symmetric;
(c) if $f \leq g$ and $g \leq h$, then $f \leq h$, transitive;
(d) if $f \leq g$ and $h \leq k$, then $f + h \leq g + k$, addition;
(e) if $0 \leq f \leq g$ and $0 \leq \alpha \leq \beta$, then $\alpha f \leq \beta g$, scalar multiplication.

Proof. Suppose $f \leq g$ and $0 \leq \alpha \leq \beta$. Then $g - f \geq 0$ and $\beta - \alpha \geq 0$. Hence $\alpha(g - f) \geq 0$ and $(\beta - \alpha)g \geq 0$. Thus $\alpha f \leq \alpha g$ and $\alpha g \leq \beta g$. Consequently, $\alpha f \leq \beta g$. This proves (e). The rest is left as an exercise. □

16-2 Lattice Structure

16-2.1. It is a matter of opinion on what is the correct definition of complex vector lattices. We use valuation instead of vee-wedge operations as in most existing textbooks and we do not require any expression of valuation of a complex element in terms of its real and complex parts. On the other hand, our version is powerful enough to provide service in the subsequent chapters of this book. For other existing versions, see [Schaefer-74; §1.8, p54; §11.1, p134; §11.2, p135], [Luxemburg; §91] and [Zaanen-97; ch 6].

16-2.2. Let $\mathcal{F}$ be an ordered vector space. For real elements, the concept of upper bounds, lower bounds, maxima, minima, suprema, infima and lattice notations $f \vee g$ and $f \wedge g$ are defined on $\mathcal{F}$ in an obvious way. It is easy to prove that they are unique if they exist. Clearly, the operations $\vee, \wedge$ are commutative and associative. A map $f \to |f|$ from $\mathcal{F}$ into $\mathcal{F}^+$ is called a *valuation* if for all $f, g \in \mathcal{F}$ and all $\alpha \in \mathbb{K}$, we have
(a) $|f + g| \leq |f| + |g|$, $|\alpha f| = |\alpha|\,|f|$ and $|f^-| = |f|$;
(b) if $|f| = 0$ then $f = 0$.
In this case, the element $|f|$ is called the *total variation*, or *variation*, or *absolute value* of f. It is obvious that $|\text{Re } f| \leq |f|$ and $|\text{Im } f| \leq |f|$ for all complex element $f \in \mathcal{F}$. An ordered vector space together with a given valuation is called a *vector lattice* if $|f|$ is the maximum of $\pm f$ for each real element $f \in \mathcal{F}^r$.

16-2.3. **Exercise** Consider the vector spaces $\mathbb{F}(X)$, $\mathbb{F}^r(X)$ of functions on a non-empty set X. Show that the map $f \to |f|$ is a valuation. Verify that $\mathbb{F}(X)$, $\mathbb{F}^r(X)$ are vector lattices under pointwise operations. Verify that a vector subspace forms a vector lattice if it is closed under formation of absolute values. See §3-8.3.

16-2.4. **Exercise** Show that the set of all real polynomials on $\mathbb{R}$ forms an ordered vector space but it is not a vector lattice under pointwise operations.

16-2.5. **Theorem** Let f, g be real elements of a vector lattice $\mathcal{F}$.
(a) $f \vee g = \frac{1}{2}(f + g + |f - g|)$ is the maximum of f, g.
(b) $f \wedge g = \frac{1}{2}(f + g - |f - g|)$ is the minimum of f, g.
(c) $f + g = f \vee g + f \wedge g$ and $|f - g| = f \vee g - f \wedge g$.

Proof. Let $h = \frac{1}{2}(f + g + |f - g|)$. Since $f - g \leq |f - g|$, we have $h \geq \frac{1}{2}(f + g + f - g) = f$. Similarly since $g - f \leq |f - g|$, we have $h \geq g$. Suppose k is a real element majorizing both f, g. Then $2k - f - g \geq 2f - f - g = f - g$ and $2k - f - g \geq -(f - g)$. Since $|f - g|$ is the least upper bound of $\pm(f - g)$, we have $2k - f - g \geq |f - g|$, i.e. $k \geq h$. Thus h is the least upper bound of f, g. Therefore h is the maximum $f \vee g$ of f, g. This proves (a). We leave the rest as an exercise. □

16-2.6. **Theorem** For all real elements f, g, h, k in a vector lattice $\mathcal{F}$, we have
(a) $h + f \vee g = (h + f) \vee (h + g)$ and $h + f \wedge g = (h + f) \wedge (h + g)$;

(b) $h - f \vee g = (h-f) \wedge (h-g)$ and $h - f \wedge g = (h-f) \vee (h-g)$;
(c) $\alpha(f \vee g) = (\alpha f) \vee (\alpha g)$ and $\alpha(f \wedge g) = (\alpha f) \wedge (\alpha g), \quad \forall\, \alpha \geq 0$;
(d) if $f \leq g$ and $h \leq k$, then $f \vee h \leq g \vee k$ and $f \wedge h \leq g \wedge k$;
(e) $(f \vee g) \wedge g = g$ and $(f \wedge g) \vee g = g$;
(f) if $f + g = h + k$, then $f + g = f \vee h + g \wedge k$.

Proof. Part (f) follows from the following calculation:

$$f + g - f \vee h = (f+g-f) \wedge (f+g-h) = g \wedge (h+k-h) = g \wedge k.$$

The other parts follow from the definitions of maximum and minimum. □

16-2.7. **Lemma** Let f_{ij} be real elements in an ordered vector space $\mathfrak{F}$. If one of the following exists, then we have $\sup\limits_{(i,j)\in I\times J} f_{ij} = \sup\limits_{i\in I}\sup\limits_{j\in J} f_{ij} = \sup\limits_{j\in J}\sup\limits_{i\in I} f_{ij}$.

16-2.8. **Theorem** Let A, B be non-empty sets of real elements in $\mathfrak{F}$ such that both $\sup A, \sup B$ exist.
(a) $\sup(f + B) = f + \sup B$ and $\sup(A + B) = \sup A + \sup B$;
(b) $\sup(\alpha A) = \alpha \sup A, \quad \forall\, \alpha \geq 0$;
(c) $\inf(-A) = -\sup A$;
(d) $\sup(f \vee B) = f \vee \sup B$ and $\sup(A \vee B) = (\sup A) \vee (\sup B)$;
(e) $\sup(f \wedge B) = f \wedge \sup B$ and $\sup(A \wedge B) = (\sup A) \wedge (\sup B)$.

Proof. We shall prove the distributive law (e) and leave the others as an exercise. Let $h = \sup B$. Since $f \wedge g \leq f \wedge h$ for all $g \in B$, $f \wedge h$ is an upper bound of the set $f \wedge B = \{f \wedge g : g \in B\}$. For any upper bound k of $f \wedge B$, we get $k + f \vee h \geq f \wedge g + f \vee g = f + g$, i.e. $g \leq k + f \vee h - f$. Taking supremum over B, we have $h = \sup B \leq k + f \vee h - f$, or $f \vee h + f \wedge h = f + h \leq k + f \vee h$. Hence $k \geq f \wedge h$. Therefore $f \wedge h$ is the supremum of $f \wedge B$. Finally,

$$\sup(A \wedge B) = \sup_{(f,g)} f \wedge g = \sup_{f}\sup_{g} f \wedge g = \sup_{f}(f \wedge \sup B) = (\sup A) \wedge (\sup B)$$

completes the proof. □

16-2.9. **Theorem** For all real elements f, g in a vector lattice $\mathfrak{F}$, we have
(a) $|f+g| \vee |f-g| = |f| + |g|$;
(b) $|f+g| \wedge |f-g| = \big|\, |f| - |g| \,\big|$;
(c) $|f+g| + |f-g| = 2(|f| \vee |g|)$;
(d) $\big|\, |f+g| - |f-g| \,\big| = 2(|f| \wedge |g|)$.

Proof. (a) $|f+g| \vee |f-g| = (f+g) \vee (-f-g) \vee (f-g) \vee (-f+g)$
$= (f+g) \vee (f-g) \vee (-f-g) \vee (-f+g)$
$= \frac{1}{2}\{(f+g)+(f-g)+|(f+g)-(f-g)|\} \vee \frac{1}{2}\{(-f-g)+(-f+g)+|(-f-g)-(-f+g)|\}$

$= (f + |g|) \vee (-f + |g|) = f \vee (-f) + |g| = |f| + |g|$.
(c) It follows from (a) by replacing f with $f + g$ and g with $f - g$. □

16-2.10. Two elements f, g in a vector lattice $\mathfrak{F}$ are said to be *mutually singular* or *disjoint* if $|f| \wedge |g| = 0$. In this case, it is denoted by $f \perp g$.

16-2.11. **Theorem** Two elements $f, g \in \mathfrak{F}$ are mutually singular iff there is an equality among any two of the following terms: $|f|+|g|$, $|f| \vee |g|$ and $\big|\, |f| - |g| \,\big|$.

Proof. It follows from the following identities: $|f| + |g| = |f| \vee |g| + |f| \wedge |g|$, $|f| \wedge |g| = \frac{1}{2}\left(|f| + |g| - \big|\, |f| - |g| \,\big| \right)$ and $\big|\, |f| - |g| \,\big| = |f| \vee |g| - |f| \wedge |g|$. □

16-2.12. **Exercise** Prove that two real elements $f, g \in \mathfrak{F}^r$ are mutually singular iff there is an equality among any two of the following terms : $|f + g|$, $|f - g|$ and $|f| \vee |g|$.

16-3 Decomposition Property

16-3.1. Let $\mathfrak{F}^+, \mathfrak{F}^r$ denote the positive and real elements of a vector lattice $\mathfrak{F}$ respectively. For every real element $f \in \mathfrak{F}^r$, the *upper variation* is defined by $f_+ = f \vee 0$ and the *lower variation* by $f_- = (-f) \vee 0$. As a simple result of the following theorem, every complex element h in a vector lattice has a decomposition into positive elements:

$$h = (\text{Re } h)_+ - (\text{Re } h)_- + i(\text{Im } h)_+ - i(\text{Im } h)_-.$$

16-3.2. **Theorem** For every real element $f \in \mathfrak{F}^r$, we have
(a) $f = f_+ - f_-$, that is $\mathfrak{F}^r = \mathfrak{F}^+ - \mathfrak{F}^+$;
(b) $|f| = f_+ + f_- = f_+ \vee f_-$;
(c) $f_+ \wedge f_- = 0$.

Proof. (a) $f = f + 0 = f \vee 0 + f \wedge 0 = f_+ - f_-$.
(b) $|f| = |f - 0| = f \vee 0 - f \wedge 0 = f_+ + f_-$
and also $|f| = f \vee (-f) \vee 0 = (f \vee 0) \vee (-f \vee 0) = f_+ \vee f_-$.
(c) By (b), we have $f_+ + f_- = f_+ \vee f_-$. Hence $f_+ \wedge f_- = 0$. □

16-3.3. **Theorem** If $f = g - h$, $g \geq 0$, $h \geq 0$ and $g \perp h$, then $g = f_+$ and $h = f_-$. In other words, the decomposition of real element into difference of positive singular vectors is unique.

Proof. Since $g \perp h$, we have $|g| + |h| = \big|\, |g| - |h| \,\big|$. By $g \geq 0$, $h \geq 0$, we get $|f| = |g - h| = g + h$. Solving for g, h with $f = g - h$, we have $g = \frac{1}{2}(|f| + f) = f_+$. Similarly, we obtain $h = f_-$. □

16-3.4. **Lemma** For positive vectors $f, g, h \in \mathcal{F}^+$, we have

$$(f+g) \wedge h \le f \wedge h + g \wedge h.$$

Proof. Clearly, $(f+g)\wedge h \le (f+g)\wedge(f+h) = f+g\wedge h$ and also $(f+g)\wedge h \le h+g\wedge h$. Hence $(f+g)\wedge h \le (f+g\wedge h)\wedge(h+g\wedge h) = f\wedge h + g\wedge h$. □

16-3.5. **Theorem** For all $f, g, h \in \mathcal{F}$, if $f \perp h$ and $g \perp h$, then $(f+g) \perp h$.

Proof. It follows from $0 \le (|f|+|g|) \wedge |h| \le |f| \wedge |h| + |g| \wedge |h| = 0$. □

16-3.6. **Exercise** Let f, g be real elements in $\mathcal{F}^r$. Prove that if $f \perp g$, then
(a) $f_+ + g_+$ and $f_- + g_-$ are mutually singular,
(b) $(f+g)_+ = f_+ + g_+$ and $(f+g)_- = f_- + g_-$.

16-3.7. **Theorem** For all real elements $f, g, h \in \mathcal{F}^r$, we have
(a) $|f-g| = |f \vee h - g \vee h| + |f \wedge h - g \wedge h|$;
(b) $|f \vee h - g \vee h| \le |f-g|$ and $|f \wedge h - g \wedge h| \le |f-g|$;
(c) $|f_+ - g_+| \le |f-g|$ and $|f_- - g_-| \le |f-g|$.

Proof. Part (a) follows from the calculation below and the rest are easy consequences:

$$\begin{aligned}
&|f \vee h - g \vee h| + |f \wedge h - g \wedge h| \\
&= |(f \vee h - g \vee h) + (f \wedge h - g \wedge h)| \vee |(f \vee h - g \vee h) - (f \wedge h - g \wedge h)| \\
&= |(f+h) - (g+h)| \vee \big||f-h| - |g-h|\big| \\
&= |f-g| \vee \{|(f-h)+(g-h)| \wedge |(f-h)-(g-h)|\} \\
&= |f-g| \vee \{|f+g-2h| \wedge |f-g|\} = |f-g|.
\end{aligned}$$

□

16-3.8. **Theorem** Let f, g, h be positive vectors in $\mathcal{F}^+$. If $h \le f+g$, then there are $p, q \in \mathcal{F}^+$ such that $h = p+q$, $p \le f$ and $q \le g$.

Proof. Let $p = h \wedge f$ and $q = h - p$. Then $h \le (h+g)\wedge(f+g) = h\wedge f + g = p + g$, that is $q = h - p \le g$. The other conditions are obvious. □

16-3.9. **Exercise** Let f, g, h be functions on $\mathbb{R}$ given by $f(x) = |x| + 1 - |x-1|$, $g(x) = |x-1| + 1 - |x|$ and $h(x) = 1 + \sin x$ for each $x \in \mathbb{R}$. Find two continuous positive functions p, q on $\mathbb{R}$ such that $h = p + q$, $p \le f$ and $q \le g$. Sketch all functions f, g, h, p, q.

16-3.10. **Theorem** Let f_i, g_j be positive vectors in $\mathcal{F}^+$. If $\sum_{i=1}^m f_i = \sum_{j=1}^n g_j$ then there are positive vectors $k_{ij} \in \mathcal{F}^+$ such that $f_i = \sum_{j=1}^n k_{ij}$ and $g_j = \sum_{i=1}^m k_{ij}$ for all i, j. It is easier to remember this theorem by writing $f_1, f_2, \cdots$ as the left column, $g_1, g_2, \cdots$ as the top row and k_{ij} as the (i,j)-cell of a table.

Proof. Assume $m = n = 2$. Then $f_1 + f_2 = g_1 + g_2$. Now $f_1 \le g_1 + g_2$. There are positive vectors $k_{11} \le g_1$, $k_{12} \le g_2$ such that $f_1 = k_{11} + k_{12}$. Define $k_{21} = g_1 - k_{11}$ and $k_{22} = g_2 - k_{12}$. Since $k_{11} \le g_1$, we have $k_{21} = g_1 - k_{11} \ge 0$. Similarly, $k_{22} \ge 0$. Finally, $k_{21} + k_{22} = g_1 + g_2 - (k_{11} + k_{12}) = f_1 + f_2 - f_1 = f_2$. This proves the case for $m = n = 2$. The general proof follows by induction. □

16-3.11. A vector lattice $\mathfrak{F}$ is said to have *decomposition property* if for every $h \in \mathfrak{F}$ and for all $f, g \in \mathfrak{F}^+$ satisfying $|h| \le f + g$, there are $p, q \in \mathfrak{F}$ such that $|p| \le f$, $|q| \le g$ and $h = p + q$. It should not be confused with the earlier condition that every complex element can be decomposed into positive elements. For convenience, a real or complex vector lattice with decomposition property is abbreviated as a *breakable* vector lattice.

16-3.12. **Theorem** Every *real* vector lattice $\mathfrak{F}$ is breakable.

Proof. Let $h \in \mathfrak{F}$ and $f, g \in \mathfrak{F}^+$ satisfy $|h| \le f + g$. Then $h_+ + h_- \le f + g$. Write $h_+ + h_- = u + v$ where $u, v \in \mathfrak{F}^+$ satisfy $u \le f$ and $v \le g$. There are $k_{ij} \in \mathfrak{F}^+$ such that $h_+ = k_{11} + k_{12}$, $h_- = k_{21} + k_{22}$, $u = k_{11} + k_{21}$ and $v = k_{12} + k_{22}$. Then $p = k_{11} - k_{21}$ and $q = k_{12} - k_{22}$ satisfy $|p| \le k_{11} + k_{21} = u \le f$, $|q| \le g$ and $p + q = k_{11} - k_{21} + k_{12} - k_{22} = k_{11} + k_{12} - k_{21} - k_{22} = h_+ - h_- = h$. □

16-4 Extension of Positive Linear Forms

16-4.1. Let $\mathfrak{F}$ be a conjugated vector space. Then a linear form $\mu : \mathfrak{F} \to \mathbb{K}$ is said to be *real* if μf is a real number whenever f is real in $\mathfrak{F}$. Obviously a linear form μ on $\mathfrak{F}$ is real iff it preserves the conjugation, i.e. $\mu(f^-) = \mu(f)^-$. For any linear form μ on $\mathfrak{F}$, the function $\mu^- : \mathfrak{F} \to \mathbb{K}$ given by $\mu^-(f) = [\mu(f^-)]^-$ for $f \in \mathfrak{F}$ is also a linear form called the *conjugate* of μ. Clearly the algebraic dual space $\mathfrak{F}^*$ of all linear forms on $\mathfrak{F}$ is a conjugated vector space under the conjugation $\mu \to \mu^-$. As a result, $\text{Re}(\mu)$ and $\text{Im}(\mu)$ are well-defined. They are actually real linear forms.

16-4.2. **Lemma** Let $\nu : \mathfrak{F}^r \to \mathbb{K}$ be a function such that for all $f, g \in \mathfrak{F}^r$ and $\alpha \in \mathbb{R}$, we have $\nu(f + g) = \nu f + \nu g$ and $\nu(\alpha f) = \alpha \nu f$. Then ν can be extended uniquely to a linear form $\mu : \mathfrak{F} \to \mathbb{K}$.

Proof. For $\mathbb{K} = \mathbb{R}$, take $\mu = \nu$. For $\mathbb{K} = \mathbb{C}$, define $\mu f = \nu(\text{Re } f) + i\nu(\text{Im } f)$, for all $f \in \mathfrak{F}$. It is routine to verify the result. □

16-4.3. **Lemma** Let h be a complex element of a conjugated vector space $\mathfrak{F}$. If for every real linear form μ on $\mathfrak{F}$, μh is a real number then h is real.

Proof. Let $h = f + ig$ where f, g are real elements. Then for every real linear form μ, $\mu h = \mu f + i\mu g$ is real, i.e. $\mu g = 0$. Thus for every complex linear form ν, we have $\nu g = (\text{Re } \nu)(g) + i(\text{Im } \nu)(g) = 0$. Therefore $g = 0$. Consequently $h = f$ is real. □

16-4.4. Let $\mathcal{F}$ be an ordered vector space. Then a linear form μ on $\mathcal{F}$ is said to be *positive* if $\mu f \geq 0$ is a positive number for every $f \geq 0$ in $\mathcal{F}$. Clearly every positive linear form is real and the set of all positive linear forms is a cone. Hence $\mathcal{F}^*$ becomes an ordered vector space but in general it is not a vector lattice. To get good results, we assume that $\mathcal{F}$ is a vector lattice.

16-4.5. **Theorem** Let $\mu \leq \nu$ be positive linear forms on $\mathcal{F}$.
(a) For every $f \leq g \in \mathcal{F}^r$ we have $\mu f \leq \mu g$.
(b) If $0 \leq f \leq g$ in $\mathcal{F}^r$, then $\mu f \leq \nu g$ in $\mathbb{R}$.
(c) For every complex element h in $\mathcal{F}$, $|\mu h| \leq \mu |h|$.

Proof. We shall prove (c) only. Write $\mu h = e^{i\theta}|\mu h|$ for some $0 \leq \theta \in \mathbb{R}$. Express $e^{-i\theta}h = f + ig$ where $f, g \in \mathcal{F}^r$. Then $f \leq |f| \leq |e^{-i\theta}h| = |h|$ and $\mu f, \mu g$ are real numbers. The proof is completed by the following calculation:

$$|\mu h| = e^{-i\theta}\mu h = \mu(e^{-i\theta}h) = \mu(f + ig) = \mu f + i\mu g = \mu f \leq \mu|h|.$$ □

16-4.6. **Lemma** Let $\nu : \mathcal{F}^+ \to \mathbb{K}$ be a map such that $\nu(f + g) = \nu f + \nu g$ and $\nu(\alpha f) = \alpha \nu f$ for all $f \in \mathcal{F}^+$ and $\alpha \geq 0$. Then ν can be extended uniquely to a linear form on $\mathcal{F}$.

Proof. Since $\mathcal{F}$ is a vector lattice, for any $f \in \mathcal{F}^r$ there are $p, q \geq 0$ such that $f = p - q$. Define $\mu f = \nu p - \nu q$. Suppose $f = x - y$ where $x, y \geq 0$ in $\mathcal{F}$. Then $p + y = q + x$. Hence $\nu p + \nu y = \nu q + \nu x$, or $\nu p - \nu q = \nu x - \nu y$. Therefore $\mu : \mathcal{F}^r \to \mathbb{K}$ is a well-defined extension of ν. Observe that

$$\begin{aligned}\mu(f + g) &= \mu(f_+ + g_+ - f_- - g_-) = \nu(f_+ + g_+) - \nu(f_- + g_-)\\ &= \nu(f_+) + \nu(g_+) - \nu(f_-) - \nu(g_-) = \mu f + \mu g.\end{aligned}$$

Next, for any $\alpha \geq 0$, we have

$$\mu(\alpha f) = \mu(\alpha f_+ - \alpha f_-) = \nu(\alpha f_+) - \nu(\alpha f_-) = \alpha\nu(f_+) - \alpha\nu(f_-) = \alpha\mu f$$

and $\mu(-f) = \mu(f_- - f_+) = \nu(f_-) - \nu(f_+) = -\mu f$. Thus $\mu : \mathcal{F}^r \to \mathbb{K}$ is real-linear. It can be extended to a linear form on $\mathcal{F}$. The uniqueness is obvious. □

16-4.7. **Theorem** Let $\mathcal{F}$ be a vector lattice and $\nu : \mathcal{F}^+ \to \mathbb{R}^+$ be a map satisfying $\nu(f + g) = \nu f + \nu g$ for all $f, g \in \mathcal{F}^+$. Then ν can be extended uniquely to a positive linear form on $\mathcal{F}$.

Proof. For integers $m, n \geq 1$, we get $\nu(ng) = \nu(g+\cdots+g) = \nu g+\cdots+\nu g = n\nu g$ or $\nu g = \nu(ng)/n$. For $g = f/n$, we obtain $\nu(f/n) = \nu(f)/n$. Combining together, $\nu(mf/n) = m\nu(f/n) = (m/n)\nu f$. Therefore $\nu(\alpha f) = \alpha\nu f$ for all rational number $\alpha \geq 0$. Next, suppose $0 \leq f \leq g$. Then $g - f \geq 0$. From $g = f + (g - f)$, we get $\nu(g - f) \geq 0$ and $\nu g = \nu f + \nu(g - f) \geq \nu f$. Thus ν is monotonic. In general, let $\alpha \geq 0$ be any positive real number. Then there are rational numbers r_n, s_n with $0 \leq r_n \leq \alpha \leq s_n$, $r_n \to \alpha$ and $s_n \to \alpha$. Since $f \geq 0$, we get $r_n f \leq \alpha f \leq s_n f$, i.e. $\nu(r_n f) \leq \nu(\alpha f) \leq \nu(s_n f)$, or $r_n \nu f \leq \nu(\alpha f) \leq s_n \nu f$. Letting $n \to \infty$, we obtain $\nu(\alpha f) = \alpha\nu f$. The result follows from last lemma.□

16-5 Order Bounded Linear Forms

16-5.1. This section provides enough coverage on order bounded linear forms so as to study the characterization of charges with finite variation in next chapter. General treatment of linear maps between two vector lattices is beyond our scope. The dual characterization of lattice operations requires Hahn-Banach Theorem.

16-5.2. Let $\mathfrak{F}$ be a vector lattice. A linear form $\mu : \mathfrak{F} \to \mathbb{K}$ is said to be *order bounded* if for every $f \in \mathfrak{F}^+$, the set $\{|\mu h| : |h| \leq f,\ h \in \mathfrak{F}\}$ is bounded in $\mathbb{K}$. Clearly, every positive linear form is order bounded. The set of all order bounded linear forms on $\mathfrak{F}$ is denoted by $\mathfrak{F}^b$. It is an ordered vector space under the cone $\mathfrak{F}^{b+}$ of all positive linear forms.

16-5.3. **Example** Draw a short line segment from the origin of xy-plane to the right of y-axis and then extend it to a function from $\mathbb{R}$ into $\mathbb{R}$. The set $\mathfrak{F}$ of all such functions form a real vector lattice. Taking the slope of the short line segment is a linear form but it is not order bounded.

16-5.4. **Theorem** Let μ be an order bounded linear form on a breakable vector lattice $\mathfrak{F}$.

(a) For all $f \geq 0$ in $\mathfrak{F}^+$ and $\mu \in \mathfrak{F}^b$, the expression

$$|\mu| f = \sup\{|\mu h| : |h| \leq f, h \in \mathfrak{F}\}$$

uniquely defines a positive linear form $|\mu|$ on $\mathfrak{F}$.

(b) For all $h \in \mathfrak{F}$, we have $|\mu h| \leq |\mu|\ |h|$.

(c) $|\mu| = \inf\{\xi \in \mathfrak{F}^{b+} : |\mu h| \leq \xi\ |h|, \quad \forall\ h \in \mathfrak{F}\}$.

(d) The map $\mu \to |\mu|$ is a valuation on $\mathfrak{F}^b$.

Proof. Let $f, g \geq 0$ in $\mathcal{F}^+$. For any $\varepsilon > 0$, there exist $p, q \in \mathcal{F}$ such that $|p| \leq f$, $|q| \leq g$, $\nu f \leq |\mu p| + \varepsilon$ and $\nu g \leq |\mu q| + \varepsilon$. There are numbers $|\alpha| = |\beta| = 1$ such that $\alpha\mu p \geq 0$ and $\beta\mu q \geq 0$. Hence we obtain $|\alpha p + \beta q| \leq f + g$ and

$$\nu f + \nu g \leq |\alpha|\,|\mu p| + |\beta|\,|\mu q| + 2\varepsilon = \mu(\alpha p) + \mu(\beta q) + 2\varepsilon \leq |\mu(\alpha p + \beta q)| + 2\varepsilon \leq \nu(f+g) + 2\varepsilon.$$

Letting $\varepsilon \to 0$, we have $\nu f + \nu g \leq \nu(f+g)$. Next, let $|h| \leq f+g$ in $\mathcal{F}^r$. There are $p, q \in \mathcal{F}$ such that $h = p+q$, $|p| \leq f$ and $|q| \leq g$. Now $|\mu h| \leq |\mu p| + |\mu q| \leq \nu f + \nu g$, i.e. $\nu(f+g) \leq \nu f + \nu g$. Therefore ν can be extended uniquely to some positive linear form $|\mu|$ on $\mathcal{F}$. Replacing f by $|h|$, we have $|\mu h| \leq \nu|h| = |\mu|\,|h|$. Suppose $\xi \in \mathcal{F}^{b+}$ satisfying $|\mu h| \leq \xi|h|$ for all $h \in \mathcal{F}$. Then for $|h| \leq f$ in $\mathcal{F}$, $|\mu h| \leq \xi|h| \leq \xi f$, that is $|\mu| f = \nu f \leq \xi f$. Since $f \in \mathcal{F}^+$ is arbitrary, $|\mu| \leq \xi$. This proves (c). It is routine to verify (d). □

16-5.5. **Theorem** Let $\mathcal{F}$ be a breakable vector lattice. Then $\mathcal{F}^b$ is also a vector lattice. Furthermore, the order interval for $f, g \in \mathcal{F}^r$ is defined by $[f, g] = \{h \in \mathcal{F}^r : f \leq h \leq g\}$. Then for all real bounded linear forms μ, ν on $\mathcal{F}$ and all $f \in \mathcal{F}^+$, we have

(a) $|\mu|(f) = \sup \mu[-f, f]$;

(b) $(\mu \vee \nu)(f) = \sup\{\mu f + \nu(f-h) : h \in [0, f]\}$;

(c) $(\mu \wedge \nu)(f) = \inf\{\mu h + \nu(f-h) : f \in [0, f]\}$;

(d) $\mu^+(f) = \sup \mu[0, f]$;

(e) $\mu^-(f) = \sup \mu[-f, 0]$.

Proof. Let $\nu : \mathcal{F} \to \mathbb{K}$ be defined by $\nu f = \sup\{\mu p : -f \leq p \leq f,\ p \in \mathcal{F}^r\}$. Suppose h is a complex element in $\mathcal{F}$ satisfying $|h| \leq f$. Write $\mu h = e^{i\theta}|\mu h|$ and $e^{-i\theta}h = p + iq$ where $p, q \in \mathcal{F}^r$. Then we have $\pm p \leq |p| \leq |h| \leq f$, that is $-f \leq p \leq f$. Now, $|\mu h| = \mu(e^{-i\theta}h) = \mu p + i\mu q$. Since μ is real, we obtain $|\mu h| = \mu p \leq |\mu p| \leq \nu f$. Therefore $|\mu| f \leq \nu f$. The reversed inequality follows immediately from definition. It is elementary to show $|\mu| = \mu \vee (-\mu)$. Consequently $\mathcal{F}^b$ is a vector lattice. Part (b) follows from the calculation:

$$\begin{aligned}
&(\mu \vee \nu)(f) = \tfrac{1}{2}(\mu + \nu + |\mu - \nu|)(f) \\
&= \tfrac{1}{2}[\mu f + \nu f + \sup\{(\mu - \nu)(p) : -f \leq p \leq f\}] \\
&= \sup\left\{\mu\left(\frac{f+p}{2}\right) + \nu\left(\frac{f-p}{2}\right) : -f \leq p \leq f\right\} \\
&= \sup\{\mu q + \nu(f-q) : 0 \leq q \leq f\}.
\end{aligned}$$

The other of the proof is left as an exercise. □

16-5.6. **Theorem** Let μ be a positive linear form on a breakable vector lattice $\mathcal{F}$. Then for every $h \in \mathcal{F}$, we have $\mu|h| = \sup\{|\nu(h)| : |\nu| \leq \mu\}$.

Proof. Let $\varphi(g) = \mu|g|$ for all $g \in \mathfrak{F}$. Since $\mu \geq 0$, φ is a seminorm on $\mathfrak{F}$. By Hahn-Banach theorem, there is a linear form ξ on $\mathfrak{F}$ such that $\xi(h) = \varphi(h)$ and $|\xi(g)| \leq \varphi(g)$ for all $g \in \mathfrak{F}$. If $|g| \leq f$ in $\mathfrak{F}$, then $|\xi(g)| \leq \varphi(g) = \mu|g| \leq \mu f$. Hence ξ is an order bounded linear form on $\mathfrak{F}$. Clearly, $|\xi| \leq \mu$. Therefore

$$\mu|h| = \varphi(h) = \xi(h) \leq \sup\{|\nu(h)| : |\nu| \leq \mu\}.$$

On the other hand, if $|\nu| \leq \mu$, then we obtain $|\nu(h)| \leq |\nu|\ |h| \leq \mu|h|$, that is $\sup\{|\nu(h)| : |\nu| \leq \mu\} \leq \mu|h|$. □

16-5.7. **Theorem** Let μ be a positive linear form on a breakable vector lattice $\mathfrak{F}$. Then for all real elements f, b in $\mathfrak{F}^r$, we have

(a) $\mu(f_+) = \sup[0, \mu](f)$;
(b) $\mu(f_-) = \sup[-\mu, 0](f)$;
(c) $\mu|f| = \sup[-\mu, \mu](f)$;
(d) $\mu(f \vee g) = \sup\{\nu(f) + (\mu - \nu)(g) : \nu \in [0, \mu]\}$;
(e) $\mu(f \wedge g) = \inf\{\nu(f) + (\mu - \nu)(g) : \nu \in [0, \mu]\}$.

Proof. Let $\varphi(h) = \mu(h_+)$ for $h \in \mathfrak{F}$. Since $\mu \geq 0$, we get $\varphi(h + k) \leq \varphi(h) + \varphi(k)$ and $\varphi(\alpha h) = \alpha\varphi(h)$ for all positive number α. By Hahn-Banach theorem, there is a real linear form ξ on $\mathfrak{F}^r$ such that $\xi(f) = \varphi(f)$ and $\xi(h) \leq \varphi(h)$ for all $h \in \mathfrak{F}^r$. Then ξ can be in turn extended to a complex linear form on the conjugated vector space $\mathfrak{F}$. Now pick any $h \geq 0$, we have

$$\xi(-h) \leq \varphi(-h) = \mu[(-h)_+] = \mu(0) = 0,$$

that is $\xi h \geq 0$. Hence $\xi \geq 0$. Next, $\xi h \leq \varphi(h) = \mu(h_+) = \mu h$ gives $\xi \leq \mu$. Therefore $\xi \in [0, \mu]$ and $\mu(f_+) = \varphi(f) = \xi(f) \leq \sup[0, \mu](f)$. On the other hand, for any $\nu \in [0, \mu]$, we get $\nu(f) \leq \nu(f_+) \leq \mu(f_+)$, i.e. $\sup[0, \mu](f) \leq \mu(f_+)$. This proves (a). Part (b) follows from

$$\mu(f_-) = \mu[(-f)_+] = \sup[0, \mu](-f) = \sup\{\xi(-f) : 0 \leq \xi \leq \mu\}$$

$$= \sup\{(-\xi)(f) : -\mu \leq -\xi \leq 0\} = \sup[-\mu, 0](f).$$

The others are left as an exercise. □

16-5.98. **Problem** Prove or disprove that if $\mathfrak{F}$ is a breakable vector lattice then so is the space $\mathfrak{F}^b$ of bounded linear forms.

16-99. **References and Further Readings** : Zaanen-97, Schaefer, Luxemburg, Aliprantis, Jacobs, Wong-76 and Filter.

Chapter 17

Vector Measures on Semirings

17-1 Semirings

17-1.1. Simple geometric objects such as semi-intervals and semi-rectangles are used as building blocks. Their essential properties are extracted to form the axioms on which a rich theory of integration is built to serve many areas of pure and applied mathematics. Semi-rectangles provide a graphical interpretation of the abstract theory.

17-1.2. A family $\mathcal{S}$ of subsets of X is called a *semiring over* a set X, or a semiring of *subsets of* X in order to be precise, if

(a) the empty set $\emptyset$ is in $\mathcal{S}$;

(b) the intersection of any two sets in $\mathcal{S}$ is in $\mathcal{S}$;

(c) for all A, B in $\mathcal{S}$; there are *disjoint* sets $B_1, B_2, \cdots, B_n$ in $\mathcal{S}$ such that

$$A \setminus B = \bigcup_{j=1}^{n} B_j.$$

17-1.3. **Example** The family of all singletons of a set X together with the empty set forms a semiring. It is called the *semiring of singletons* of X.

17-1.4. **Example** Consider the real line $\mathbb{R}$. A *semi-interval* is a set of the form $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ where $a < b$. The family of all semi-intervals together with the empty set forms a semiring. It is called the *semiring of semi-intervals* of $\mathbb{R}$.

17-1.5. **Exercise** The sets of the form $(a, b] \times (c, d]$, where $a < b$ and $c < d$, are called *semi-rectangles* of $\mathbb{R}^2$. Sketch the semi-rectangles: $A = (0, 3] \times (2, 4]$, $B = (2, 5] \times (1, 3]$ and $C = (1, 4] \times (1, 5]$. Identify $A \cap B$ and express $A \setminus B$, $C \setminus A$ and $C \setminus (A \cup B)$ as unions of disjoint semi-rectangles. Show that the family of all semi-rectangles together with the empty set forms a semiring over $\mathbb{R}^2$.

17-1.6. **Lemma** Let $\mathcal{S}$ be a semiring over a set X. Then for all $A, B_1, B_2, \cdots, B_n$ in $\mathcal{S}$, there are disjoint sets $C_1, C_2, \cdots, C_m$ in $\mathcal{S}$ such that

$$A \setminus \bigcup_{i=1}^{n} B_i = \bigcup_{j=1}^{m} C_j.$$

Proof. For $n = 1$ it follows immediately from the definition. Assume that it is true for n by induction. Observe $A \setminus \bigcup_{i=1}^{n+1} B_i = \bigcup_{j=1}^{m}(C_j \setminus B_{n+1})$. There are disjoint sets $D_{j1}, D_{j2}, \cdots, D_{jp(j)}$ in $\mathcal{S}$ such that $C_j \setminus B_{n+1} = \bigcup_{k=1}^{p(j)} D_{jk}$. Then $\{D_{jk} : 1 \le j \le m, 1 \le k \le p(j)\}$ is a finite family of disjoint sets in $\mathcal{S}$ such that $A \setminus \bigcup_{i=1}^{n+1} B_i = \bigcup_{j=1}^{m} \bigcup_{k=1}^{p(j)} D_{jk}$. This completes the proof. □

17-1.7. Let A be a subset of X. The *characteristic function* of A is defined by $\rho_A(x) = 1$ if $x \in A$ and $\rho_A(x) = 0$ if $x \in X \setminus A$. The community uses χ instead.

17-1.8. **Exercise** For all subsets A, B of X, prove that $\rho_{A\cap B} = \rho_A \wedge \rho_B = \rho_A \rho_B$ and $\rho_{A\cup B} = \rho_A \vee \rho_B = \rho_A + \rho_B - \rho_{A\cap B}$. Prove that if $\{A_n\}$ is a sequence, finite or infinite, of *disjoint* subsets of X and if $B = \bigcup_n A_n$, then $\rho_B = \sum_n \rho_{A_n}$.

17-1.9. **Semiring Formula** For all $A_1, A_2, \cdots, A_n$ in a semiring $\mathcal{S}$, there are disjoint sets $B_1, B_2, \cdots, B_m$ in $\mathcal{S}$ such that $\rho_{A_i} = \sum_{j=1}^{m} \alpha_{ij} \rho_{B_j}$ for all $1 \le i \le n$ where α_{ij} are either zero or one.

Proof. Inductively assume it is true for $n \ge 1$. Write $B_j \setminus A_{n+1} = \bigcup_p C_{jp}$ where $\{C_{j1}, C_{j2}, \cdots\}$ is a finite family of disjoint sets in $\mathcal{S}$. Again write $A_{n+1} \setminus \bigcup_{j=1}^{m} B_j = \bigcup_k D_k$ where $\{D_1, D_2, \cdots\}$ is also a finite family of disjoint sets in $\mathcal{S}$. Define $E_j = B_j \cap A_{n+1}$. Clearly the totality of C_{jp}, D_k, E_j is a finite family of disjoint sets in $\mathcal{S}$. Simple calculations gives $\rho_{A_i} = \sum_j \alpha_{ij} \rho_{E_j} + \sum_j \sum_p \alpha_{ij} \rho_{C_{jp}}$ for all $1 \le j \le n$ and $\rho_{A_{n+1}} = \sum_j \rho_{E_j} + \sum_k \rho_{D_k}$. The induction is completed by inserting zero coefficients. □

17-1.10. **Corollary** Every finite union of sets in $\mathcal{S}$ can be written as a finite union of *disjoint* sets in $\mathcal{S}$.

17-1.11. **Exercise** Let $A = (0,3] \times (2,4]$, $B = (2,5] \times (1,3]$ and $C = (1,4] \times (1,5]$. Express their characteristic functions explicitly as described in the Semiring Formula. Also write $A \cup B \cup C$ as a finite union of disjoint semi-rectangles.

17-2 Charges and Associated Integrals

17-2.1. Let $\mathbb{K}$ denote the set of scalars which is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let E, F, FE be vector spaces over $\mathbb{K}$ with a bilinear map $\varphi : F \times E \to FE$. Note that we have used two letters FE to denote one Banach space for convenience. If E or F is the scalar field, we understand that φ is the scalar multiplication unless it is specified explicitly. Beginners should consider E, F, FE to be the scalar field $\mathbb{K}$ and $\varphi(u, v)$ the product of two numbers u, v.

17-2.2. Let $\mathcal{S}$ be a semiring over X. An *$\mathcal{S}$-step F-map* $g : X \to F$ is of the form $g = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ where $\alpha_j \in F$ and $A_j \in \mathcal{S}$. For convenience, we may drop the prefixes $\mathcal{S}$ and F.

17-2.3. **Step Mapping Theorem** For each $1 \leq i \leq n$, let f_i be a step map on X into a vector space F_i. Then there are disjoint sets $B_1, B_2, \cdots, B_m$ in $\mathcal{S}$ and β_{ij} in F_i such that $f_i = \sum_{j=1}^{m} \beta_{ij} \rho_{B_j}$ for all $1 \leq i \leq n$. Furthermore,
(a) if f_i is real then β_{ij} is real for each $1 \leq j \leq m$;
(b) if $f_i \geq 0$ then $\beta_{ij} \geq 0$ for each $1 \leq j \leq m$.

Proof. Given the following finite sums $f_i = \sum_k \alpha_{ik} \rho_{A_{ik}}$ where $\alpha_{ik} \in F_i$ and $A_{ik} \in \mathcal{S}$, there are disjoint set $B_1, B_2, \cdots, B_m$ in $\mathcal{S}$ such that $\rho_{A_{ik}} = \sum_j \gamma_{ijk} \rho_{B_j}$ where all summations are finite and the numbers γ_{ijk} are either zero or one. It follows by substitution that $f_i = \sum_{j=1}^{m} \beta_{ij} \rho_{B_j}$ where $\beta_{ij} = \sum_k \alpha_{ik} \gamma_{ijk}$. Furthermore, assume $f_i \geq 0$. If $B_j = \emptyset$ then let $\beta_{ij} = 0$. If $B_j \neq \emptyset$ then evaluation at any point $x \in B_j$ gives $\beta_{ij} = f_i(x) \geq 0$. Similarly we can prove the case when f_i is real. □

17-2.4. **Exercise** Let $A = (1,3]$, $B = (2,5]$ and $C = (4,6]$ be semi-intervals in $\mathbb{R}$. Sketch the functions f, g, h given by $f(x) = 2\rho_A - \rho_B$, $g(x) = 2\rho_A - \rho_B$, $g(x) = 4\rho_B - 3\rho_C$ and $h(x) = 3\rho_C - 2\rho_A$. Write f, g, h as linear combinations of characteristic functions of the same finite family of disjoint semi-intervals as in Step Mapping Theorem.

17-2.5. **Exercise** Show that the set $\mathcal{F}$ of all step functions on X is a vector lattice under pointwise operations. Prove that the product of two step functions is a step function.

17-2.6. A map $\mu : \mathcal{S} \to E$ is said to be *additive*, or is called an *E-charge*, or a finitely additive set map if for every finite disjoint union $A = \bigcup_{j=1}^{n} B_j$ where $A, B_1, B_2, \cdots, B_n$ are in $\mathcal{S}$, we have $\mu A = \sum_{j=1}^{n} \mu B_j$. Since $\emptyset = \emptyset \cup \emptyset$, we must have $\mu\emptyset = 0$. Furthermore μ is said to be *positive* or *real* if for each $A \in \mathcal{S}$, μA is a positive (≥ 0) or real number respectively. By a *vector* charge, we merely emphasize that E need not be the scalar field. By a *scalar* charge or simply a charge, we mean that $E = \mathbb{K}$. By a *complex* charge, we merely emphasize that μ need not be but might be positive or real.

17-2.7. **Example** Let $\mathcal{S}$ be the semiring of singletons of X. Let $\xi : X \to \mathbb{K}$ be a function. For each x in X, let $\mu\{x\} = \xi(x)$ and $\mu\emptyset = 0$. Then μ is a charge on $\mathcal{S}$. It is called the *counting measure* on X *weighted* by ξ. If $\mu\{x\} = 1$ for all $x \in X$, then it is simply called the *counting measure*.

17-2.8. **Example** Let $\mathcal{S}$ be a semiring over X and let $p \in X$ be a point. For every A in $\mathcal{S}$ let $\mu A = 1$ if $p \in A$ and $\mu A = 0$ otherwise. Then μ is additive on $\mathcal{S}$. It is called the *point measure* at p.

17-2.9. **Lemma** Let μ be an E-charge on $\mathcal{S}$. If $\sum_{i=1}^m \alpha_i \rho_{A_i} = \sum_{j=1}^n \beta_j \rho_{B_j}$ where $\alpha_i, \beta_j \in E$ and A_i, B_j in $\mathcal{S}$, then $\sum_{i=1}^m \varphi(\alpha_i, \mu A_i) = \sum_{j=1}^n \varphi(\beta_j, \mu B_j)$.

Proof. There are disjoint sets $D_1, D_2, \cdots, D_p$ in $\mathcal{S}$ such that $\rho_{A_i} = \sum_{k=1}^p \gamma_{ik} \rho_{D_k}$ and $\rho_{B_j} = \sum_{k=1}^p \delta_{jk} \rho_{D_k}$ where γ_{ik}, δ_{jk} are either zero or one. It is crucial to observe that each A_i is a finite disjoint union of some D_k selected by the values of $\gamma_{ik} = 1$. Hence we have $\mu A_i = \sum_{k=1}^p \gamma_{ik} \mu D_k$. Similarly, we have $\mu B_j = \sum_{k=1}^p \delta_{jk} \mu D_k$. Expressing the given condition in terms of D_k, we get $\sum_i \alpha_i \sum_k \gamma_{ik} \rho_{D_k} = \sum_j \beta_j \sum_k \delta_{jk} \rho_{D_k}$. If $D_k \neq \emptyset$ then evaluation of last expression at any point $x \in D_k$ gives $\sum_i \alpha_i \gamma_{ik} = \sum_j \beta_j \delta_{jk}$ for $D_1, D_2, \cdots$ are disjoint. Hence we obtain $\varphi(\sum_i \alpha_i \gamma_{ik}, \mu D_k) = \varphi(\sum_j \beta_j \delta_{jk}, \mu D_k)$. If $D_k = \emptyset$ then $\mu D_k = 0$ and last equation also holds. Because $\varphi : F \times E \to FE$ is bilinear, we have

$$\sum_i \varphi(\alpha_i, \mu A_i) = \sum_i \varphi(\alpha_i, \sum_k \gamma_{ik} \mu D_k) = \sum_k \varphi(\sum_i \alpha_i \gamma_{ik}, \mu D_k)$$
$$= \sum_k \varphi(\sum_j \beta_j \delta_{jk}, \mu D_k) = \sum_j \varphi(\beta_j, \sum_k \delta_{jk} \mu D_k) = \sum_j \varphi(\beta_j, \mu B_j). \qquad \square$$

17-2.10. For every step map $f = \sum_{i=1}^n \alpha_i \rho_{A_i} : X \to F$, the *integral* of f *with respect to* the vector charge μ is defined by $I_\varphi(f) = \sum_{i=1}^n \varphi(\alpha_i, \mu A_i)$. It follows from last lemma that the integral is independent of the representation of f and hence it is well defined. The map $I_\varphi : \mathcal{F}(\mathcal{S}, F) \to FE$ is called the *integration associated with* μ where $\mathcal{F}(\mathcal{S}, F)$ denote the vector space of all $\mathcal{S}$-step F-maps. For convenience, we write $\varphi(u, v) = uv$ for all $u \in F$, $v \in E$ and drop the symbol φ unless precision is required. Therefore we have

$$I\left(\sum_{i=1}^n \alpha_i \rho_{A_i}\right) = \sum_{i=1}^n \alpha_i \mu A_i.$$

Because the integrands are on the left of the measure, we also call $I(f)$ the *left* integral. The integral after lots of hard work in subsequent chapters should be denoted by $\int_\varphi (f, d\mu)$. We assume that our theory works for bilinear map $\psi : E \times F \to EF$ producing *right* integrals $\int_\psi (d\mu, f)$ without any further specification. We do not need this notational precision except only in §22-3.12 on repeated integrals.

17-2.11. **Theorem** (a) The integration $I : \mathcal{F}(\mathcal{S}, F) \to FE$ associated with μ on $\mathcal{S}$ is a linear map.
(b) If μ is real, then for every real step function f the integral $I(f)$ is real.
(c) Suppose μ is positive. Then $I(f) \geq 0$ for every positive step function $f \geq 0$. Hence I is a positive linear form on the vector lattice $\mathcal{F}$ of step functions. If $f \leq g$ in $\mathcal{F}$, then $I(f) \leq I(g)$.

17-2.12. **Theorem** Let μ be a *positive* charge on a semiring $\mathcal{S}$ over X.
(a) Let $B_1, B_2, \cdots,$ be a sequence, finite or infinite, of *disjoint* sets in $\mathcal{S}$. If $A \in \mathcal{S}$ satisfies $\bigcup_i B_i \subset A$, then $\sum_i \mu B_i \leq \mu A$.
(b) Let $A, B_1, B_2, \cdots, B_n$ be sets in $\mathcal{S}$. If $A \subset \bigcup_{i=1}^n B_i$, then $\mu A \leq \sum_{i=1}^n \mu B_i$.
(c) If $A \subset B$ in $\mathcal{S}$, then $\mu A \leq \mu B$.

Proof. (a) Since $B_1, \cdots, B_n$ are disjoint, we have $\sum_{i=1}^n \rho_{B_i} \leq \rho_A$. By $\mu \geq 0$, we obtain $\sum_{i=1}^n \mu B_i = \sum_{i=1}^n I(\rho_{B_i}) = I(\sum_{i=1}^n \rho_{B_i}) \leq I(\rho_A) = \mu A$. The case for infinite sequence follows by letting $n \to \infty$.
(b) Since $\rho_A \leq \sum_{i=1}^n \rho_{B_i}$, we have $\mu A = I(\rho_A) \leq I(\sum_{i=1}^n \rho_{B_i}) = \sum_{i=1}^n \mu B_i$. □

17-3 Finite Variation

17-3.1. In this section, we restrict ourselves to scalar charges in order to link up with positive linear forms. Vector charges of finite variation will be studied in next section.

17-3.2. Let $\mathcal{C}$ be the set of all complex charges on a semiring $\mathcal{S}$ over a set X and $\mathcal{F}$ the vector space of all step functions on X. For all $\mu, \nu \in \mathcal{C}$, $\alpha \in \mathbb{C}$ and $A \in \mathcal{S}$, let $(\mu + \nu)(A) = \mu A + \nu A$, $(\alpha\mu)(A) = \alpha\mu A$ and $\mu^-(A) = \mu(A)^-$. Now $\mathcal{C}$ becomes a conjugated vector space ordered by the cone of positive charges. The *real* and *imaginary* parts of μ are defined by Re $\mu = \frac{1}{2}(\mu + \mu^-)$ and Im $\mu = \frac{1}{2i}(\mu - \mu^-)$ respectively. Let I_μ denote the integration associated with μ. Clearly the map $\mu \to I_\mu : \mathcal{C} \to \mathcal{F}^*$ is a *natural isomorphism* preserving the order and conjugation. In fact, if I is a linear form on $\mathcal{F}$, then $\mu(A) = I(\rho_A)$ defines a charge on $\mathcal{S}$ such that I is the integration associated with μ. For next lemma, the sign of a complex number α is defined by sgn$(\alpha) = 0$ if $\alpha = 0$ and sgn$(\alpha) = \alpha/|\alpha|$ if $\alpha \neq 0$. Obviously if f is a step function, then so is sgn(f).

17-3.3. **Lemma** $\mathcal{F}$ is a breakable vector lattice.

Proof. Clearly $\mathcal{F}$ is a complex vector lattice. Let $|h| \leq f + g$ where $f, g, h \in \mathcal{F}$. Write $|h| = u + v$ where $u, v \in \mathcal{F}^+$ with $u \leq f$ and $v \leq g$. Then $p = u$ sgn(h),

$q = v\ \mathrm{sgn}(h)$ are step functions satisfying $h = p + q$, $|p| \le f$ and $|q| \le g$. Therefore $\mathcal{F}$ is breakable. □

17-3.4. As a result of last lemma, the order dual $\mathcal{F}^b$ of order bounded linear forms on $\mathcal{F}$ is also a vector lattice. We want to study the subspace of $\mathcal{C}$ corresponding to $\mathcal{F}^b \subset \mathcal{F}^* \simeq \mathcal{C}$. Let μ be a charge on $\mathcal{S}$. The *variation* $|\mu|$ of μ, or the total variation in order to be precise, is defined by

$$|\mu|(A) = \sup_{P(A)} \sum_{D\in P(A)} |\mu D|, \qquad \forall\ A \in \mathcal{S}$$

where the supremum is taken over all finite partitions $P(A)$ of A by sets in $\mathcal{S}$. We say that μ is of *finite variation* if $|\mu|(A) < \infty$ for every $A \in \mathcal{S}$.

17-3.5. **Exercise** Prove that every positive charge is of finite variation.

17-3.6. **Theorem** Let μ be a charge on $\mathcal{S}$ and I its associated integration. Then μ is of finite variation iff I is order bounded on $\mathcal{F}$. In this case, $|\mu|$ is also a charge. Furthermore, $|I|$ is the integration associated with $|\mu|$.

Proof. For every $f \in \mathcal{F}^+$, we have $|I|(f) = \sup\{|I(g)| : |g| \le f, g \in \mathcal{F}\}$. Assume that μ is of finite variation. Take any step functions $0 \ne |g| \le \rho_A$ where $A \in \mathcal{S}$. Write $g = \sum_{j=1}^n \beta_j \rho_{B_j}$ where B_j are disjoint sets in $\mathcal{S}$ and all $\beta_j \ne 0$. Then $|g| = \sum_{j=1}^n |\beta_j| \rho_{B_j} \le \rho_A$. It is easy to verify that all $|\beta_j| \le 1$ and $B_j \subset A$. Let $A \setminus \bigcup_{j=1}^n B_j = \bigcup_{j=n+1}^m B_j$ be a disjoint union of sets in $\mathcal{S}$. Thus

$$|I(g)| \le \sum_{j=1}^n |\beta_j||\mu B_j| \le \sum_{j=1}^n |\mu B_j| \le \sum_{j=1}^m |\mu B_j| \le |\mu|(A) < \infty.$$

Taking supremum over $|g| \le \rho_A$, we get $|I|(\rho_A) \le |\mu|(A)$. In general, let $|g| \le f = \sum_{j=1}^n \alpha_j \rho_{A_j}$ where A_j are disjoint sets in $\mathcal{S}$ and all $\alpha_j > 0$. Let $\gamma = \max\{\alpha_j : 1 \le j \le n\}$. Then $|(1/\gamma) g \rho_{A_j}| \le \rho_{A_j}$. Hence we obtain $|I[(1/\gamma) g \rho_{A_j}]| \le |\mu|(A_j)$, that is $|I(g\rho_{A_j})| \le \gamma |\mu|(A_j)$, or $|I(g)| \le \gamma \sum_{j=1}^n |\mu|(A_j)$. Therefore we have

$$|I|(f) = \sup\{|I(g)| : |g| \le f\} \le \gamma \sum_{j=1}^n |\mu|(A_j) < \infty.$$

Conversely assume that I is of finite variation. Let $A = \bigcup_{j=1}^n B_j$ where A, B_j are sets in $\mathcal{S}$ and $B_1, B_2, \cdots,$ are disjoint. There are numbers $|\alpha_j| = 1$ such that $\alpha_j \mu B_j = |\mu B_j|$. Let $g = \sum_{j=1}^n \alpha_j \rho_{B_j}$. Then $|g| \le \rho_A$. Hence

$$\sum_{j=1}^n |\mu(B_j)| = I(g) = |I(g)| \le |I|(\rho_A) < \infty.$$

Therefore we have $|\mu|(A) \le |I|(\rho_A)$. Combining with the result of the first part, we get $|\mu|(A) = |I|(\rho_A)$ for all $A \in \mathcal{S}$. Because $|I|$ is a linear form on $\mathcal{F}$, $|\mu|$ is additive on $\mathcal{S}$. Clearly, $|I|$ is the integration associated with $|\mu|$. □

17-3.7. **Corollary** If μ is of finite variation, then $|\mu|$ is the smallest positive charge on $\mathcal{S}$ satisfying $|\mu(A)| \leq |\mu|(A)$ for all $A \in \mathcal{S}$. The set $\mathcal{C}^b$ of all charges of finite variation forms a vector lattice. The natural isomorphism $\mathcal{C}^b \to \mathcal{F}^b$ preserves the valuation and hence all lattice operations.

17-3.8. **Theorem** Let μ, ν be *real* charges of finite variation.
(a) $\mu \vee \nu = \frac{1}{2}(\mu + \nu + |\mu - \nu|)$ and $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. Furthermore,

$$(\mu \vee \nu)(A) = \sup_{P(A)} \sum_{D \in P(A)} \mu(D) \vee \nu(D), \quad \forall\ A \in \mathcal{S}.$$

(b) $\mu_+ = \frac{1}{2}(|\mu| + \mu)$, $\mu_- = \frac{1}{2}(|\mu| - \mu)$. They are called *upper* and *lower* variations of μ respectively. Furthermore, we have $\mu_+(A) = \sup_{P(A)} \sum_{D \in P(A)} \mu(D)_+, \quad \forall\ A \in \mathcal{S}.$

(c)Both μ_+ and μ_- are unique positive charges such that $\mu = \mu_+ - \mu_-$ and $|\mu| = \mu_+ + \mu_-$.

Proof. Most results follow immediately from general treatment of vector lattice while the others can be obtained by simple calculation. The following is an example. For each A in $\mathcal{S}$, observe that

$$\begin{aligned}
\mu_+(A) &= \frac{1}{2}(\mu + |\mu|)(A) = \frac{1}{2}[\mu A + |\mu|(A)] = \frac{1}{2}\left(\mu A + \sup_{P(A)} \sum_{D \in P(A)} |\mu D|\right) \\
&= \frac{1}{2} \sup_{P(A)} \left(\mu A + \sum_{D \in P(A)} |\mu D|\right) = \frac{1}{2} \sup_{P(A)} \left(\sum_{D \in P(A)} \mu D + \sum_{D \in P(A)} |\mu D|\right) \\
&= \sup_{P(A)} \sum_{D \in P(A)} \frac{1}{2}(\mu D + |\mu D|) = \sup_{P(A)} \sum_{D \in P(A)} \mu(D)_+.
\end{aligned}$$

□

17-3.9. **Corollary** Every complex charge μ of finite variation can be decomposed into linear combination of positive ones as follows:

$$\mu = (\text{Re } \mu)_+ - (\text{Re } \mu)_- + i(\text{Im } \mu)_+ - i(\text{Im } \mu)_-.$$

17-3.10. **Exercise** Prove that the upper variation μ_+ of a *real* charge μ of finite variation is the smallest positive charge such that $\mu A \leq \mu_+(A)$ for every A in $\mathcal{S}$.

17-3.11. **Exercise** State and prove a formula for $\mu \wedge \nu$ and μ_- respectively.

17-4 Absolutely Convergent Charges

17-4.1. Complex charges of finite variation were motivated by order bounded linear forms in §17-3.6 but the expression in §17-3.4 can be carried over to vector charges as follow. Let E be a Banach space and $\mu : \mathcal{S} \to E$ a vector charge. The *variation* of μ is defined by

$$|\mu|(A) = \sup_{P(A)} \sum_{D \in P(A)} \|\mu D\| \text{ for all } A \in \mathcal{S}$$

where the supremum is taken over all finite partitions $P(A)$ of A by sets in $\mathcal{S}$. We say that μ is of *finite variation* if $|\mu|(A) < \infty$ for every $A \in \mathcal{S}$. It is an exercise to prove $|\mu|(A) = \sup_{Q(A)} \sum_{D \in Q(A)} \|\mu D\|$ for all $A \in \mathcal{S}$ where the supremum is taken over all finite families $Q(A)$ of disjoint subsets $D \in \mathcal{S}$ of A. This can be used to shorten proofs slightly. Functions of finite variation are introduced at the end of this section.

17-4.2. Let μ be an E-charge over $\mathcal{S}$. Then μ is said to be *absolutely convergent* or strongly additive if for $\bigcup_{j=1}^{\infty} B_j \subset A$ where all A, B_j are in $\mathcal{S}$ and $B_1, B_2, \cdots$ are disjoint, the series $\sum_{j=1}^{\infty} \|\mu B_j\|$ converges. We shall prove that absolute convergence is equivalent to being of finite variation.

17-4.3. **Lemma** Let $A \in \mathcal{S}$ be given. If there is a finite partition $P(A)$ of A by sets in $\mathcal{S}$ such that $|\mu|(B) < \infty$ for each B in $P(A)$, then $|\mu|(A) < \infty$.

Proof. Take any finite partition $Q(A)$ of A by sets in $\mathcal{S}$. Then the family $\{D \cap B : B \in P(A)\}$ is a finite partition of any $D \in Q(A)$ by sets in $\mathcal{S}$. Since μ is additive, $\mu D = \sum\{\mu(D \cap B) : B \in P(A)\}$. Now observe that

$$\sum_{D \in Q(A)} \|\mu D\| \le \sum_{D \in Q(A)} \sum_{B \in P(A)} \|\mu(D \cap B)\|$$
$$= \sum_{B \in P(A)} \sum_{D \in Q(A)} \|\mu(D \cap B)\| \le \sum_{B \in P(A)} |\mu|(B).$$

Taking the supremum over all $Q(A)$, we have $|\mu|(A) \le \sum_{B \in P(A)} |\mu|(B) < \infty$. □

17-4.4. **Lemma** If $A \in \mathcal{S}$ satisfies $|\mu|(A) = \infty$, then there exist a finite partition $P(A)$ of A by set in $\mathcal{S}$ and also a set B in $P(A)$ such that $|\mu|(B) = \infty$ and $\sum\{\|\mu D\| : D \in P(A) \text{ and } D \neq B\} \ge 1$.

Proof. Since $|\mu|(A) = \infty$, there is a finite partition $P(A)$ of A by sets in $\mathcal{S}$ such that $\sum\{\|\mu D\| : D \in P(A)\} \ge \|\mu A\| + 2$. By last lemma, there is B in $P(A)$ such that $|\mu|(B) = \infty$. Since μ is additive, we have

$$\mu A = \mu B + \sum\{\mu D : D \in P(A), D \neq B\}.$$

Hence,
$$\|\mu B\| \le \|\mu A\| + \sum\{\|\mu D\| : D \in P(A), D \neq B\}.$$

Now,
$$\|\mu A\| + 2 \le \|\mu B\| + \sum\{\|\mu D\| : D \in P(A), D \ne B\}$$
$$\le \|\mu A\| + 2\sum\{\|\mu D\| : D \in P(A), D \ne B\}.$$

Deleting $\|\mu A\| < \infty$ and dividing by 2, we have the required inequality. □

17-4.5. **Theorem** A vector charge μ on a semiring $\mathcal{S}$ over X is of finite variation iff it is absolutely convergent.

Proof. Assume that μ is absolutely convergent. Suppose to the contrary that there is A_0 in $\mathcal{S}$ such that $|\mu|(A_0) = \infty$. By last lemma, there is a finite partition $P(A_0)$ and a set A_1 in $P(A_0)$ such that $|\mu|(A_1) = \infty$ and also
$$\sum\{\|\mu D\| : D \in P(A_0), D \ne A_1\} \ge 1.$$

Repeating to A_1, there exist a finite partition $P(A_1)$ and a set A_2 in $P(A_1)$ such that $|\mu|(A_2) = \infty$ and also $\sum\{\|\mu D\| : D \in P(A_1), D \ne A_2\} \ge 1$. By induction, suppose that sequences of set A_n and partitions $P(A_n)$ have been constructed in similar way. Then $\bigcup_{n=0}^{\infty} \cup\{D : D \in P(A_n), D \ne A_{n+1}\}$ is a countable disjoint union of subsets of A_0. Since μ it is absolutely convergent, the series
$$\sum_{n=0}^{\infty} \sum\{\|\mu D\| : D \in P(A_n), D \ne A_{n+1}\} < \infty.$$

converges. However for each n we have $\sum\{\|\mu D\| : D \in P(A_n), D \ne A_{n+1}\} \ge 1$. This contradiction shows that $|\mu|$ is finite valued on $\mathcal{S}$. Conversely, assume that μ is of finite variation. Suppose $\bigcup_{i=1}^{\infty} B_i \subset A$ where all A, B_i are in $\mathcal{S}$ and $B_1, B_2, \cdots$ are disjoint. For each n, write $A \setminus \bigcup_{i=1}^{n} B_i = \bigcup_{j=1}^{m} D_j$ where D_j are disjoint sets in $\mathcal{S}$. Then $P(A) = \{B_1, \cdots, B_n, D_1, \cdots, D_m\}$ is a partition of A by sets in $\mathcal{S}$ and $\sum_{i=1}^{n} \|\mu B_i\| \le \sum_{E \in P(A)} \|\mu E\| \le |\mu|(A)$, that is $\sum_{i=1}^{\infty} \|\mu B_i\| \le |\mu|(A) < \infty$. Therefore $\sum_{i=1}^{\infty} \mu B_i$ is absolutely convergent. □

17-4.6. **Theorem** The variation $|\mu|$ of a vector charge $\mu : \mathcal{S} \to E$ of finite variation is the smallest positive charge satisfying $\|\mu A\| \le |\mu|A$, $\forall\, A \in \mathcal{S}$.

Proof. Let $A = \bigcup_{j=1}^{n} B_j$ be a disjoint union where $A, B_j \in \mathcal{S}$. For every $\varepsilon > 0$, choose a partition $P_j(B_j)$ of B_j by sets in $\mathcal{S}$ such that $|\mu|B_j \le \sum_{D \in P_j(B_j)} \|\mu D\| + \varepsilon/n$. Then $P(A) = \bigcup_{j=1}^{n} P_j(B_j)$ is a partition of A and we have
$$\sum_{j=1}^{n} |\mu|B_j \le \sum_{j=1}^{n} \sum_{D \in P_j(B_j)} \|\mu B_j\| + \varepsilon \le \sum_{D \in P(A)} \|\mu D\| + \varepsilon \le |\mu|A + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we get $\sum_{j=1}^{n} |\mu|B_j \le |\mu|A$. On the other hand, suppose that $P(A)$ is a partition of A by sets in $\mathcal{S}$. Then $P_j(B_j) = \{D \cap B_j : D \in P(A)\}$ is a partition of B_j by sets in $\mathcal{S}$. Observe that

$$\sum_{D\in P(A)} \|\mu D\| = \sum_{D\in P(A)} \left\| \sum_{j=1}^{n} \mu(D\cap B_j) \right\| \le \sum_{D\in P(A)} \sum_{j=1}^{n} \|\mu(D\cap B_j)\|$$

$$= \sum_{j=1}^{n} \sum_{D\in P(A)} \|\mu(D\cap B_j)\| = \sum_{j=1}^{n} \sum_{Q\in P_j(B_j)} \|\mu Q\| \le \sum_{j=1}^{n} |\mu| B_j.$$

Taking the supremum over $P(A)$, we obtain $|\mu|A = \sum_{j=1}^{n} |\mu|B_j$. Hence $|\mu|$ is a positive charge on $\mathcal{S}$. It is an exercise to prove $\|\mu A\| \le |\mu|A$, $\forall\, A \in \mathcal{S}$. □

17-4.7. **Exercise** Prove that $|\mu + \nu| \le |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha|\ |\mu|$ on $\mathcal{S}$.

17-4.8. Let E, F, FE be Banach spaces with a bilinear map $F \times E \to FE$ which is *admissible*, that is $\|uv\| \le \|u\|\ \|v\|$ for all $(u,v) \in F\times E$. Clearly every admissible bilinear map is continuous. For every map $f : X \to F$, its *modulus*, or *absolute value* is the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) = \|f(x)\|$ for every $x \in X$.

17-4.9. **Exercise** Let E, G be Banach spaces. Show that the evaluation map $\varphi : L(E,G) \times E \to G$ defined by $\varphi(u,v) = u(v)$ is an admissible bilinear map. Specify the case when $G = \mathbb{K}$.

17-4.10. **Theorem** $\|I_\mu(f)\| \le I_{|\mu|}(|f|)$ for every step map $f : X \to F$ and every E-charge μ of finite variation.

Proof. For $f = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ where $\alpha_j \in F$ and $A_j \in \mathcal{S}$, the result follows from

$$\|I_\mu(f)\| = \left\| \sum_{j=1}^{n} \alpha_j \mu A_j \right\| \le \sum_{j=1}^{n} \|\alpha_j \mu A_j\|$$
$$= \sum_{j=1}^{n} |\alpha_j|\ \|\mu A_j\| \le \sum_{j=1}^{n} |\alpha_j|\ |\mu|(A_j) \le I_{|\mu|}(|f|).$$ □

17-4.11. **Observation** Let μ be a complex charge. If there is a positive charge ν such that $|I_\mu(f)| \le I_\nu(|f|)$ for every step function f, then μ is of finite variation. This explains why our measures in this book are of finite variation.

Proof. Let $P(A)$ be a partition of $A \in \mathcal{S}$ by sets in $\mathcal{S}$. For each $D \in P(A)$, choose $|\alpha_D| = 1$ with $\alpha_D \mu D = |\mu D|$. For the step function $f = \sum_D \alpha_D \rho_D$, we have $\sum_D |\mu D| = \sum_D \alpha_D \mu D = |I_\mu(f)| \le I_\nu(|f|) = \sum_D |\alpha_D| \nu D = \sum \nu D = \nu A$. Therefore $|\mu|A = \sup_{P(A)} \sum_D |\mu D| \le \nu A < \infty$. □

17-5 Countable Additivity on Rings

17-5.1. A family $\mathcal{R}$ of subsets of a set X is called a *ring over* X, or a ring of *subsets* of X in order to be precise, if

(a) the empty set is in $\mathcal{R}$;
(b) for all A, B in $\mathcal{R}$, both $A \cup B$ and $A \setminus B$ are in $\mathcal{R}$.
Since $A \cap B = A \cup B \setminus [(A \setminus B) \cup (B \setminus A)]$, every ring of subsets is a semiring. Consequently previous results on charges are applicable to rings.

17-5.2. **Theorem** Let $\mathcal{S}$ be a semiring over X. Then the family $\mathcal{R}$ of all finite unions of disjoint sets in $\mathcal{S}$ forms the smallest ring containing $\mathcal{S}$. In this case, $\mathcal{R}$ is called the *ring generated by* $\mathcal{S}$.

Proof. It is obvious that $\mathcal{R}$ contains the empty set. Let A_1, A_2 be in $\mathcal{R}$. It follows from the Semiring Formula, we may write $\rho_{A_i} = \sum_{j=1}^{m} \alpha_{ij}\rho_{B_j}$ where $B_1, B_2, \cdots, B_m$ are disjoint sets in $\mathcal{S}$ and α_{ij} are either zero or one. Since the coefficients of $\rho_{A_1 \cup A_2} = \sum_j (\alpha_{1j} \vee \alpha_{2j})\rho_{B_j}$ and $\rho_{A_1 \setminus A_2} = \sum_j [\alpha_{1j} - (\alpha_{1j} \wedge \alpha_{2j})]\rho_{B_j}$ are either zero or one, $\mathcal{R}$ is a ring. Clearly it is the smallest among all rings containing $\mathcal{S}$. □

17-5.3. Let E, F, FE be vector spaces with a bilinear map $F \times E \to FE$. Let $\mathcal{R}$ be the ring generated by a semiring $\mathcal{S}$ over a set X and $\mu : \mathcal{S} \to E$ a vector charge. Let $\mathcal{F}(\mathcal{S}, F)$ denote the vector spaces of $\mathcal{S}$-step F-maps and $\mathcal{F}(\mathcal{R}, F)$ be similarly defined.

17-5.4. **Algebraic Extension Theorem** Both $\mathcal{S}, \mathcal{R}$ have the same step maps. A charge μ on $\mathcal{S}$ induces uniquely a linear map I on $\mathcal{F}(\mathcal{S}, F) = \mathcal{F}(\mathcal{R}, F)$ which gives a unique charge ν on $\mathcal{R}$. Clearly ν is the unique extension of μ from $\mathcal{S}$ to $\mathcal{R}$. Explicitly, for every $A = \bigcup_{j=1}^{n} B_j$ in $\mathcal{R}$ where $B_1, B_2, \cdots, B_n$ are disjoint sets in $\mathcal{S}$, we have $\nu A = \sum_{j=1}^{n} \mu B_j$. We identify μ, I, ν and write $\mu(f) = I(f)$, $\mu A = \nu A$ for all $f \in \mathcal{F}(\mathcal{R}, F)$ and $A \in \mathcal{R}$.

17-5.5. **Exercise** Show that for all $A \subset B$ in $\mathcal{R}$, we have $\mu(B \setminus A) = \mu B - \mu A$. Prove that for all A, B in $\mathcal{R}$, we have $\mu(A \cup B) + \mu(A \cap B) = \mu A + \mu B$.

17-5.6. Let $\mathcal{S}$ be a semiring over a set X, E a Banach space and $\mu : \mathcal{S} \to E$ a vector charge. Then μ is said to be *countably additive* if for every countable disjoint union $A = \bigcup_{j=1}^{\infty} B_j$ where all A, B_j are in $\mathcal{S}$, we have $\mu A = \sum_{j=1}^{\infty} \mu B_j$. Let $\mathcal{R}$ be a ring. Write $B_n \uparrow A$ in $\mathcal{R}$ if all A, B_n are in $\mathcal{R}$; $B_n \subset B_{n+1}$ and $A = \bigcup_{n=1}^{\infty} B_n$. Similarly $B_n \downarrow A$ in $\mathcal{R}$ is defined.

17-5.7. **Theorem** Let μ be a vector charge on a *ring* $\mathcal{R}$ over X. Then following statements are equivalent.
(a) μ is countably additive on $\mathcal{R}$.
(b) If $B_n \uparrow A$ in $\mathcal{R}$ then $\mu B_n \to \mu A$.

(c) If $B_n \downarrow A$ in $\mathcal{R}$ then $\mu B_n \to \mu A$.

(d) If $B_n \downarrow \emptyset$ in $\mathcal{R}$ then $\mu B_n \to 0$.

Proof. $(a \Rightarrow b)$ Let $B_n \uparrow A$ in $\mathcal{R}$. Define $B_0 = \emptyset$. Then $A = \bigcup_{j=1}^{\infty}(B_j \setminus B_{j-1})$ and $B_n = \bigcup_{j=1}^{n}(B_j \setminus B_{j-1})$ are disjoint unions of sets in $\mathcal{R}$. By (a), we have

$$\mu A = \sum_{j=1}^{\infty} \mu(B_j \setminus B_{j-1}) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j \setminus B_{j-1}) = \lim_{n\to\infty} \mu B_n.$$

$(b \Rightarrow c)$ Let $B_n \downarrow A$ in $\mathcal{R}$. Then $(B_1 \setminus B_n) \uparrow (B_1 \setminus A)$ in $\mathcal{R}$. By (b), we have $\mu(B_1 \setminus B_n) \to \mu(B_1 \setminus A)$ as $n \to \infty$. Hence we get $\mu B_1 - \mu B_n \to \mu B_1 - \mu A$, that is $\mu B_n \to \mu A$.

$(c \Rightarrow d)$ It is a special case when $A = \emptyset$.

$(d \Rightarrow a)$ Let $A = \bigcup_{j=1}^{\infty} B_j$ be a disjoint union where A, B_j are in $\mathcal{R}$. Define $D_n = A \setminus \bigcup_{j=1}^{n} B_j$. Then $D_n \downarrow \emptyset$ in $\mathcal{R}$. By (d), we get $\mu D_n \to 0$. Since $A = (\bigcup_{j=1}^{n} B_j) \cup D_n$ is a finite disjoint union of sets in $\mathcal{R}$, we have $\mu A = \sum_{j=1}^{n} \mu B_j + \mu D_n$. Letting $n \to \infty$ we obtain $\mu A = \sum_{j=1}^{\infty} \mu B_j$. The proof is complete. □

17-5.8. **Theorem** If μ is a *positive* countably additive charge on a semiring $\mathcal{S}$, then it is *countably subadditive*. More precisely, let $A, B_1, B_2, \cdots$ be sets in $\mathcal{S}$. If $A \subset \bigcup_{i=1}^{\infty} B_i$, then $\mu A \le \sum_{i=1}^{\infty} \mu B_i$.

Proof. Observe $A = \bigcup_{i=1}^{\infty}(A \cap B_i)$ and $A \cap B_i \in \mathcal{S}$. Let $D_{11} = A \cap B_1$ and for each $i > 1$ let $(A \cap B_i) \setminus \bigcup_{k=1}^{i-1}(A \cap B_k) = \cup D_{ij}$ where $\{D_{i1}, D_{i2}, \cdots\}$ is a finite number of disjoint sets in $\mathcal{S}$. Then $\bigcup_j D_{ij} \subset B_i$ and hence $\sum_j \mu D_{ij} \le \mu B_i$ since μ is positive. Now we have the countable union $A = \bigcup_{ij} D_{ij}$ of disjoint sets in $\mathcal{S}$. By countable additivity, we get $\mu A = \sum_{i=1}^{\infty} \sum_j \mu D_{ij} \le \sum_{i=1}^{\infty} \mu B_i$. □

17-5.9. Write $f_n \uparrow g$ if all f_n, g are real functions, $f_n(x) \le f_{n+1}(x)$ and $\lim f_n(x) = g(x)$ as $n \to \infty$ at every point $x \in X$. Likewise, $f_n \downarrow g$ is defined. The following characterizes countable additivity in terms of *order continuity*.

17-5.10. **Theorem** Let μ be a *positive* charge on a ring $\mathcal{R}$ over X and I the associated integral. Then μ is countably additive iff for step functions $f_n \downarrow 0$, we have $I(f_n) \downarrow 0$.

Proof. Since f_1 is a step function, write $0 \le f_1 = \sum_{j=1}^{m} \alpha_j \rho_{A_j}$ where A_j are disjoint sets in $\mathcal{R}$ and $\alpha_j \ge 0$. Define $\beta = \max\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. For any $\varepsilon > 0$, let $D_{nj} = \{x \in A_j : f_n(x) \ge \varepsilon\}$. Since every f_n is a step function, D_{nj} is a finite union of sets in the ring $\mathcal{R}$ and hence it is in $\mathcal{R}$. Since $f_n \downarrow 0$ we have $D_{nj} \downarrow \emptyset$, i.e. $\mu D_{nj} \to 0$ as $n \to \infty$. Observe that

$$0 \leq f_n\rho_{A_j} = f_n\rho_{A_j\setminus D_{nj}} + f_n\rho_{D_{nj}} \leq \varepsilon\rho_{A_j\setminus D_{nj}} + \beta\rho_{D_{nj}} \leq \varepsilon\rho_{A_j} + \beta\rho_{D_{nj}}.$$

Integrating on both sides, we have $0 \leq I(f_n\rho_{A_j}) \leq \varepsilon\mu A_j + \beta\mu D_{nj}$. Summing over $1 \leq j \leq m$, we have

$$0 \leq I(f_n) = I\left(f_n \sum_{j=1}^m \rho_{A_j}\right) = \sum_{j=1}^m I\left(f_n\rho_{A_j}\right) \leq \varepsilon \sum_{j=1}^m \mu A_j + \beta \sum_{j=1}^m \mu D_{nj}$$

which can be made arbitrarily small. Therefore $I(f_n) \to 0$ as required. The converse is obvious. □

17-5.11. **Exercise** Let μ be a *positive* countably additive charge on a ring $\mathcal{R}$ and f_n, f be step functions. Prove that if $f_n \uparrow f$, then $I(f_n) \uparrow I(f)$.

17-5.12. **Exercise** Let E be the Banach space of bounded functions on $\mathbb{R}$ under the sup-norm. Let $\mu A = \rho_A$ for every semi-interval A. Show that μ is a vector charge but it is neither countably additive nor of finite variation.

17-6 Vector Measures

17-6.1. Let $\mathcal{S}$ be a semiring over a set X and E a Banach space. A vector charge $\mu : \mathcal{S} \to E$ is a *measure* if it is countably additive and of finite variation. Since every positive charge is of finite variation, every positive countably additive charge is a measure. We also say measures and charges *on the space* X whenever $\mathcal{S}$ is well understood.

17-6.2. **Example** Let $h(x) = xe^{ix}$ for each $x \in \mathbb{R}$. Then the counting measure weighted by h is a complex measure on the semiring of singletons of $\mathbb{R}$.

17-6.3. **Example** The point measures are positive measures.

17-6.4. **Theorem** If $\mu : \mathcal{S} \to E$ is a vector measure, then so is the unique extension ν of μ to a charge on the ring $\mathcal{R}$ generated by $\mathcal{S}$.

Proof. Through the associated integration, ν is additive on $\mathcal{R}$. To prove the countable additivity, let $A = \bigcup_{i=1}^\infty B_i$ be a disjoint union where $A, B_1, B_2, \cdots$ are sets in $\mathcal{R}$. Since $\mathcal{S}$ generates $\mathcal{R}$, write $A = \bigcup_k A_k$ and $B_i = \bigcup_j B_{ij}$ where $\{A_1, A_2, \cdots\}$ and $\{B_{i1}, B_{i2}, \cdots\}$ are finite sequences of disjoint sets in $\mathcal{S}$. By definition of ν we have $\nu A = \sum_k \mu A_k$ and $\nu B_i = \sum_j \mu B_{ij}$. Observe that both

$$A_k = A_k \cap A = A_k \cap (\textstyle\bigcup_i B_i) = \bigcup_{ij}(A_k \cap B_{ij})$$

and $B_{ij} = A \cap B_{ij} = \bigcup_k(A_k \cap B_{ij})$ are countable disjoint unions of sets $A_k \cap B_{ij}$ in $\mathcal{S}$. Since μ is countably additive, we have $\mu A_k = \sum_{ij} \mu(A_k \cap B_{ij})$ and

$\mu B_{ij} = \sum_k \mu(A_k \cap B_{ij})$. The unconditional convergence allows us to change the order of summation in the following calculation:

$$\nu A = \sum_k \mu A_k = \sum_k \sum_{ij} \mu(A_k \cap B_{ij})$$
$$= \sum_i \sum_j \sum_k \mu(A_k \cap B_{ij}) = \sum_i \sum_j \mu B_{ij} = \sum_i \nu B_i.$$

Similar argument shows that ν is of finite variation on $\mathcal{R}$. The details are left as an exercise. □

17-6.5. **Theorem** The variation $|\mu|$ of a vector measure $\mu : \mathcal{S} \to E$ is a positive measure on $\mathcal{S}$.

Proof. Let $A = \bigcup_{i=1}^{\infty} B_i$ be a disjoint union where A, B_j belong to $\mathcal{S}$. Since $|\mu|$ is positive additive, we have $|\mu|(A) \geq \sum_{i=1}^{\infty} |\mu|(B_i)$. To prove the reversed inequality, let $P(A) = \{D_1, D_2, \cdots, D_m\}$ be a finite partition of A by sets D_k in $\mathcal{S}$. Applying $|\mu|$ to the disjoint union $B_i = \sum_{k=1}^{m}(B_i \cap D_k)$, we have $|\mu|(B_i) = \sum_{k=1}^{m} |\mu|(B_i \cap D_k)$. For the countable partition $D_k = \sum_{i=1}^{\infty}(B_i \cap D_k)$, the countable additivity of μ gives $\mu D_k = \sum_{i=1}^{\infty} \mu(B_i \cap D_k)$, i.e.

$$|\mu D_k| \leq \sum_{i=1}^{\infty} |\mu(B_i \cap D_k)| \leq \sum_{i=1}^{\infty} |\mu|(B_i \cap D_k).$$

Summing over k, we obtain

$$\sum_{k=1}^{m} |\mu D_k| \leq \sum_{k=1}^{m} \sum_{i=1}^{\infty} |\mu|(B_i \cap D_k)$$
$$= \sum_{i=1}^{\infty} \sum_{k=1}^{m} |\mu|(B_i \cap D_k) = \sum_{i=1}^{\infty} |\mu|(B_i).$$

Taking the supremum over all partitions $P(A)$ we have $|\mu|(A) \leq \sum_{i=1}^{\infty} |\mu|(B_i)$. Hence $|\mu|$ is countably additive and therefore it is a measure. □

17-6.6. **Corollary** The set of all complex measures forms a vector sublattice induced by the vector lattice of all complex charges with finite variation.

17-6.7. **Exercise** Show that the family $\mathcal{S}$ of all finite subsets of $\mathbb{R}$ together with their complements forms a semiring over $\mathbb{R}$. For each finite subset A of $\mathbb{R}$, let μA be the total number of elements in A and $\mu(\mathbb{R} \setminus A) = -\mu A$. Prove that μ is countably additive on $\mathcal{S}$ but *not* of finite variation.

17-6.8. **Exercise** Let μ be a charge of finite variation on a *ring* $\mathcal{R}$. Prove that the following statements are equivalent.

(a) μ is a measure on $\mathcal{R}$.

(b) Both Re μ and Im μ are measures.

(c) All (Re $\mu)_\pm$ and (Im $\mu)_\pm$ are measures.

17-7 Lebesgue-Stieltjes Measures

17-7.1. For every map $g : \mathbb{R} \to E$ let $dg(a,b] = g(b) - g(a)$ for all $a < b$. It is easy to verify that dg is additive. It is called the charge *induced by* g. Conversely, for every vector charge ν on the semi-intervals into E, the map $M\nu : \mathbb{R} \to E$ given by $M\nu(x) = \nu(0,x]$ for all $x \geq 0$ and $M\nu(x) = -\nu(x,0]$ for $x < 0$ is called the map *induced by* ν. It is simple to verify that $dM\nu = \nu$ and $Mdg(x) = g(x) - g(0)$ for all $x \in \mathbb{R}$.

17-7.2. **Theorem** Let $\mathbb{M}$ denote the vector space of all maps $g : \mathbb{R} \to E$ with $g(0) = 0$ and $\mathcal{C}$ the vector space of all charges on the semi-intervals of $\mathbb{R}$ into E. Then $g \to dg : \mathbb{M} \to \mathcal{C}$ is an algebraic isomorphism. Furthermore for the scalar case $E = \mathbb{K}$, $g \to dg$ preserves the complex conjugates; dg is real iff g is real-valued; and also dg is positive iff g is increasing.

17-7.3. Let $g : \mathbb{R} \to E$ be a map. Then the variation $|dg|$ of the charge dg is given by $|dg|(a,b] = \sup \sum_{j=1}^{n} \|g(x_j) - g(x_{j-1})\|$ for all $a < b$ where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$. The map g is *of finite variation* if $|dg|(a,b] < \infty$ for all $a < b$. In this case, $|dg|$ is also a charge on the semi-intervals. The map induced by $|dg|$ is called the *total variation* of g, denoted by Vg. Therefore we have the following explicit formula
(a) $Vg(0) = 0$;
(b) for $x > 0$, $Vg(x) = \sup \sum_{j=1}^{n} \|g(x_j) - g(x_{j-1})\|$ where the supremum is taken over all partitions $0 = x_0 < x_1 < \cdots < x_n = x$;
(c) for $x < 0$, $Vg(x) = -\sup \sum_{j=1}^{n} \|g(x_j) - g(x_{j-1})\|$ where the supremum is taken over all partitions $x = x_0 < x_1 < \cdots < x_n = 0$.

17-7.4. **Theorem** Let g be of finite variation. Then the variation of the charge induced by g is the charge of the total variation of g, that is $|dg| = dVg$.

Proof. For $0 \leq a < b$, $|dg|(a,b] = |dg|(0,b] - |dg|(0,a] = Vg(b) - Vg(a)$. For $a < 0 \leq b$, $|dg|(a,b] = |dg|(a,0] + |dg|(0,b] = Vg(b) - Vg(a)$. Finally for $a < b < 0$, $|dg|(a,b] = |dg|(a,0] - |dg|(b,0] = Vg(b) - Vg(a)$. Therefore $|dg|(a,b] = Vg(b) - Vg(a) = dVg(a,b]$ for all $a < b$. □

17-7.5. **Exercise** Prove that linear combinations of maps of finite variation are of finite variation.

17-7.6. **Exercise** Prove that a complex function on $\mathbb{R}$ is of finite variation iff both its real and imaginary parts are of finite variation.

17-7.7. **Theorem** Let $g : \mathbb{R} \to E$ be a map of finite variation. Then g is right continuous, that is $\lim g(x) = g(a)$ as $x \downarrow a$ for every $a \in \mathbb{R}$, iff its total variation Vg is right continuous.

Proof. Let $a < x$ be given. If Vg is right continuous at a, then $\|g(x)-g(a)\| \le |dg|(a,x] = Vg(x) - Vg(a) \to 0$ as $x \downarrow a$ implies the right continuity of g at a. Conversely, suppose that g is right continuous at a. For every $\varepsilon > 0$, there is $\delta > 0$ such that for all $x - a \le \delta$, we have $\|g(x) - g(a)\| \le \varepsilon$. Pick any $b > a + \delta$. There is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $|dg|(a,b] \le \sum_{j=1}^{n} \|f(x_j) - f(x_{j-1})\| + \varepsilon$. Insert x so that $x_0 < x < x_1$ and $x - x_0 \le \delta$. Then we have

$$
\begin{aligned}
&Vg(b) - Vg(a) = |dg|(a,b] \le \textstyle\sum_{j=1}^{n} \|f(x_j) - f(x_{j-1})\| + \varepsilon \\
&\le \|f(x) - f(x_0)\| + \|f(x_1) - f(x)\| + \textstyle\sum_{j=2}^{n} \|f(x_j) - f(x_{j-1})\| + \varepsilon \\
&\le \varepsilon + |dg|(x,b] + \varepsilon = Vg(b) - Vg(x) + 2\varepsilon,
\end{aligned}
$$

that is $0 \le Vg(x) - Vg(a) \le 2\varepsilon$ for all $0 < x - a \le \delta$. Therefore Vg is right continuous at a. □

17-7.8. **Theorem** Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function and ν the charge of g given by $\nu(a,b] = g(b) - g(a)$ for all $a < b$. Then ν is a positive measure on the semi-intervals if g is right continuous on $\mathbb{R}$. In particular, the *Lebesgue measure* given by $g(x) = x$ is a positive measure.

Proof. Clearly ν is a positive charge on the semiring of semi-intervals. Let $A = \bigcup_{j=1}^{\infty} B_j$ be a disjoint union where A, B_j are semi-intervals. Since ν is a positive charge, we have $\sum_{j=1}^{\infty} \nu B_j \le \nu A$. To show the reverse inequality, we may assume that all B_j are non-empty. Let $A = (a,b]$ and $B_j = (a_j, b_j]$. Suppose $\varepsilon > 0$ is given. Because g is right continuous, there is $0 < \delta < b - a$ such that $g(a+\delta) - g(a) \le \varepsilon$. Similarly for each j, there is $\delta_j > 0$ satisfying $g(b_j + \delta_j) - g(b_j) \le \varepsilon/2^j$. Then $\{(a_j, b_j + \delta_j) : j \ge 1\}$ is an open cover of the compact set $[a+\delta, b]$. There is a finite subcover, i.e. for some n, we get $[a+\delta, b] \subset \bigcup_{j=1}^{n}(a_j, b_j + \delta_j)$, or $(a+\delta, b] \subset \bigcup_{j=1}^{n}(a_j, b_j + \delta_j]$. Since ν is a positive charge, we have $\nu(a+\delta, b] \le \sum_{j=1}^{n} \nu(a_j, b_j + \delta_j]$, that is

$$g(b) - g(a+\delta) \le \sum_{j=1}^{n} [g(b_j + \delta_j) - g(a_j)],$$

or,

$$g(b) - g(a) \le 2\varepsilon + \sum_{j=1}^{n} [g(b_j) - g(a_j)],$$

equivalently,

$$\nu A \le 2\varepsilon + \sum_{j=1}^{n} \nu B_j \le 2\varepsilon + \sum_{j=1}^{\infty} \nu B_j.$$

Because $\varepsilon > 0$ is arbitrary, we have $\nu A \le \sum_{j=1}^{\infty} \nu B_j$. Therefore ν is countably additive and consequently it is a measure. □

17-7.9. **Theorem** Let $g : \mathbb{R} \to E$ be a map with finite variation and $\nu = dg$ the charge given by $\nu(a,b] = g(b) - g(a)$ for all $a < b$. Then ν is a vector measure on the semi-intervals iff g is right continuous on $\mathbb{R}$. In this case, ν is called the *Stieltjes' measures* induced by g.

Proof. ($\Rightarrow$) Let $x_n \downarrow a$. Then $(a, x_n] \downarrow \emptyset$ and hence $\nu(a, x_n] \to 0$, that is $g(x_n) - g(a) \to 0$, or $g(x_n) \to g(a)$. Since the sequence $\{x_n\}$ is arbitrary, g is right continuous.
($\Leftarrow$) Since g is right continuous, so is its total variation Vg. Thus its induced charge $|\nu|$ is a measure on the semiring of semi-intervals. It is also a measure on the ring $\mathcal{R}$ generated by the semi-intervals. Let $A_n \to \emptyset$ in $\mathcal{R}$ as $n \to \infty$. Then $\|\nu A_n\| \le |\nu|(A_n) \to 0$. Therefore ν is countably additive. □

17-7.10. **Example** Let $h : \mathbb{R} \to \mathbb{R}$ be a real function and $\mu = dh$ the charge induced by h. The *upper variation* of h *on* $(a, b]$ is defined as

$$\mu_+(a,b] = \sup \sum_{j=1}^{n} [h(x_j) - h(x_{j-1})]_+$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$. Similarly, the lower variation is defined. Suppose h is of finite variation. Let V_+h, V_-h be functions induced by the charges μ_+ and μ_- respectively. Since both $\mu_\pm$ are positive, $V_\pm h$ are increasing functions. From $\mu(a,b] = \mu_+(a,b] - \mu_-(a,b]$, we have $h(x) = V_+h(x) - V_-h(x) + h(0)$ by setting $a = 0$ and $b = x > 0$. The same formula holds for $a = x < 0$ and $b = 0$. Therefore every complex function g of finite variation can be decomposed into increasing functions: $g = V_+(\text{Re } g) - V_-(\text{Re } g) + iV_+(\text{Im } g) - iV_-(\text{Im } g) + g(0)$.

17-7.11. **Exercise** Prove that every continuously differentiable map on $\mathbb{R}$ into E is of finite variation.

17-7.12. **Exercise** Show that the function $g = \rho_{(-\infty,0)}$ is of finite variation but its induced charge is not countably additive.

17-7.13. **Exercise** Show that the function g given by $g(x) = x \sin \frac{1}{x}$ for $x \ne 0$ and $g(0) = 0$ is a bounded continuous function on $\mathbb{R}$ but is not of finite variation. Find a sequence of semi-intervals A_n such that $\sum_{n=1}^{\infty} |dgA_n| = \infty$ where dg is the charge induced by g.

17-7.14. **Exercise** Let $g : \mathbb{R} \to \mathbb{K}$ be a function of finite variation. Prove that the following statements are equivalent.

(a) g is right continuous.

(b) The total variation Vg is right continuous.

(c) Both Re(g) and Im(g) are right continuous.

(d) All upper and lower variations $V_{\pm}(\text{Re } g)$ and $V_{\pm}(\text{Im } g)$ are right continuous.

17-7.15. **Exercise** Show that maps of finite variation are bounded on bounded sets.

17-7.16. **Exercise** Prove that if $f : \mathbb{R} \to \mathbb{K}$ and $g : \mathbb{R} \to E$ are of finite variation, then so is the product fg.

17-99. **References and Further Readings** : Zaanen-59, Garnir-72, Munster, Apostol, Hawkins, Fell, Chuaqui, Huggins, DeLucia, Haluska, Olejeck, Kelly, Forelli, Brooks, Argabright, Stewart, Holland, Zelichenok, Byers and Jurzak.

Chapter 18

Extensions of Positive Measures

18-1 Uniqueness of Extension

18-1.1. Let $f : [a, b] \to [\alpha, \beta]$ be a function. For Riemann integral, we chop up the domain $[a, b]$ into subintervals by partitions $a = x_0 < x_1 < \cdots < x_n = b$; select points $c_j \in [x_{j-1}, x_j]$; construct step function $\sum_{j=1}^n f(c_j)\rho_{(x_{j-1},x_j]}$ and define the integral of f as the limit of the sum $\sum_{j=1}^n f(c_j)(x_j - x_{j-1})$ interpreted as the areas of histograms. It is simple and it can cope with most of the engineering problem but it cannot handle the convergence $f_n \to f$ of functions required in harmonic and Fourier analysis. For Lebesgue integral, we chop up the range $[\alpha, \beta]$ into subintervals by partitions $\alpha = y_0 < y_1 < \cdots < y_n = \beta$; select points $\gamma_j \in [y_{j-1}, y_j]$; construct simple function $s = \sum_{j=1}^n \gamma_j \rho_{A_j}$ and define the integral of as the limit of the sum $\sum_{j=1}^n \gamma_j \lambda A_j$ but the Lebesgue measures λA_j of the sets $A_j = f^{-1}(y_{j-1}, y_j]$ are not yet available. It is more complicate but Monotonic Convergence Theorem, Fatou's Lemma and Dominated Convergence Theorem developed in later chapters are some milestones providing service better than Riemann integrals. This motivates the need to extend measures from semi-intervals to larger classes of sets. In this section, we shall prove the uniqueness of such extension.

18-1.2. A ring over a set X is called a *δ-ring* if it is closed under formation of countable intersections. A *δ-space* or a *measurable space modeled on δ-ring* consists of a δ-ring $\mathbb{D}$ over a set X. Sets in $\mathbb{D}$ are called *decent sets*.

18-1.3. **Exercise** Let A, B_n be decent sets in a δ-space $(X, \mathbb{D})$. Prove that if all $B_n \subset A$, then $\bigcup_{n=1}^\infty B_n$ is also a decent set.

18-1.4. **Theorem** Every countably additive complex charge μ on a δ-ring $\mathbb{D}$ is of finite variation and hence is a measure. This may explain why finite variation does not play an important role in the scalar measure theory.

Proof. Let A, B_j be decent sets such that $\bigcup_{j=1}^\infty B_j \subset A$ and that $B_1, B_2, \cdots$ are disjoint. Then $B = \bigcup_{j=1}^\infty B_j$ is a decent set. By countable additivity, the convergence $\mu B = \bigcup_{j=1}^{\infty} \mu B_j$ is unconditional in $\mathbb{K}$, that is invariant under permutation of indices. Hence the series $\sum_{j=1}^{\infty} |\mu B_j|$ converges. Therefore μ is of finite variation. □

18-1.5. Note that the above result also holds for vector measures into Banach spaces in which unconditional convergence of series implies absolute convergence. Characterization of this class of Banach spaces in terms of subspace c_0 is beyond our scope.

18-1.6. **Lemma** Let $\mathcal{K}$ be a family of subsets of a set X.
(a) The smallest δ-ring $\mathbb{D}$ containing $\mathcal{K}$ is the intersection of all δ-rings containing $\mathcal{K}$. It is called the *δ-ring generated by $\mathcal{K}$.*
(b) Every set in $\mathbb{D}$ is contained in a finite union of sets in $\mathcal{K}$.

Proof. (b) The family $\mathbb{F}$ of all subsets of finite unions of sets in $\mathcal{K}$ forms a δ-ring. Since $\mathcal{K} \subset \mathbb{F}$, we have $\mathbb{D} \subset \mathbb{F}$. □

18-1.7. **Exercise** Decent sets of the real line $\mathbb{R}$ are sets in the δ-ring generated by the semiring of semi-intervals. Show that every decent set is bounded. Prove that every bounded interval is a decent set. Conclude that the family of finite unions of semi-intervals forms a ring but not a δ-ring.

18-1.8. For the rest of this section, assume that $\mathcal{S}$ is a semiring over X, $\mathcal{R}$ the ring generated by $\mathcal{S}$ and $\mathbb{D}$ the δ-rings generated by $\mathcal{S}$. A *monotone family* $\mathbb{F}$ of $\mathbb{D}$ is a subfamily of $\mathbb{D}$ such that
(a) if $A_n \in \mathbb{F}$ and $A_n \downarrow A$ then $A \in \mathbb{F}$,
(b) if $A_n \in \mathbb{F}$ and $A_n \uparrow A \subset D \in \mathbb{D}$ then $A \in \mathbb{F}$.

18-1.9. **Lemma** For every $M \in \mathbb{D}$ and for every monotone family $\mathbb{F}$ of $\mathbb{D}$, the family $\mathbb{F}(M) = \{N \in \mathbb{D} : \text{all } M \cup N, M \setminus N, N \setminus M \in \mathbb{F}\}$ is also a monotone family of $\mathbb{D}$.

Proof. We shall verify the second condition of monotone family only. Assume $A_n \in \mathbb{F}(M)$ and $A_n \uparrow A \subset D \in \mathbb{D}$. Then $A_n \in \mathbb{D}$ and $M \cup A_n \in \mathbb{F}$ by definition of $\mathbb{F}(M)$. Because the δ-ring $\mathbb{D}$ is monotone, we have $A \in \mathbb{D}$. Clearly,

$$M \cup A_n \uparrow M \cup A \subset M \cup D \in \mathbb{D}.$$

Since $\mathbb{F}$ is monotone we have $M \cup A \in \mathbb{F}$. Similarly we obtain $M \setminus A, A \setminus M \in \mathbb{F}$. Consequently $A \in \mathbb{F}(M)$. It is an exercise to verify the first condition of monotone family and to complete the proof. □

18-1.10. **Lemma** Let $\mathbb{F}$ be a monotone family of $\mathbb{D}$. If $\mathbb{F}$ contains $\mathcal{R}$, then we have $\mathbb{F} = \mathbb{D}$.

Proof. Clearly the intersection of all monotone families of $\mathbb{D}$ containing $\mathcal{R}$ is a monotone family of $\mathbb{D}$ satisfying $\mathcal{R} \subset \mathbb{E} \subset \mathbb{F} \subset \mathbb{D}$. Because $\mathcal{R}$ is a ring, for all $A, B \in \mathcal{R}$ we have $B \in \mathbb{E}(A)$ which was defined in last lemma. Hence $\mathcal{R}$ is contained in the monotone family $\mathbb{E}(A)$. Since $\mathbb{E}$ is the smallest, we obtain $\mathbb{E} \subset \mathbb{E}(A)$. For every $M \in \mathbb{E}$ we have $M \in \mathbb{E}(A)$. The symmetry in the definition of $\mathbb{E}(A)$ gives $A \in \mathbb{E}(M)$. Since $\mathbb{E}(M)$ is another monotone family containing $\mathcal{R}$, we have $\mathbb{E} \subset \mathbb{E}(M)$. We conclude that for arbitrary $M, N \in \mathbb{E}$, the sets $M \cup N$, $M \setminus N$, $N \setminus M$ are all in $\mathbb{E}$. In other words, $\mathbb{E}$ is a ring. Since $\mathbb{E}$ is monotone, it is a δ-ring containing $\mathcal{R}$. Because $\mathbb{D}$ is the δ-ring generated by $\mathcal{R}$ we have $\mathbb{D} \subset \mathbb{E}$. Consequently $\mathbb{E} = \mathbb{F} = \mathbb{D}$. This completes the proof. □

18-1.11. **Theorem** A vector measure μ on a semiring $\mathcal{S}$ into a Banach space E has at most one extension to a measure on the δ-ring $\mathbb{D}$ generated by $\mathcal{S}$.

Proof. Let φ, π be two extensions of μ from $\mathcal{S}$ to measures on $\mathbb{D}$. Clearly $\varphi = \pi$ on $\mathcal{R}$. Now $\mathbb{F} = \{D \in \mathbb{D} : \varphi D = \pi D\}$ is a monotone family of $\mathbb{D}$ containing $\mathcal{R}$. Consequently $\mathbb{F} = \mathbb{D}$, i.e. $\varphi = \pi$. □

18-1.12. **Exercise** Let $X = \{a, b\}$, $\mathcal{S} = \{\emptyset, \{a\}\}$. Show that $\mu\{a\} = 1$ and $\mu\emptyset = 0$ is a measure on the semiring $\mathcal{S}$. Construct two different extensions of μ on the power set $\mathbb{P}$ of X. Show that $\mathbb{P}$ is a δ-ring containing $\mathcal{S}$ but not generated by $\mathcal{S}$.

18-2 Outer Measures

18-2.1. To define the area of a disk which is obviously not a finite union of semi-rectangles, we cut the plane with horizontal and vertical lines and then count the total area of semi-rectangles which barely cover the disk. As we cut the plane finer and finer, we have better and better approximations to the area of the disk. This idea is formalized as the outer measure.

18-2.2. Let $\{a_n\}$ be a sequence in $[0, \infty]$. Define $\sum_n a_n = \infty$ if one of them is ∞. Clearly if all $0 \leq a_n < \infty$, the series $\sum_n a_n$ may also diverge to ∞.

18-2.3. Let $\mathcal{K}$ be an *arbitrary* family of subsets of a set X with $\emptyset \in \mathcal{K}$. Let $\mu : \mathcal{K} \to [0, \infty)$ be a *finite-valued* function satisfying $\mu\emptyset = 0$. Note that $\mathcal{K}$ need not be a semiring. In fact, for a locally compact space, $\mathcal{K}$ should be taken as the family of all compact sets. For every subset H of X, let $C(H)$ denote the family of all sequential covers $\{A_n\}$ of H by sets in $\mathcal{K}$. The *outer measure* of H is defined by

$$\mu^* H = \begin{cases} \inf \sum_{n=1}^{\infty} \mu A_n, & \text{if } C(H) \neq \emptyset; \\ \infty, & \text{if } C(H) = \emptyset. \end{cases}$$

18-2.4. **Theorem** The outer measure is a function on the power set of X into $[0, \infty]$. It satisfies the following conditions.

(a) If $H \in \mathcal{K}$ then $\mu^* H \leq \mu H$.

(b) If $H \subset K$ then $\mu^* H \leq \mu^* K$, monotonic.

(c) $\mu^* \left(\bigcup_{n=1}^{\infty} H_n\right) \leq \sum_{n=1}^{\infty} \mu^* H_n$, countably subadditive.

Proof. (a) Let $A_1 = H$ and $A_n = \emptyset$ for all $n \geq 2$. Then $\mu^* H \leq \sum_{n=1}^{\infty} \mu A_n = \mu H$.

(b) It follows from the definition immediately.

(c) If $\sum_{n=1}^{\infty} \mu^* H_n = \infty$ then the inequality is obviously true. So we may assume $\sum_{n=1}^{\infty} \mu^* H_n < \infty$. Then all $\mu^* H_n < \infty$ and in particular, $C(H_n) \neq \emptyset$. For every $\varepsilon > 0$, there is a sequential cover $\{A_{nj} : j \geq 1\}$ in $C(H_n)$ such that $\sum_{j=1}^{\infty} \mu A_{nj} \leq \mu^* H_n + \varepsilon/2^n$. Now $\{A_{nj} : n, j \geq 1\}$ is a sequential cover in $C\left(\bigcup_{n=1}^{\infty} H_n\right)$. The result follows by letting $\varepsilon \downarrow 0$ in the inequality below:

$$\mu^* \left(\bigcup_{n=1}^{\infty} H_n\right) \leq \sum_n \sum_j \mu A_{nj}$$

$$\leq \sum_{n=1}^{\infty} (\mu^* H_n + \varepsilon/2^n) = \left(\sum_{n=1}^{\infty} \mu^* H_n\right) + \varepsilon. \qquad \square$$

18-2.5. A subset M of X is said to be *μ^*-measurable* if

$$\mu^*(H \cap M) + \mu^*(H \setminus M) = \mu^* H$$

for every subset H of X. Geometrically, M cuts every set H nicely as the sum of inside and outside of M. It is important to note that μ^*-measurable set depends on μ while the measurable sets introduced in next chapter has nothing to do with μ. The following lemma gives a nice criterion for μ^*-measurability.

18-2.6. **Lemma** A subset M of X is μ^*-measurable if for every A in $\mathcal{K}$ we have $\mu^*(A \cap M) + \mu^*(A \setminus M) \leq \mu A$.

Proof. Let H be any subset of X. By countable subadditivity, we obtain

$$\mu^*(H \cap M) + \mu^*(H \setminus M) \geq \mu^* H.$$

If $\mu^* H = \infty$ then the equality holds. Hence we may assume $\mu^* H < \infty$. Then for every $\varepsilon > 0$, let $\{A_n\}$ be a sequence of sets in $\mathcal{K}$ such that $H \subset \bigcup_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} \mu A_n \leq \mu^* H + \varepsilon$. Observe that

$$\mu^*(H \cap M) + \mu^*(H \setminus M) \le \mu^* \left[\left(\bigcup_{n=1}^{\infty} A_n \right) \cap M \right] + \mu^* \left[\left(\bigcup_{n=1}^{\infty} A_n \right) \setminus M \right]$$
$$= \mu^* \left[\bigcup_{n=1}^{\infty} (A_n \cap M) \right] + \mu^* \left[\bigcup_{n=1}^{\infty} (A_n \setminus M) \right] \le \sum_{n=1}^{\infty} \mu^*(A_n \cap M) + \sum_{n=1}^{\infty} \mu^*(A_n \setminus M)$$
$$= \sum_{n=1}^{\infty} [\mu^*(A_n \cap M) + \mu^*(A_n \setminus M)] \le \sum_{n=1}^{\infty} \mu A_n \le \mu^* H + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have $\mu^*(H \cap M) + \mu^*(H \setminus M) \le \mu^* H$. Therefore the set M is μ^*-measurable. □

18-2.7. **Exercise** Prove that if $\mu^* M = 0$, then M is μ^*-measurable.

18-2.8. **Exercise** Consider the counting measure μ on X. Find a formula for the outer measure $\mu^* M$ of any subset M of X. Prove that every subset of X is μ^*-measurable. State and prove similar result for point measures.

18-2.9. A family $\mathcal{A}$ of subsets of X is called an *algebra* if
(a) the empty set is in $\mathcal{A}$;
(b) for every A in $\mathcal{A}$, the complement $X \setminus A$ is in $\mathcal{A}$;
(c) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.
An algebra is called a *σ-algebra* if it is also closed under formation of countable unions. Clearly, every σ-algebra is closed under countable intersections.

18-2.10. **Exercise** Let $\mathcal{A}$ be an *algebra* of subsets of X. Prove that if the union $\bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{A}$ for all *disjoint* sequences $\{A_n\}$ of sets in $\mathcal{A}$, then $\mathcal{A}$ is a σ-algebra.

18-2.11. **Theorem** The family $\mathcal{A}$ of all μ^*-measurable sets forms a σ-algebra. Furthermore μ^* is countably additive on μ^*-measurable sets.

Proof. Simple verification shows that $\mathcal{A}$ contains the empty set and is closed under formation of complements. We shall prove that $\mathcal{A}$ is closed under formation of countable unions in several steps. Let M, N be μ^*-measurable sets. We claim that $M \cup N$ is μ^*-measurable. For simplicity, write $M' = X \setminus M$. Now observe that for every subset H of X,

$$\mu^* H \le \mu^*[H \cap (M \cup N)] + \mu^*[H \setminus (M \cup N)]$$
$$= \mu^*[H \cap (M \cup N) \cap M] + \mu^*[H \cap (M \cup N) \setminus M] + \mu^*[H \cap (M \cup N)']$$
$$= \mu^*(H \cap M) + \mu^*(H \cap N \cap M') + \mu^*(H \cap M' \cap N')$$
$$= \mu^*(H \cap M) + \mu^*[(H \cap M') \cap N] + \mu^*[(H \cap M') \setminus N]$$
$$= \mu^*(H \cap M) + \mu^*(H \cap M') = \mu^*(H \cap M) + \mu^*(H \setminus M) = \mu^* H.$$

Therefore we have $\mu^* H = \mu^*[H \cap (M \cup N)] + \mu^*[H \setminus (M \cup N)]$ and hence $M \cup N$ is μ^*-measurable. Suppose further that M, N are disjoint. Then we have

$$\begin{aligned}\mu^*[H \cap (M \cup N)] &= \mu^*[H \cap (M \cup N) \cap M] + \mu^*[H \cap (M \cup N) \setminus M] \\ &= \mu^*(H \cap M) + \mu^*(H \cap N).\end{aligned}$$

By induction, it follows that finite unions of μ^*-measurable sets are μ^*-measurable. Furthermore if $M_1, M_2, \cdots M_p$ are disjoint μ^*-measurable sets then for every subset H of X, we have

$$\mu^* \left(H \cap \bigcup_{j=1}^{p} M_j\right) = \sum_{j=1}^{p} \mu^*(H \cap M_j). \qquad \#1$$

In particular when $H = \bigcup_{j=1}^{p} M_j$, we get $\mu^* \left(\bigcup_{j=1}^{p} M_j\right) = \sum_{j=1}^{p} \mu^*(M_j)$. Next, let $\{M_j\}$ be an infinite sequence of μ^*-measurable sets. Assume that the sets M_j are disjoint. Let $N_p = \bigcup_{j=1}^{p} M_j$ and $N = \bigcup_{j=1}^{\infty} M_j$. Then for every subset H of X, since N_p is μ^*-measurable we obtain by #1

$$\begin{aligned}\mu^* H &= \mu^*(H \cap N_p) + \mu^*(H \setminus N_p) \\ &\geq \mu^*(H \cap N_p) + \mu^*(H \setminus N) = \sum_{j=1}^{p} \mu^*(H \cap M_j) + \mu^*(H \setminus N).\end{aligned}$$

Letting $p \to \infty$ and using countable subadditivity, we get

$$\begin{aligned}\mu^* H &\geq \sum_{j=1}^{\infty} \mu^*(H \cap M_j) + \mu^*(H \setminus N) \geq \mu^* \left[\bigcup_{j=1}^{\infty} (H \cap M_j)\right] + \mu^*(H \setminus N) \\ &= \mu^*(H \cap N) + \mu^*(H \setminus N) \geq \mu^* H.\end{aligned}$$

Therefore $N = \bigcup_{j=1}^{\infty} M_j$ is μ^*-measurable. Letting $H = N$, we have

$$\mu^* \left(\bigcup_{j=1}^{\infty} M_j\right) = \sum_{j=1}^{\infty} \mu^* M_j.$$

Since $\mathcal{A}$ is a ring which is closed under formations of complements and countable disjoint unions, it follows that $\mathcal{A}$ is a σ-algebra. □

18-2.12. **Exercise** Let $\mathcal{B}$ denote the family of μ^*-measurable sets with *finite* measures. Show that $\mathcal{B}$ is a δ-ring and the restriction of μ^* to $\mathcal{B}$ is a measure. Furthermore prove that if $A \subset B$ are μ^*-measurable with $\mu^* A < \infty$, then $\mu^*(B \setminus A) = \mu^* B - \mu^* A$.

18-2.13. **Exercise** Let $X = \{a, b, c\}$, $A = \{a\}$, $B = \{b\}$, $C = \{a, b\}$, $\mathcal{K} = \{\emptyset, A, B, C\}$, $\mu A = \mu B = \mu C = 1$ and of course $\mu\emptyset = 0$. Find the outer measures of every subset of X. List all μ^*-measurable sets. Is A a non-measurable set? Show that $\mu^* \neq \mu$ on $\mathcal{K}$.

18-2.14. **Exercise** Let λ be the Lebesgue measure on the semi-intervals of the real line $\mathbb{R}$. Prove that every singleton is λ^*-measurable and its outer measure is zero. Then show that the outer measure of every countable set is zero. Prove that the set of irrational numbers in the interval $[\sqrt{2}, 3)$ is λ^*-measurable and find its outer measure. It can be proved that there exists a subset of $\mathbb{R}$ which is *not* Lebesgue measurable but we do not need this result.

18-3 Extension to Decent Sets

18-3.1. Let μ be a *finite-valued positive* measure on a semiring $\mathcal{S}$ over a set X and $\mathbb{D}$ the δ-ring generated by $\mathcal{S}$. Sets in $\mathbb{D}$ are called *decent sets*. In order to avoid ∞ taken by the outer measure μ^*, we have to cut back from μ^*-measurable sets to decent sets.

18-3.2. **Theorem** The outer measure μ^* is an extension of μ from $\mathcal{S}$ to a (finite-valued) measure on decent sets. For convenience, write μH instead of $\mu^* H$ for every decent set H.

Proof. Let M be a set in $\mathcal{S}$. Then for every A in $\mathcal{S}$, write $A \setminus M = \bigcup_{j=1}^{m} D_j$ where $D_1, D_2, \cdots, D_m$ are disjoint sets in $\mathcal{S}$. Because μ is additive, we have

$$\mu^*(A \cap M) + \mu^*(A \setminus M) \leq \mu^*(A \cap M) + \sum_{j=1}^{m} \mu^* D_j \leq \mu(A \cap M) + \sum_{j=1}^{m} \mu D_j = \mu A.$$

Thus M is μ^*-measurable. Next, let $\{A_j\}$ be a sequential cover of M. By countable subadditivity, we have $\mu M \leq \sum_{j=1}^{\infty} \mu A_j$. Taking infimum over all $\{A_j\}$ in $C(M)$ we obtain $\mu M \leq \mu^* M$. Therefore μ^* agrees with μ on $\mathcal{S}$. Next, let N be a decent set. Choose $A_1, \cdots, A_n \in \mathcal{S}$ such that $N \subset \bigcup_{j=1}^{n} A_j$. Then $\mu^* N \leq \sum_{j=1}^{n} \mu^* A_j = \sum_{j=1}^{n} \mu A_j < \infty$. Finally, since the family of all μ^*-measurable sets is a σ-algebra, it is a δ-ring containing $\mathcal{S}$ and therefore it also contains $\mathbb{D}$. $\square$

18-3.3. **Theorem** For each decent set H and for each $\varepsilon > 0$ there is a sequence $\{A_n\}$ of *disjoint* sets in $\mathcal{S}$ such that $H \subset \bigcup_{n=1}^{\infty} A_n$, $\sum_{n=1}^{\infty} \mu A_n \leq \mu H + \varepsilon$ and $\bigcup_{n=1}^{\infty} A_n$ is a decent set.

Proof. Let H be covered by a finite union of disjoint sets $H_1, H_2, \cdots, H_p$ in $\mathcal{S}$. Since $\mu H = \mu^* H$, for every $\varepsilon > 0$ there is a sequence $\{B_i\}$ of sets in $\mathcal{S}$ such that $H \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \mu B_i \leq \mu H + \varepsilon$. Define $B_0 = \emptyset$, $D_{11} = B_1$, $m(1) = 1$ and for each $i > 1$, write $B_i \setminus B_{i-1} = \bigcup_{j=1}^{m(i)} D_{ij}$ where $D_{i1}, D_{i2}, \cdots, D_{im(i)}$ are disjoint sets in $\mathcal{S}$. Then

$$H \subset \left(\bigcup_{i=1}^{\infty} B_i\right) \cap \left(\bigcup_{k=1}^{p} H_k\right) = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m(i)} \bigcup_{k=1}^{p} D_{ij} \cap H_k$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} \sum_{k=1}^{p} \mu[D_{ij} \cap H_k] = \sum_{i=1}^{\infty} \mu \left[\bigcup_{j=1}^{m(i)} D_{ij} \cap \left(\bigcup_{k=1}^{p} H_k\right)\right]$$

$$\leq \sum_{i=1}^{\infty} \mu(B_i \setminus B_{i-1}) \leq \sum_{i=1}^{\infty} \mu B_i \leq \mu H + \varepsilon.$$

Therefore an enumeration of $\{D_{ij} \cap H_k : 1 \leq k \leq p, 1 \leq j \leq m(i), i = 1, 2, \cdots\}$ is the required sequence. Their union is a decent set by §18-1.3. □

18-3.4. **Theorem** For each decent set H and for each $\varepsilon > 0$ there are disjoint sets $A_1, A_2, \cdots, A_n$ in $\mathcal{S}$ such that $\mu(H \triangle B) \leq \varepsilon$ for $B = \bigcup_{j=1}^{n} A_j$ where $H \triangle B = (H \setminus B) \cup (B \setminus H)$ is called the *symmetric difference* of H and B.

Proof. Let $\varepsilon > 0$ be given. Choose $\{A_j\}$ by last theorem. There is $n \geq 1$ such that $\sum_{j=n+1}^{\infty} \mu A_j \leq \varepsilon$. Define $B = \bigcup_{j=1}^{n} A_j$. Then $H \setminus B \subset \bigcup_{j=n+1}^{\infty} A_j$ and $B \setminus H \subset \bigcup_{j=1}^{\infty} A_j \setminus H$. Hence we obtain

$$\mu(H \triangle B) = \mu[(H \setminus B) \cup (B \setminus H)]$$

$$\leq \sum_{j=n+1}^{\infty} \mu A_j + \left[\mu\left(\bigcup_{j=1}^{\infty} A_j\right) - \mu H\right] \leq \varepsilon + \left(\sum_{j=1}^{\infty} \mu A_j - \mu H\right) \leq 2\varepsilon. \quad \square$$

18-3.5. A decent set which is a countable union of sets in $\mathcal{S}$ is called a *σ-set.* A countable intersection of σ-sets is called a *σδ-set*. Clearly every σ-set is a countable union of *disjoint* sets in $\mathcal{S}$ and every σ-set is a σδ-set.

18-3.6. **Corollary** Let H be a decent set.

(a) For every $\varepsilon > 0$, there is a σ-set $A \supset H$ with $\mu A \leq \mu H + \varepsilon$.

(b) There is a σδ-set B containing H such that $\mu(B \setminus H) = 0$.

Proof. Part (a) follows from §18-3.3. To prove (b), for each $n \geq 1$, there is a σ-set A_n containing H such that $\mu A_n \leq \mu H + 1/n$. Let $B = \bigcap_{n=1}^{\infty} A_n$. Then B is a σδ-set. Clearly $H \subset B$. From $\mu H \leq \mu B \leq \mu A_n \leq \mu H + 1/n$, we get $\mu H = \mu B < \infty$. Hence $\mu(B \setminus H) = \mu B - \mu H = 0$. □

18-3.7. **Exercise** Prove that finite intersections of σ-sets are σ-sets. Prove that every σδ-set is the intersection of a *decreasing* sequence of σ-sets. Show that every σδ-set is a decent set.

18-3.8. **Exercise** Let μ, ν be real measures on a semiring $\mathcal{S}$ over X and μ_e, ν_e their extension over the δ-ring $\mathbb{D}$ generated by $\mathcal{S}$. Prove that if $\mu \leq \nu$ on $\mathcal{S}$, then $\mu_e \leq \nu_e$ on $\mathbb{D}$.

18-99. **References and Further Readings** : Guainua, Fox, Uhl, Kluvanek, Ohba, Kuo, Niculescu-77, Christensen, Luo, Munroe, Kolmogorov, Rana, Wang, Dobrakov and Kandilakis.

Chapter 19

Measurable Objects

19-1 Measurable Sets

19-1.1. Let $(X, \mathbb{D})$ be a δ-space. Sets in $\mathbb{D}$ are called *decent sets*. A subset M of X is said to be ***measurable***, or $\mathbb{D}$-measurable to be precise, if for each decent set A, the intersection $M \cap A$ is a decent set. It is important to realize that measurable sets in this chapter are independent of any measure while μ^*-measurable sets in previous chapter rely heavily on a particular measure μ. In later chapters, vector measures of decent sets of X are always defined and form a vector space. Positive measures are defined on all measurable sets.

19-1.2. **Example** Let $X = \{x, y, z\}$, $A = \{x, y\}$, $B = \{z\}$, $C = \{x\}$ and $\mathbb{D} = \{\emptyset, A\}$. Then $\mathbb{D}$ is a δ-ring. The measurable sets are $\emptyset, A, B$ and X. On the other hand, the function $\mu : \mathbb{D} \to \mathbb{R}$ given by $\mu\emptyset = \mu A = 0$ is a measure on $\mathbb{D}$. Since $\mu^* C = 0$, the set C is μ^*-measurable but not $\mathbb{D}$-measurable.

19-1.3. **Lemma** Let $\mathcal{K}$ be a family of subsets of a set X and $\mathbb{D}$ the δ-ring generated by $\mathcal{K}$. Then a subset M of X is measurable if for each $A \in \mathcal{K}$, the intersection $A \cap M$ is a decent set. Two important cases are when $\mathcal{K}$ is a semiring or the family of all compact sets in a locally compact space.

Proof. Let $\mathbb{F}$ be the family of all decent sets A such that $A \cap M$ is a decent set. Since $(A \backslash B) \cap M = (A \cap M) \backslash (B \cap M)$, we have $A \backslash B \in \mathbb{F}$, $\forall\, A, B \in \mathbb{F}$. Similarly, it is easy to verify other conditions showing that $\mathbb{F}$ is a δ-ring containing $\mathcal{K}$. Therefore $\mathbb{F} = \mathbb{D}$ which means M is measurable. $\square$

19-1.4. **Theorem** The family $\mathbb{M}$ of all measurable sets in a δ-space X forms a σ-algebra containing $\mathbb{D}$.

Proof. Since $\mathbb{D}$ is a ring, every decent set is measurable, i.e. $\mathbb{D} \subset \mathbb{M}$. In particular, $\emptyset \in \mathbb{M}$. Let M, M_j be measurable sets and let A be any decent set. Because $\mathbb{D}$ is a δ-ring, we have $(X \setminus M) \cap A = A \setminus M \in \mathbb{D}$ and

$$\left(\bigcup\nolimits_{j=1}^{\infty} M_j\right) \cap A = A \setminus \bigcap\nolimits_{j=1}^{\infty} [A \setminus (M_j \cap A)] \in \mathbb{D}.$$

Therefore $\mathbb{M}$ is a σ-algebra. □

19-1.5. **Exercise** Show that a set is measurable iff its intersection with every decent set is measurable.

19-1.6. The measurable sets of $\mathbb{R}$, without any δ-ring explicitly indicated, are defined in terms of the δ-ring generated by the semi-intervals and have nothing to do with the Lebesgue measure. Similarly the measurable sets of $\mathbb{R}^2$ are obtained from the semi-rectangles. Since the complex plane $\mathbb{C}$ is identified with $\mathbb{R}^2$, its measurable sets are also defined in terms of the semi-rectangles.

19-1.7. **Exercise** Prove that every singleton in $\mathbb{R}$ is a countable intersection of semi-intervals. Hence show that every countable set in $\mathbb{R}$ is measurable.

19-1.8. For the rest of this section, we assume that $\mathcal{S}$ is a semiring over X, $\mathcal{R}$ the ring generated by $\mathcal{S}$ and $\mathbb{D}$ the common δ-ring generated by $\mathcal{S}$ and $\mathcal{R}$. Suppose $\mathcal{A}$ is a family of subsets of X. The intersection $\mathcal{B}$ of all σ-algebra containing $\mathcal{A}$ is called the ***σ-algebra generated*** by $\mathcal{A}$. Clearly $\mathcal{B}$ is the smallest σ-algebra containing $\mathcal{A}$.

19-1.9. **Exercise** Prove that the σ-algebras generated by $\mathcal{S}, \mathcal{R}, \mathbb{D}$ are identical.

19-1.10. **Theorem** If X is covered by a sequence of decent sets, then the family $\mathbb{M}$ of all measurable sets is the σ-algebra generated by the δ-ring $\mathbb{D}$ and hence by $\mathcal{S}$. Consequently the measurable sets of $\mathbb{R}$ are generated by semi-intervals and of $\mathbb{R}^2$ by semi-rectangles.

Proof. Let $X = \bigcup_n A_n$ where $A_n \in \mathbb{D}$ and let $\mathbb{N}$ be the σ-algebra generated by $\mathbb{D}$. Since $\mathbb{N}$ is the smallest, it follows that $\mathbb{N} \subset \mathbb{M}$. Conversely take any $Q \in \mathbb{M}$. Then $Q \cap A_n \in \mathbb{D} \subset \mathbb{N}$. Since $\mathbb{N}$ is a σ-algebra, we have $Q = \bigcup_n (Q \cap A_n) \in \mathbb{N}$. Thus $\mathbb{M} \subset \mathbb{N}$. Therefore $\mathbb{M}$ is generated by $\mathbb{D}$. It follows that $\mathbb{M}$ is also generated by $\mathcal{S}$. The last statement follows from the fact that $\mathbb{R}$ can be covered by $(-n, n]$ and $\mathbb{R}^2$ by $(-n, n]^2$ for $n = 1, 2, 3, \cdots$. □

19-1.11. **Theorem** Consider the real line $\mathbb{R}$. The family $\mathbb{M}$ of all measurable sets is the σ-algebra generated by any one of the following families of sets:

(a) the semi-intervals $(a, b]$ for $a < b$;
(b) the open intervals (a, b) for $a < b$;
(c) the upper open rays (a, ∞) for $a \in \mathbb{R}$;
(d) the lower closed rays $(-\infty, a]$ for $a \in \mathbb{R}$;
(e) the lower open rays $(-\infty, a)$ for $a \in \mathbb{R}$;

(f) the upper closed rays $[a, \infty)$ for $a \in \mathbb{R}$;

(g) the open sets of $\mathbb{R}$.

Proof. Let $\mathbb{M}(a), \mathbb{M}(b), \cdots$ be σ-algebras generated by sets of $(a), (b), \cdots$ respectively. It follows from last theorem that $\mathbb{M} = \mathbb{M}(a)$. For any $a < b$ we have $(a, b) = \bigcup_{n=1}^{\infty}(a, b_n]$ where $a < b_n \uparrow b$. Hence $\mathbb{M}(a)$ is a σ-algebra containing all (a, b). Since $\mathbb{M}(b)$ is the smallest, we get $\mathbb{M}(b) \subset \mathbb{M}(a)$. Similarly $(a, \infty) = \bigcup_{n=1}^{\infty}(a, a+n)$ gives $\mathbb{M}(c) \subset \mathbb{M}(b)$. Next, $\mathbb{R} \setminus (a, \infty) = (-\infty, a]$ gives $\mathbb{M}(d) \subset \mathbb{M}(c)$. Now it is easy to prove $\mathbb{M}(f) \subset \mathbb{M}(e) \subset \mathbb{M}(d)$. Next for all $a < b$, we have $(a, b] = \left\{\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty\right)\right\} \cap \left\{\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \left[b + \frac{1}{n}, \infty\right)\right\}$ establishes $\mathbb{M}(a) \subset \mathbb{M}(f)$. We have proved $\mathbb{M} = \mathbb{M}(a) = \mathbb{M}(b) = \cdots = \mathbb{M}(f)$. Since every open interval is an open set we have $\mathbb{M}(b) \subset \mathbb{M}(g)$. Finally given any open set A, there is a sequence of bounded open intervals B_i such that $A = \bigcup_{i=1}^{\infty} B_i$. Hence $\mathbb{M}(g) \subset \mathbb{M}(b)$. This proves $\mathbb{M}(b) = \mathbb{M}(g)$. □

19-1.12. **Exercise** Prove that every semi-rectangle in $\mathbb{R}^2$ is a countable intersection of open sets. Show that every non-empty open set in $\mathbb{R}^2$ is a countable union of semi-rectangles. Deduce that the family of all measurable sets in $\mathbb{R}^2$ are generated by the open sets.

19-1.13. Subsets of $\mathbb{R}^n$ in the σ-algebra generated by open sets are usually called *Borel sets* by the community. Our decent sets are bounded Borel sets. Clearly every Borel set in $\mathbb{R}$ is Lebesgue measurable. Borel sets are independent of any measure but Lebesgue measurable sets depend on the Lebesgue measure.

19-2 Measurable Functions

19-2.1. **Lemma** Let X, Y be δ-spaces; $f : X \to Y$ a map and $\mathbb{M}(X), \mathbb{M}(Y)$ the families of measurable subsets of X, Y respectively. Suppose that $\mathbb{M}(Y)$ is generated by a subfamily $\mathcal{K}$. If $f^{-1}(A) \in \mathbb{M}(X)$ for each $A \in \mathcal{K}$, then $f^{-1}(B) \in \mathbb{M}(X)$ for each $B \in \mathbb{M}(Y)$

Proof. Suppose that $\mathbb{F} = \{B \subset Y : f^{-1}(B) \in \mathbb{M}(X)\}$. It is given that $\mathcal{K} \subset \mathbb{F}$. Since taking inverse images preserves complements and unions, $\mathbb{F}$ is a σ-algebra over Y. Because $\mathbb{M}(\mathbb{K})$ is generated by $\mathcal{K}$ we have $\mathbb{M}(Y) \subset \mathbb{F}$. This completes the proof. □

19-2.2. Let X be a δ-space. A function $f : X \to \mathbb{K}$ is said to be *measurable* if the inverse image of every measurable set is measurable. It is important to note that we use the Borel sets in $\mathbb{K}$ rather than the larger class of Lebesgue measurable sets. Observe that in this book the target spaces are restricted to $\mathbb{K}$ or Banach spaces only.

19-2.3. **Theorem** Let f be a real function on a δ-space X. Then the following statements are equivalent.
(a) f is measurable.
(b) $f^{-1}(a,\infty) = \{x \in X : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$.
(c) $f^{-1}(-\infty,a] = \{x \in X : f(x) \le a\}$ is measurable for all $a \in \mathbb{R}$.
(d) $f^{-1}(-\infty,a) = \{x \in X : f(x) < a\}$ is measurable for all $a \in \mathbb{R}$.
(e) $f^{-1}[a,\infty) = \{x \in X : f(x) \ge a\}$ is measurable for all $a \in \mathbb{R}$.
(f) The inverse image $f^{-1}(A)$ of every open set A in $\mathbb{R}$ is measurable.

19-2.4. **Corollary** Let f, g be real functions on a δ-space X. If both f, g are measurable then the following sets $A = \{x \in X : f(x) < g(x) + a\}$, $B = \{x \in X : f(x) \ge g(x) + a\}$, $C = \{x \in X : f(x) = g(x) + a\}$ and $D = \{x \in X : f(x) \ne g(x) + a\}$ are all measurable for every $a \in \mathbb{R}$.

Proof. Let $\mathbb{Q}$ denote the countable set of all rational numbers. Since

$$A = \bigcup_{r\in\mathbb{Q}}\{f^{-1}(-\infty, r) \cap g^{-1}(r-a,\infty)\}$$

is a countable union of intersections of two measurable sets, it is measurable. Thus $B = X \setminus A$ is also measurable. Next, the set

$$C = \{x \in X : f(x) \ge g(x) + a\} \cap \{x \in X : g(x) \ge f(x) - a\}$$

is measurable. Finally $D = X \setminus C$ is measurable. □

19-2.5. **Theorem** A function $f : X \to \mathbb{K}$ is measurable iff the inverse image $f^{-1}(A)$ of every open subset A of $\mathbb{K}$ is measurable.

Proof. The measurable sets in $\mathbb{K}$ are generated by open sets. □

19-2.6. **Theorem** Let $\mathbb{K}_i$ denote $\mathbb{R}$ or $\mathbb{C}$. If $f : X \to \mathbb{K}_1$ is a measurable function and if $\varphi : \mathbb{K}_1 \to \mathbb{K}_2$ is a continuous function, then the composite function φf is a measurable function on X.

Proof. Let A be an open set in $\mathbb{K}_2$. Since φ is continuous, $\varphi^{-1}(A)$ is open in $\mathbb{K}_1$. Since f is measurable, $f^{-1}\varphi^{-1}(A)$ is measurable. Hence $(\varphi f)^{-1}(A)$ is measurable for every open subset A of $\mathbb{K}_2$. Consequently φf is measurable. □

19-2.7. **Corollary** If $f : X \to \mathbb{K}$ is measurable then $|f|^p$ is measurable for every $p > 0$.

Proof. The function $\varphi(t) = |t|^p : \mathbb{K} \to \mathbb{R}$ is continuous. The result follows immediately from last theorem. □

19-2.8. **Theorem** A complex function $f : X \to \mathbb{C}$ is measurable iff both real and imaginary parts are measurable real functions.

Proof. Since $\varphi(z) = \text{Re } z : \mathbb{C} \to \mathbb{R}$ is a continuous function, Re $f = \varphi f$ is measurable. Similarly Im f is also measurable. Conversely assume both Re f and Im f are measurable *real* functions. Let $A = (a, b] \times (c, d]$ be a semi-rectangle of $\mathbb{C}$ where $a < b$ and $c < d$. Then the set

$$f^{-1}(A) = \{x \in X : a < \text{Re } f(x) \leq b,\ c < \text{Im } f(x) \leq d\}$$
$$= (\text{Re } f)^{-1}(a, b] \cap (\text{Im } f)^{-1}(c, d]$$

is a measurable subset of X. Since the measurable sets of $\mathbb{C}$ are generated by the semi-rectangles, f is a measurable complex function. □

19-2.9. **Theorem** Linear combinations of measurable functions are measurable.

Proof. Let f, g be real measurable functions on a δ-space X and let k be a real number. If $k = 0$ then the constant function kf is measurable. If $k > 0$ then for all $a \in \mathbb{R}$, the set $\{x \in X : kf(x) < a\} = \{x \in X : f(x) < a/k\}$ is measurable and if $k < 0$ then $\{x \in X : kf(x) < a\} = \{x \in X : f(x) > a/k\}$ is again measurable. Therefore kf is measurable. Finally for each $a \in \mathbb{R}$, the set $\{x \in X : f(x) + g(x) < a\} = \{x \in X : f(x) < -g(x) + a\}$ is measurable. Consequently $f + g$ is measurable. The complex case is reduced to the real case by last theorem and the use of an explicit formula to express the complex linear combination in terms of the real and imaginary parts. □

19-2.10. **Theorem** The product $f \cdot g$ of two measurable functions $f, g : X \to \mathbb{K}$ is measurable.

Proof. For real case, $f \cdot g = \frac{1}{4}(|f + g|^2 - |f - g|^2)$ is measurable. For complex case, $f \cdot g = (\text{Re } f)(\text{Re } g) - (\text{Im } f)(\text{Im } g) + i(\text{Re } f)(\text{Im } g) + i(\text{Im } f)(\text{Re } g)$ is measurable by reduction to real case. □

19-2.11. **Exercise** Prove that every continuous function on $\mathbb{K}$ is measurable.

19-2.12. **Exercise** Prove that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$ is measurable.

19-3 Limits of Measurable Functions

19-3.1. Given a sequence of real functions $f_n : X \to \mathbb{R}$, their supremum $\sup f_n$ may take ∞ as values. For this reason, we have to consider the extended real line $[-\infty, \infty]$. In this section, we shall prove that measurable functions are closed under limiting process.

19-3.2. It is easy to prove that the family of all semi-intervals of $\mathbb{R}$ together with two singletons $\{\infty\}$, $\{-\infty\}$ and the empty set forms a semiring over $[-\infty, \infty]$. For convenience, we also call this the *semiring of semi-intervals*. It follows that the measurable sets of $[-\infty, \infty]$ are members of the σ-algebra generated by semi-intervals and two singletons $\{\infty\}$, $\{-\infty\}$. Since $\{\infty\} = \bigcap_{n=1}^{\infty}(n, \infty]$, $\{-\infty\} = [-\infty, \infty] \setminus \bigcup_{n=1}^{\infty}(-n, \infty]$, and $(a, b] = \{[-\infty, \infty] \setminus (b, \infty]\} \cap (a, \infty]$, the measurable sets of $[-\infty, \infty]$ are generated by the sets $(a, \infty]$ for all $a \in \mathbb{R}$ and similarly also by sets $[\infty, b)$ for all $b \in \mathbb{R}$. Given a subset M of $\mathbb{R}$, it follows that M is measurable in $\mathbb{R}$ iff it is measurable in $[-\infty, \infty]$.

19-3.3. Let X be a δ-space and $f : X \to [-\infty, \infty]$ be an extended function. Now f is measurable iff $\{x \in X : f(x) > a\}$ is measurable for every $a \in \mathbb{R}$. Consequently, §19-2.4 also holds for measurable extended functions. Measurable extended positive (≥ 0) functions are also called *upper* functions. By a measurable function, we always mean a real or complex function without taking $\pm\infty$ as its value.

19-3.4. **Exercise** Prove that for any upper function f and $\alpha \in \mathbb{R}$, the sets $A = \{x \in X : f(x) > \alpha\}$, $B = \{x \in X : f(x) \geq \alpha\}$ and $C = \{x \in X : f(x) = \infty\}$ are measurable.

19-3.5. **Exercise** Let f be the extended function on $\mathbb{R}$ given by $f(x) = \infty$ if x is irrational and $f(x) = x$ if x is rational. Show that f is an upper function.

19-3.6. Let $\{x_n\}$ be a sequence of extended real numbers in $[-\infty, \infty]$. Its *lower limit* is defined by $\liminf_{n\to\infty} x_n = \sup_{k\geq 1} \inf_{n\geq k} x_n$. Similarly its *upper limit* is defined by $\limsup_{n\to\infty} x_n = \inf_{k\geq 1} \sup_{n\geq k} x_n$. Clearly both upper and lower limits exist in $[-\infty, \infty]$. Standard properties will be used without specification. See §2-3.3.

19-3.7. For any sequence $\{f_n\}$ of real functions on X, define

$$\big(\liminf_{n\to\infty} f_n\big)(x) = \liminf_{n\to\infty}\,[f_n(x)] \quad \text{and} \quad \big(\limsup_{n\to\infty} f_n\big)(x) = \limsup_{n\to\infty}\,[f_n(x)].$$

19-3.8. **Theorem** Let f_n, f, g be measurable extended functions on a δ-space X. Then the following functions are measurable.

(a) $\sup f_n$, $\inf f_n$ where n runs over a countable index set I.

(b) $\liminf f_n$, $\limsup f_n$.

(c) $\lim f_n$ if it exists.

(d) $f \vee g$, $f \wedge g$, $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$.

Proof. As a countable union of measurable sets, for every $a \in \mathbb{R}$ the set $\{x \in X : \sup f_n(x) > a\} = \bigcup_n \{x \in X : f_n(x) > a\}$ is measurable. Hence $\sup f_n$ is measurable. Similarly, $\inf f_n$ is measurable. Repeated application of these results to $\sup_{k\geq 1} \inf_{n\geq k} f_n$ shows that $\liminf f_n$ is measurable. Similarly the function $\limsup f_n$ is measurable. If $\lim f_n$ exists then $\lim f_n = \liminf f_n$ is measurable. Both $f \vee g$, $f \wedge g$ are measurable when the index set I in (a) consists of two elements. The last result follows from the fact that the zero function is measurable. □

19-3.9. **Exercise** Prove that the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ differentiable everywhere is measurable.

19-3.10. **Exercise** Let $f_n : \mathbb{R} \to [-\infty, \infty]$ be a sequence of extended functions and let M be the set of $x \in \mathbb{R}$ at which $\lim f_n(x)$ exists and is finite. Prove that M is measurable.

19-4 Approximations by Simple Functions

19-4.1. We started off with a semiring $\mathcal{S}$ over a set X. Then we constructed more semirings $\mathcal{R}, \mathbb{D}, \mathbb{M}$. For convenience, a function of the form $f = \sum_{i=1}^{n} \alpha_i \rho_{A_i}$, where $\alpha_i \in \mathbb{K}$, is called a *step function* if all A_i are in $\mathcal{S}$ or $\mathcal{R}$, a *decent function* if all A_i are in $\mathbb{D}$ and a *simple function* if all A_i are in $\mathbb{M}$. Clearly every simple function is measurable. For every function f on X, the sup-norm is defined by $\|f\| = \sup\{|f(x)| : x \in X\}$. Other norms usually have a subscript such as $\|f\|_p$.

19-4.2. **Theorem** Every positive bounded measurable function is a uniform limit of some increasing sequence of positive simple functions.

Proof. Let $f : X \to [0, \infty)$ be a bounded measurable function. Firstly we claim there is a simple function $0 \leq g \leq f$ such that $\|f - g\| \leq \frac{1}{2}\|f\|$. In fact, let $\beta = \frac{1}{2}\|f\|$ and $A = \{x \in X : f(x) > \beta\}$. Then $g = \beta\rho_A$ is a simple function. Clearly $0 \leq g \leq f$. If $x \in A$ then $|f(x) - g(x)| = f(x) - \beta \leq 2\beta - \beta = \beta$ and if

$x \notin A$ then $|f(x) - g(x)| = f(x) \le \beta$. Therefore $\|f - g\| \le \beta = \frac{1}{2}\|f\|$. Now we apply this machine to prove the theorem. There is a simple function g_1 such that $0 \le g_1 \le f$ and $\|f - g_1\| \le \frac{1}{2}\|f\|$. By induction there is a simple function g_n such that $0 \le g_n \le f - (g_1 + g_2 + \cdots + g_{n-1})$ and

$$\|f - (g_1 + g_2 + \cdots + g_n)\| \le \tfrac{1}{2}\|f - (g_1 + g_2 + \cdots + g_{n-1})\| \le \tfrac{1}{2^n}\|f\|.$$

Then $f_n = g_1 + g_2 + \cdots + g_n$ is a simple function such that $0 \le f_n \uparrow f$ and $\|f - f_n\| \le \frac{1}{2^n}\|f\|$. □

19-4.3. **Theorem** Every upper function is a pointwise limit of some increasing sequence of positive simple function.

Proof. Let $f : X \to [0, \infty]$ be a measurable function. For each $n \ge 1$, let $g_n = f \wedge n$. Then g_n is bounded measurable and $0 \le g_n \uparrow f$. There is a simple function h_n such that $0 \le h_n \le g_n$ and $\|g_n - h_n\| \le \frac{1}{n}$. Now for each $x \in X$, if $f(x) < \infty$, then $|h_n(x) - f(x)| = |h_n(x) - g_n(x)| \le \frac{1}{n}$ for all $n > f(x)$ otherwise $h_n(x) \ge g_n(x) - \frac{1}{n} \ge n - 1$. Therefore $h_n \to f$ pointwise and $h_n \le g_n \le f$. Define $f_n = h_1 \vee h_2 \vee \cdots \vee h_n$. Then f_n are positive simple functions satisfying $0 \le f_n \uparrow f$. □

19-4.4. **Theorem** For every measurable function f, there is a sequence of simple functions f_n such that $|f_n| \le |f_{n+1}| \le |f|$ and $f_n \to f$ on X as $n \to \infty$. Furthermore if f is bounded then we may assume uniform convergence on X. Finally, if f is real, then all f_n can be chosen as real.

Proof. Assume f to be real. There are simple functions $0 \le g_n \uparrow f_+$, $0 \le h_n \uparrow f_-$. Then all $f_n = g_n - h_n$ are simple functions. Clearly $f_n \to f$ on X. Now $|f_n| = g_n + h_n \le f_+ + f_- = |f|$. Similarly $|f_n| \le |f_{n+1}|$. Next, for complex function f, apply the above result to both Re f and Im f respectively and then assemble them together. The case of bounded function is left as an exercise. □

19-4.5. **Corollary** For every extended real measurable function f, there is a sequence of real simple functions f_n such that $|f_n| \le |f_{n+1}| \le |f|$ and $f_n \to f$ pointwise on X as $n \to \infty$.

19-4.6. **Exercise** Let f be a real function given by $f(x) = x + i(2 - |x|), \forall x \in \mathbb{R}$. Construct explicitly a sequence of simple functions f_n on $\mathbb{R}$ such that $f_n \to f$ pointwise and $|f_n| \le |f_{n+1}| \le |f|$ for all n.

19-5 Measurable Maps

19-5.1. It is tempting to define a vector map to be measurable if it can be approximated by a sequence of simple maps as motivated by 19-4.4, 3.8. This is inappropriate because in general it excludes continuous maps on locally compact spaces. Let $(X, \mathbb{D})$ be a δ-space, M a measurable subset of X, F a Banach space and $f : X \to F$ a map. A map $f : X \to F$ is *simply approximable on* M if there is a sequence of simple maps convergent pointwise to f on M. The proofs in this section can be simplified slightly with the assumption $M = X$ without loss of generality although M is normally a decent set in applications.

19-5.2. **Lemma** If f is simply approximable on M, then $f(M)$ is separable.

Proof. Let $f_n : X \to F$ be simple maps with $f_n \to f$ on M. Since $f_n(M)$ is a finite set, the set $K = \bigcup_{n=1}^{\infty} f_n(M)$ is countable and its closure $\overline{K}$ is separable. Therefore the subset $f(M)$ of $\overline{K}$ is separable. □

19-5.3. **Example** The Banach space $F = \ell_\infty$ is not separable. With discrete metric, $X = \ell_\infty$ is a locally compact space and the identity map $f : X \to F$ is continuous. No matter what δ-ring $\mathbb{D}$ is assigned to X, the set $f(X)$ is not separable. Consequently, it cannot be approximated by sequences of simple maps but we want continuous maps to be measurable. See §27-1.4.

19-5.4. **Theorem** Let $f(M)$ be separable in F. Then the following statements are equivalent.

(a) f is simply approximable on M.
(b) $M \cap f^{-1}(B)$ is measurable for every *open ball* B in F.
(c) $M \cap f^{-1}(W)$ is measurable for every *open set* W in F.
(d) $M \cap f^{-1}\left(\overline{\mathbb{B}}\right)$ is measurable for every *closed ball* $\overline{\mathbb{B}}$ in F.
(e) $M \cap f^{-1}(K)$ is measurable for every *closed set* K in F.

Proof. We shall prove two loops: $(c \Rightarrow b \Rightarrow a \Rightarrow c)$ and $(c \Rightarrow d \Rightarrow e \Rightarrow c)$.
$(c \Rightarrow b)$ It is trivial because open balls are open sets.
$(b \Rightarrow a)$ For the open ball $B_{nk} = \{\alpha \in F : \|\alpha - a_k\| < 1/n\}$, the set $A_{nk} = M \cap f^{-1}(B_{nk})$ is measurable. Let $D_{n1} = A_{n1}$ and $D_{nk} = A_{nk} \setminus \bigcup_{j=1}^{k-1} A_{nj}$. Then all D_{nk} are measurable. Since $f(M)$ is separable, there is a countable dense subset $\{a_n\}$ of $f(M)$. The maps $f_n = \sum_{j=1}^{n} a_j \rho_{D_{nj}}$ are simple with $f_n(X \setminus M) = 0$. To show $f_n \to f$ on M, take any $x \in M$ and $\varepsilon > 0$. Choose $m \geq 1/\varepsilon$ and pick any $n \geq m$. Since $\{a_n\}$ is dense in $f(M)$, we have $\|f(x) - a_k\| < 1/n$ for some k. Select the smallest index k so that

$\|f(x)-a_j\| \geq 1/n$ for all $j < k$. Thus $x \in A_{nk} \setminus \bigcup_{j=1}^{k-1} A_{nj} = D_{nk}$, or $f_n(x) = a_k$. Therefore $\|f(x) - f_n(x)\| < 1/n \leq 1/m \leq \varepsilon$.

($a \Rightarrow c$) The distance function $\varphi(e) = d(\alpha, F \setminus W)$ is continuous in $\alpha \in F$. The set $V_m = \varphi^{-1}(\frac{1}{m}, \infty)$ is open in F. Since $F \setminus W$ is closed, $W = \bigcup_{m=1}^{\infty} V_m$. Also $\overline{V}_m \subset \varphi^{-1}[\frac{1}{m}, \infty) \subset V_{m+1} \subset W$. Choose simple maps $f_n \to f$ on M. Let $Q = \bigcup_{m,n=1}^{\infty} \bigcap_{j \geq n} f_j^{-1}(V_m)$. We claim $M \cap f^{-1}(W) = M \cap Q$. In fact, take any $x \in M \cap f^{-1}(W)$. Then $x \in \bigcup_{m=1}^{\infty} f^{-1}(V_m)$. Hence $\exists\ m$, $f(x) \in V_m$. Since $f_j(x) \to f(x)$; $\exists\ n$, $\forall\ j \geq n$, $f_j(x) \in V_m$; that is $x \in Q$. On the other hand, let $x \in M \cap Q$. Then $\exists\ m, n$, $\forall\ j \geq n$, $f_j(x) \in V_m$; hence $f(x) \in \overline{V}_m \subset W$; that is $x \in f^{-1}(W)$. This establishes what we claim. Since each f_j is a simple map, all $f_j^{-1}(V_m)$ are measurable. Consequently, the set

$$M \cap f^{-1}(W) = \bigcup_{m,n=1}^{\infty} \bigcap_{j \geq n} M \cap f_j^{-1}(V_m)$$

is also measurable.

($e \Rightarrow c$) It follows from $M \cap f^{-1}(W) = M \setminus \left[M \cap f^{-1}(F \setminus W)\right]$.

($c \Rightarrow d$) It follows from $M \cap f^{-1}\left(\mathbb{B}\right) = M \setminus \left[M \cap f^{-1}\left(F \setminus \mathbb{B}\right)\right]$.

($d \Rightarrow e$) Take any closed set K in F. Then the subset $\overline{f(M)} \cap K$ of the separable set $\overline{f(M)}$ is separable. Hence there is a countable set D which is dense in the closed set $\overline{f(M)} \cap K$. From §1-5.15, it follows $f^{-1}(K) = f^{-1}\left(\overline{f(M)} \cap K\right) = \bigcap_{n=1}^{\infty} \bigcup_{x \in D} f^{-1}\left[\,\mathbb{B}(x, 1/n)\right]$. Since inverses of closed balls are measurable, then the inverse $f^{-1}(K)$ is also measurable. □

19-5.5. A map $f : X \to F$ is *measurable* or *strongly measurable* if f is simply approximable on every decent set. Clearly the set of all measurable maps forms a vector space under the pointwise operations.

19-5.6. **Exercise** Let E, F, FE be Banach spaces with a continuous bilinear map $F \times E \to FE$. Prove that if $f : X \to F$ and $g : X \to E$ are measurable maps, then the product $fg : X \to FE$ is also measurable.

19-5.7. **Theorem** Let $f : X \to F$ be a map such that $f(A)$ is separable for every decent set A. Then the following statements are equivalent.

(a) f is a measurable map.

(b) f is simply approximable on X.

(c) Inverse images of open sets (respectively open balls, closed sets, closed balls) are measurable.

Since $\mathbb{K}$ is separable, the new definition of measurable maps by (a) agrees with the old definition of measurable functions by (b), (c).

Proof. To prove $(a \Rightarrow c)$, let W be an open subset of F and A a decent subset of X. Then the subset $f(A)$ of the separable set $f(X)$ is separable. Since f is a measurable map, it is simply approximable on A. Thus $A \cap f^{-1}(W)$ is measurable. Because A is arbitrary, $f^{-1}(W)$ is measurable. It is an exercise to prove $(c \Rightarrow b \Rightarrow a)$. □

19-5.8. **Corollary** The variation $|f| : X \to \mathbb{R}$ of a measurable map $f : X \to F$ is measurable.

19-5.9. **Exercise** Show that f is simply approximable on M iff $f\rho_M$ is simply approximable on X.

19-5.10. **Exercise** Let $f : X \to F$ be a map such that $f(X)$ is relatively compact. Prove that if the inverses of open balls are measurable, then f is measurable.

19-5.11. **Exercise** Let $\mathcal{S}$ be a semiring over a set X and $\mathbb{D}$ the δ-ring generated by $\mathcal{S}$. Show that if f is simply approximable on every set in $\mathcal{S}$, then f is measurable.

19-6 More Properties

19-6.1. In this section, we develop some properties of measurable maps such as monotone approximation required by integration and weak measurability to ensure Radon-Nikodym property for duality theory. Let $(X, \mathbb{D})$ be a δ-space, M a measurable subset of X, F a Banach space and $f : X \to F$ a map.

19-6.2. **Example** For every measurable map $f : X \to F$, the map given by $\mathrm{sgn}[f(x)] = 0$ if $f(x) = 0$, and $\mathrm{sgn}[f(x)] = \dfrac{f(x)}{\|f(x)\|}$ otherwise, is measurable.

Proof. For every decent set A, choose simple maps $g_n \to f\rho_A$. Then $h_n = \dfrac{g_n}{(1/n) + |g_n|}$ are simple maps convergent to $\mathrm{sgn}(f)\rho_A$. □

19-6.3. **Lemma** If f is simply approximable on a measurable set M, then there are simple maps f_n such that $|f_n| \uparrow |f\rho_M|$ and $f_n \to f\rho_M$ pointwise.

Proof. Choose simple functions $0 \le s_n \uparrow |f\rho_M|$ and simple maps $g_n \to f\rho_M$. Select any $\alpha \in F$ with $\|\alpha\| = 1$. Since the sets $N = \{x \in X : f(x) \ne 0\}$ and $B_n = \{x \in X : g_n(x) \ne 0\}$ are measurable, $h_n = \mathrm{sgn}(g_n)\rho_{N \cap B_n} + \alpha\rho_{N \setminus B_n}$ is simple with $|h_n| = \rho_N$. Thus $f_n = s_n h_n$ are simple maps with $|f_n| \uparrow |f\rho_M|$. If $x \notin N$, then $0 \le s_n(x) \le |f\rho_M|(x) = 0$ and hence $f_n(x) \to f(x)\rho_M(x)$. Suppose

$x \in N$. There is k, for all $n \geq k$, $g_n(x) \neq 0$, i.e. $x \in N \cap B_n$. Therefore we have $f_n(x) = s_n(x)h_n(x) = s_n(x)\,\dfrac{g_n(x)}{\|g_n(x)\|} \to f(x)\rho_M(x)$. □

19-6.4. **Theorem** Let $g_n : X \to F$ be a sequence of simple maps. If $g_n \to f$ uniformly on a measurable set M, then $f(M)$ is relatively compact.

Proof. For every $\varepsilon > 0$, there is an integer n such that $\|f(x) - g_n(x)\| < \varepsilon$ for all $x \in M$. Write $g_n = \sum_{j=1}^{k} \alpha_j \rho_{A_j}$ where $\alpha_j \in F$ and $\{A_j\}$ are disjoint measurable sets in X. Let $\alpha_0 = 0$. If $x \in M \cap A_j$, then $\|f(x) - \alpha_j\| = \|f(x) - g_n(x)\| < \varepsilon$. If $x \in M \setminus A_j$ for all j, then $\|f(x) - \alpha_0\| = 0 < \varepsilon$. Hence $f(M) \subset \bigcup_{j=0}^{k} \mathbb{B}(\alpha_j, \varepsilon)$. Therefore $f(M)$ is precompact in the Banach space F. Consequently, $f(M)$ is relatively compact. □

19-6.5. **Theorem** Let $f : X \to F$ be a measurable map. If $f(M)$ is relatively compact, then there is a sequence of simple maps g_n such that $|g_n| \uparrow |f|$ on M and $g_n \to f$ uniformly on M.

Proof. Let $g_1 = \alpha\rho_M$ where $\alpha = 0 \in F$. Inductively, suppose that $g_{n-1} = \sum_{k=1}^{m} \alpha_k \rho_{A_k}$ is a simple map so that $\|f(x) - g_{n-1}(x)\| \leq t/2^{n-1}$ for all $x \in M$ where $A_1, \cdots, A_m$ are disjoint measurable sets with $M = \bigcup_{k=1}^{m} A_k$; all $\alpha_k \in F$ with $\|\alpha_k\| \leq \inf \|f(A_k)\|$ and $t = \sup \|f(M)\|$. By compactness, write $\overline{f(A_k)} \subset \bigcup_{j=1}^{p} \mathbb{B}(a_j, t/2^{n+1})$. The sets $B_j = A_k \cap f^{-1}[\mathbb{B}(a_j, t/2^{n+1})]$ are measurable. Let $D_1 = B_1$ and $D_j = B_j \setminus \bigcup_{i=1}^{j-1} B_i$. Then $D_1, \cdots, D_p$ are disjoint measurable sets satisfying $A_k = \bigcup_{j=1}^{p} B_j = \bigcup_{j=1}^{p} D_j$. If $\overline{f(D_j)} \cap \overline{\mathbb{B}}(a_j, t/2^{n+1})$ is not empty, choose β_j in the compact set $\overline{f(D_j)} \cap \overline{\mathbb{B}}(a_j, t/2^{n+1})$ with minimum norm; otherwise let $\beta_j = \alpha_k$. Then $\varphi_k(x) = \sum_{j=1}^{p} \beta_j \rho_{D_j}$ and $g_n = \sum_{k=1}^{m} \varphi_k$ are simple maps. Take any $x \in A_k$. We have $x \in D_i \subset B_i$ for some i. Thus $f(x) \in f(D_i) \cap \mathbb{B}(a_i, t/2^{n+1})$. Hence we obtain $\beta_i \in \overline{\mathbb{B}}(a_i, t/2^{n+1})$, or $\|f(x) - \varphi_k(x)\| = \|f(x) - \beta_i\| \leq t/2^n$. From $\|\alpha_k\| \leq \inf \|f(A_k)\|$, we have $\|g_{n-1}(x)\| = \|\alpha_k\| \leq \|\beta_i\| = \|\varphi_k(x)\| = \|g_n(x)\|$. Clearly the sequence $\{g_n\}$ fulfils all the requirements. □

19-6.6. Next lemma links up strong measurability with weak and weak-star measurability respectively. It will be used again at the end of Chapter 25.

19-6.7. **Lemma** A map $f : X \to F$ is measurable if the image of every decent set is separable and there is a subset W of the closed unit ball B of F' such that $\|e\| = \sup_{u \in W} |u(e)|$ for all $e \in F$ and that the function $uf : X \to \mathbb{K}$ is measurable for all $u \in W$.

Proof. Let $g = f - \alpha : X \to H$ be the translate of f by any $\alpha \in H$. We claim that $|g|$ is a measurable function. Indeed, it suffices to show that for each decent set A, the function $|g|\rho_A$ is measurable; or for every $r \in \mathbb{R}$, the set $P = \{x \in A : \|g(x)\| \le r\}$ is measurable. Since $g(A) = f(A) - \alpha$ is separable, the closed vector subspace H generated by $g(A)$ is also separable. Let $\{a_n\}$ be a countable dense subset of H. For each n, the subset Wa_n of $\mathbb{K}$ is separable. There is a countable subset V_n of W such that $V_n a_n$ is dense in Wa_n. Now $V = \bigcup_{n=1}^{\infty} V_n$ is a countable subset of W. Since $v_n \in B$, clearly $P \subset \bigcap_{n\ge 1} Q_n$ where $Q_n = \{x \in A : |v_n g(x)| \le r\}$. Next, suppose $x \in \bigcap_{n \ge 1} Q_n$. For every $\varepsilon > 0$, there is $u \in W$ such that $\|g(x)\| \le |ug(x)| + \varepsilon$. Choose n such that $\|g(x) - a_n\| \le \varepsilon$. Select $v_n \in V_n$ such that $|u(a_n) - v_n(a_n)| \le \varepsilon$. So we get

$$\begin{aligned} |ug(x) - v_n g(x)| &\le |ug(x) - u(a_n)| + |u(a_n) - v_n(a_n)| + |v_n(a_n) - v_n g(x)| \\ &\le \|u\|\ \|g(x) - a_n\| + |u(a_n) - v_n(a_n)| + \|v_n\|\ \|g(x) - a_n\| \le 3\varepsilon. \end{aligned}$$

Hence $\|g(x)\| \le |v_n g(x)| + 4\varepsilon \le r + 4\varepsilon$. Letting $\varepsilon \downarrow 0$, we have $\|g(x)\| \le r$, that is $x \in P$. Therefore $P = \bigcap_{n \ge 1} Q_n$. Because all functions $v_n g = v_n f - v_n \alpha$ are measurable, all sets Q_n are also measurable and so is P. This proves that $|g|\rho_A$ is measurable. For the closed ball $\overline{\mathbb{B}} = \{e \in H : \|e - \alpha\| \le r\}$ in H, the set $A \cap f^{-1}(\overline{\mathbb{B}}) = (|g|\rho_A)^{-1}[0, r]$ is measurable. Therefore, f can be approximated by simple maps on A into H, and hence also into F. Consequently f is simply approximable on every decent set A, i.e. strongly measurable. □

19-6.8. **Theorem** A map $f : X \to F$ is a measurable if the image of every decent set is separable and if f is *weakly measurable* i.e. for every $v \in F'$, the function vf is measurable.

Proof. Apply last lemma when $W = B$ is the closed unit ball of F'. □

19-6.9. **Theorem** A map $f : X \to F'$ is a measurable if the image of every decent set is separable and if f is *weak-star measurable* i.e. for every $e \in F$, the function $x \to f(x)e$ is measurable.

Proof. Let $W = \{J(x) : x \in F,\ \|x\| \le 1\}$ where $J : F \to F''$ is the natural embedding. The result follows immediately from last lemma. □

19-6.10. **Exercise** Let $f : X \to F$ be a measurable map and T a continuous linear map from F into a Banach space G. Prove that the composite map $Tf : X \to G$ is measurable.

19-6.11. **Theorem** A sequential limit f of measurable maps $g_n : X \to F$ is measurable.

Proof. Let A be any decent set. Then $g_n(A)$ is separable. Let D_n be a countable dense subset of $g_n(A)$. Then the set $D = \bigcup_{n=1}^{\infty} D_n$ is countable. Let $k > 0$ and $x \in A$. There is n such that $\|f(x) - g_n(x)\| \leq \frac{1}{2k}$. There is $\alpha_k \in D_n$ with $\|g_n(x) - \alpha_k\| \leq \frac{1}{2k}$. Hence $\|f(x) - \alpha_k\| \leq \frac{1}{k}$. Thus $\alpha_k \in D$ and $\alpha_k \to f(x)$. Hence $f(x) \in \overline{D}$. Since $x \in A$ is arbitrary, $f(A) \subset \overline{D}$. Therefore $f(A)$ is separable. Since $g_n(x) \to f(x)$ for every $x \in X$, we have $vg_n(x) \to vf(x)$ for every $v \in F'$. Since vg_n are measurable functions, so is vf. Therefore f is weakly measurable. Consequently f is strongly measurable. □

19-99. **References and Further Readings** : Xia, Kuttler, Puglisi, Folland, Dudley-89, Diestel-83 and Tucker.

Chapter 20

Integrals of Upper Functions

20-1 Upper Functions

20-1.1. Throughout this chapter, let μ be a *positive* measure on a δ-space $(X, \mathbb{D})$. Unless specified explicitly, let f, f_n, g, g_n be upper functions which are extended positive measurable functions on X into $[0, \infty]$.

20-1.2. The vector space of all decent functions of the form $f = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ for some $\alpha_j \in \mathbb{K}$ and some $A_j \in \mathbb{D}$ is denoted by $\mathfrak{F}$ and their integrals has been defined by $I(f) = \sum_{j=1}^{n} \alpha_j \mu A_j$ where $\mathbb{D}$ is considered as a semiring. If $f \leq g$ are real in $\mathfrak{F}$, then $I(f) \leq I(g)$. Furthermore if $f_n \downarrow 0$ in $\mathfrak{F}$, then $I(f_n) \downarrow 0$. It is easy to show that if $f_n \uparrow f$ in $\mathfrak{F}$, then $I(f_n) \uparrow I(f)$. The following modification allows us to extend the integral to simple functions.

20-1.3. Let f be a positive simple function given by $f = \sum_{j=1}^{n} \alpha_j \rho_{H_j}$ where all α_j are positive (≥ 0) and all H_j are measurable. Hence for each $A \in \mathbb{D}$, the function $f\rho_A = \sum_{j=1}^{n} \alpha_j \rho_{A \cap H_j}$ is a decent function. Define the integral of f by $J(f) = \sup\{I(f\rho_A) : A \in \mathbb{D}\}$.

20-1.4. **Theorem** Let f, f_n, g be positive simple functions.
(a) If f is a positive decent function, then $J(f) = I(f)$.
(b) If $f \leq g$, then $J(f) \leq J(g)$.
(c) If $f_n \uparrow f$, then $J(f_n) \uparrow J(f)$.

Proof. (a) Let $f = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ where $\alpha_j \geq 0$ and $A_j \in \mathbb{D}$. Then $A = \bigcup_{j=1}^{n} A_j$ is in $\mathbb{D}$. Hence $I(f) = I(f\rho_A) \leq J(f)$. Next taking any $B \in \mathbb{D}$, we have $f\rho_B \leq f$ and $J(f) = \sup\{I(f\rho_B) : B \in \mathbb{D}\} \leq I(f)$.
(b) It follows by taking supremum of $I(f\rho_A) \leq I(g\rho_A)$ when A runs over $\mathbb{D}$.
(c) For each $A \in \mathbb{D}$ we have $0 \leq f_n\rho_A \uparrow f\rho_A$ in $\mathfrak{F}$ and hence $I(f_n\rho_A) \uparrow I(f\rho_A)$. The following calculation gives the result:

$$\begin{aligned} J(f) &= \sup_{A \in \mathbb{D}} I(f\rho_A) = \sup_{A \in \mathbb{D}} \lim_{n \to \infty} I(f_n\rho_A) = \sup_{A \in \mathbb{D}} \sup_{n \geq 1} I(f_n\rho_A) \\ &= \sup_{n \geq 1} \sup_{A \in \mathbb{D}} I(f_n\rho_A) = \sup_{n \geq 1} J(f_n) = \lim_{n \to \infty} J(f_n). \end{aligned}$$

□

20-1.5. **Lemma** Let f, g be upper functions. Suppose that $0 \leq f_n \uparrow f$ and $0 \leq g_n \uparrow g$ where f_n, g_n are simple functions.
(a) If $f \leq g$, then $\lim J(f_n) \leq \lim J(g_n)$.
(b) If $f = g$, then $\lim J(f_n) = \lim J(g_n)$.

Proof. Since $J(f_n)$ is a monotonic sequence in $[0, \infty]$, its limit is well defined by the supremum. Observe that $f_m \wedge g_n$ is a simple function. For $n \to \infty$ in $f_m \wedge g_n \leq g_n$ we have $f_m \wedge g_n \uparrow f_m \wedge g = f_m$. Hence we have $J(f_m) = \lim_{n\to\infty} J(f_m \wedge g_n) \leq \lim_{n\to\infty} J(g_n)$. Now part (a) follows by letting $m \to \infty$. Part (b) is obtained by symmetry. □

20-1.6. Let f be an upper function. Choose simple functions f_n with $0 \leq f_n \uparrow f$. The *integral* $\int f d\mu = \lim J(f_n)$ as $n \to \infty$ is well-defined because it is independent of the choice of the sequence $\{f_n\}$. An upper function f is *integrable* if $\int f d\mu < \infty$.

20-1.7. **Theorem** Let f, g be upper functions.
(a) If f is simple, then $\int f d\mu = J(f)$.
(b) If $f \leq g$, then $\int f d\mu \leq \int g d\mu$.
(c) $\int (f+g) d\mu = \int f d\mu + \int g d\mu$.
(d) If $f \leq g$ and if $\int f d\mu < \infty$, then $\int (g-f) d\mu = \int g d\mu - \int f d\mu$.
(e) For every $\alpha \geq 0$, $\int \alpha f d\mu = \alpha \int f d\mu$.
(f) $\int f d\mu = \sup_{A \in \mathbb{D}} \int f \rho_A d\mu$.

Proof. We only prove part (c) while the rest are left as exercises. Let f_n, g_n be simple functions satisfying $0 \leq f_n \uparrow f$ and $0 \leq g_n \uparrow g$. Then $f_n + g_n$ are simple functions satisfying $0 \leq (f_n + g_n) \uparrow (f+g)$. For every n, observe that

$$J(f_n + g_n) = \sup_{A \in \mathbb{D}} I[(f_n + g_n)\rho_A] = \sup_{A \in \mathbb{D}} [I(f_n \rho_A) + I(g_n \rho_A)]$$
$$\leq \sup_{A \in \mathbb{D}} I(f_n \rho_A) + \sup_{A \in \mathbb{D}} I(g_n \rho_A) = J(f_n) + J(g_n) \leq \int f d\mu + \int g d\mu.$$

Letting $n \to \infty$, we have $\int (f+g) d\mu \leq \int f d\mu + \int g d\mu$. If the left hand side of this inequality is ∞, then the equality holds. Assume $\int (f+g) d\mu < \infty$. Since

$$\sup_n \sup_A I(f_n \rho_A) = \int f d\mu \leq \int (f+g) d\mu < \infty,$$

for every $\varepsilon > 0$, we have $\int f d\mu \leq I(f_m \rho_A) + \varepsilon$ for some integer m and $A \in \mathbb{D}$. Similarly $\int g d\mu \leq I(g_n \rho_B) + \varepsilon$ for some integer n and $B \in \mathbb{D}$. Then $D = A \cup B$ is in the ring $\mathbb{D}$. For $f_m \leq f_k$ and $g_n \leq g_k$ where $k = m+n$, we get
$\int f d\mu + \int g d\mu \leq I(f_k \rho_D) + I(g_k \rho_D) + 2\varepsilon = I[(f_k + g_k)\rho_D] + 2\varepsilon \leq \int (f+g) d\mu + 2\varepsilon$.
The result follows by letting $\varepsilon \downarrow 0$. □

20-1.8. **Theorem** Let f_n be upper functions. If $f_n \uparrow f$, then f is also an upper function with $\int f_n d\mu \uparrow \int f d\mu$. Consequently if $\sup \int f_n d\mu < \infty$, then f is integrable.

Proof. Since f is measurable extended positive function, it is an upper function. For each n, there are positive simple functions $g_{mn} \uparrow f_n$ as $m \to \infty$. Define $h_m = g_{m1} \vee g_{m2} \vee \cdots \vee g_{mm}$. Then $h_m \leq h_{m+1}$ and $g_{mn} \leq h_m \leq f_m$ for all $n \leq m$. For $m \to \infty$, we have $f_n \leq \lim h_m \leq f$. Next letting $n \to \infty$, we get $h_m \uparrow f$. Hence f is an upper function approximated by the simple functions h_m. For all $n \leq m$, we get $\int g_{mn} d\mu \leq \int h_m d\mu \leq \int f_m d\mu$, or $J(g_{mn}) \leq J(h_m) \leq \int f_m d\mu$. As $m \to \infty$, $\int f_n d\mu \leq \int f d\mu \leq \lim \int f_m d\mu$. Finally letting $n \to \infty$ we obtain $\int f d\mu = \lim \int f_n d\mu$. □

20-1.9. **Corollary** $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$ for all upper functions f_n.

Proof. All $g_n = \sum_{j=1}^{n} f_j$ and $g = \sum_{j=1}^{\infty} f_j$ are upper functions. Furthermore,

$$\int \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \int g d\mu = \lim_{n\to\infty} \int g_n d\mu$$
$$= \lim_{n\to\infty} \int \left(\sum_{j=1}^{n} f_j\right) d\mu = \lim_{n\to\infty} \sum_{j=1}^{n} \int f_j d\mu = \sum_{j=1}^{\infty} \int f_j d\mu. \qquad \square$$

20-1.10. **Exercise** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 0$ if $x \leq 1$ and $f(x) = \frac{1}{n(n+1)}$ if $n < x \leq n+1$ for each integer $n \geq 1$. Show that f is an upper function and find its Lebesgue integral $\int f d\lambda$.

20-1.11. **Exercise** Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions given by $f(x) = n$, $g(x) = \frac{1}{n+2}$ if $\frac{1}{n+1} < x \leq \frac{1}{n}$ for some integer $n \geq 1$ and $f(x) = g(x) = 0$ for all other x. Show that f, g are upper functions. Find their Lebesgue integrals $\int f d\lambda$ and $\int g d\lambda$.

20-1.12. **Exercise** Let $f : \mathbb{R} \to [0, \infty]$ be given by $f(x) = \infty$ for rational x and $f(x) = 0$ for irrational x. Show that f is an upper function. Find its integral.

20-1.13. **Fatou's Lemma** For all upper functions f_n, we have

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. Clearly, $g_n = \inf\{f_k : k \geq n\}$ are upper functions satisfying $g_n \uparrow \liminf f_n$. Consequently, we have

$$\int \liminf_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int g_n d\mu = \liminf_{n\to\infty} \int g_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu. \qquad \square$$

20-1.14. **Exercise** Applying Fatou's Lemma to the counting measure on a set of two elements, show that $\liminf \alpha_n + \liminf \beta_n \leq \liminf(\alpha_n + \beta_n)$ for all $\alpha_n, \beta_n \geq 0$. Let $\alpha_n = 1 + (-1)^n$ and $\beta_n = 1 + (-1)^{n+1}$. Show that the equality of Fatou's Lemma may fail.

20-2 Almost Everywhere

20-2.1. Let μ be a *positive* measure on a δ-space $(X, \mathbb{D})$. For every measurable set H, the integral $\mu H = \int \rho_H d\mu$ of simple function ρ_H is well-defined although it may be ∞. It provides an extension of μ from decent sets to all measurable sets. Properties of this extension follow immediately from integrals of upper functions. Note that positive measures are defined on measurable sets while real, complex and later vector measures are defined on decent sets.

20-2.2. **Theorem** Let μ be a *positive* measure and H, K, H_n measurable sets.
(a) $\mu H = \sup\{\mu A : H \supset A \in \mathbb{D}\}$, inner regularity.
(b) If $H \subset K$, then $\mu H \leq \mu K$.
(c) If $H_n \uparrow K$, then $\mu H_n \uparrow \mu K$.
(d) $\mu(\bigcup_{n=1}^{\infty} H_n) \leq \sum_{n=1}^{\infty} \mu H_n$, countable subadditivity.
(e) If $\{H_n\}$ are disjoint, then $\mu(\bigcup_{n=1}^{\infty} H_n) = \sum_{n=1}^{\infty} \mu H_n$, countable additivity.
(f) If $\mu H_n < \infty$ for some n and if $H_n \downarrow K$, then $\mu H_n \downarrow \mu K$.

20-2.3. **Example** For all measurable sets H, K we have

$$\begin{aligned}\mu(H \cap K) &= \sup\{\mu(A \cap K) : H \supset A \in \mathbb{D}\} \\ &= \sup\{\mu(A \cap B) : H \supset A \in \mathbb{D},\ K \supset B \in \mathbb{D}\}.\end{aligned}$$

Proof. Let $s = \sup\{\mu(A \cap K) : H \supset A \in \mathbb{D}\}$. If $H \supset A \in D$, then we have $A \cap K \subset H \cap K$ and hence $\mu(A \cap K) \leq \mu(H \cap K)$. Since A is arbitrary, we obtain $s \leq \mu(H \cap K)$. If $s = \infty$, then $s = \mu(H \cap K)$. Suppose $s < \infty$. For every $\varepsilon > 0$, choose $D \in \mathbb{D}$ such that $D \subset H \cap K$ and $\mu(H \cap K) \leq \nu D + \varepsilon$. Hence $D \subset H$ and $\mu(H \cap K) \leq \mu(D \cap K) + \varepsilon \leq s + \varepsilon$. Letting $\varepsilon \downarrow 0$, we get $\mu(H \cap K) \leq s$. This proves the first formula. The second one is left as an exercise. □

20-2.4. **Exercise** Let $H_n = (n, \infty)$ and $K = \emptyset$ be subsets of the real line. Show that $H_n \downarrow K$ but $\lambda H_n \downarrow \lambda K$ is false where λ denotes the Lebesgue measure.

20-2.5. **Example** Let $X = \{x, y, z\}$, $A = \{x, y\}$, $B = \{z\}$, $\mathcal{S} = \{\emptyset, A\}$ and $\mu\emptyset = 0$, $\mu A = 1$. Obviously μ is a measure on the semiring $\mathcal{S}$. The δ-ring $\mathbb{D}$

generated by $\mathcal{S}$ is $\mathcal{S}$ itself. The family of $\mathbb{D}$-measurable sets is $\mathbb{M} = \{\emptyset, A, B, X\}$. It is also the family of all μ^*-measurable. Clearly $\mu^*\{x\} = \mu^*\{y\} = \mu^* A = 1$ and $\mu^* B = \infty$ but $\mu B = 0$ by regularity. The function f defined by $f(x) = 1$ and $f(y) = f(z) = -1$ is not measurable but $|f|$ is. The functions φ, ξ on $\mathbb{M}$ defined by $\varphi\emptyset = \xi\emptyset = 0$, $\varphi A = \xi A = \xi B = \varphi X = 1$ but $\varphi B = 0$, $\xi X = 2$ are different extensions of μ to measures on $\mathbb{M}$.

20-2.6. A measurable set N is said to be ***null*** if $\mu N = 0$. By countable subadditivity, countable unions of null sets are null. Clearly a measurable set N is null iff $\mu(A \cap N) = 0$ for every $A \in \mathbb{D}$.

20-2.7. **Exercise** Prove that every singleton of the real line is a null set with respect to Lebesgue measure. Prove that every countable subset of the real line is a null set but the set of all rational numbers is not a decent set.

20-2.8. **Exercise** Give an example to show that a subset of a null set need not be measurable and hence not a null set in this book. See §23-4.6 on completion.

20-2.9. A proposition $p(x)$ about each x in X said to be true *μ-almost everywhere* (briefly μ-ae) if there exists a null set N such that $p(x)$ is true for each $x \in X \setminus N$. Clearly if $\{p_n(x)\}$ is a countable family of propositions which are true μ-ae, then there is a null set N such that all $p_n(x)$ are true for every $x \in X \setminus N$. The following convention will be used: $0\infty = \infty 0 = 0$, $\alpha\infty = \infty\alpha = \infty$ if $\alpha > 0$, and $\alpha\infty = \infty\alpha = -\infty$ if $\alpha < 0$. However, $\infty - \infty$ is not defined.

20-2.10. **Exercise** Construct two functions f, g on X such that f is measurable, g is not measurable but $f = g$ almost everywhere.

20-2.11. **Theorem** $\int f d\mu = 0$ iff $f = 0$, μ-ae for every upper function f.

Proof. ($\Rightarrow$) For each $n \geq 1$, the set $H_n = \{x \in X : f(x) > 1/n\}$ is measurable. Integrating $\rho_{H_n} \leq nf$, we have $0 \leq \mu H_n = \int \rho_{H_n} d\mu \leq n \int f d\mu = 0$. Hence each H_n is a null set. Consequently $N = \bigcup_{n=1}^{\infty} H_n$ is also a null set. Since $f = 0$ on $X \setminus N$, we have $f = 0$, μ-ae.

($\Leftarrow$) Assume $f = 0$, μ-ae. Then the set $H = \{x \in X : f(x) > 0\}$ is null. All $g_n = n\rho_H$ are simple functions satisfying $0 \leq g_n \uparrow g = \infty\rho_H$. Hence g is an upper function. Since $0 \leq f \leq g$, we have

$$0 \leq \int f d\mu \leq \int g d\mu = \lim \int g_n d\mu = \lim n\mu H = 0.$$

Consequently, $\int f d\mu = 0$. □

20-2.12. **Exercise** Prove that $X \setminus \cup \mathbb{D}$ is a null set.

20-2.13. **Exercise** Prove the following statements for upper functions.
(a) Let $f \le g$, μ-ae. Then $\int f d\mu \le \int g d\mu$. If g is integrable, then so is f.
(b) Let $f = g$, μ-ae. Then $\int f d\mu = \int g d\mu$. If f is integrable, then so is g.

20-2.14. Since $f = g$, μ-ae is an equivalence relation, for a *specific* measure we are dealing with equivalence classes. For example, suppose that $N = f^{-1}(0)$ is a null set and $g = f + \rho_N$. Then $f = g$ and $1/g$ is defined everywhere. We interpret $1/f$ as the equivalence class containing $1/g$. See §22-3.4.

20-3 Seeds of the Theory

20-3.1. This section contains the most primitive form of the important results of the whole theory.

20-3.2. **Positive Monotone Convergence Theorem** Let f_n, f be upper functions. If $f_n \uparrow f$, μ-ae; then $\int f_n d\mu \uparrow \int f d\mu$. Moreover if $\sup \int f_n d\mu < \infty$, then f is integrable.

Proof. Let N be a null set such that $f_n \uparrow f$ on $M = X \setminus N$. Then $f_n \rho_M \uparrow f \rho_M$ everywhere. Also $f = f\rho_M$ and $f_n = f_n \rho_M$, μ-ae. Hence $\int f_n d\mu = \int f_n \rho_M d\mu \uparrow \int f \rho_M d\mu = \int f d\mu$. Furthermore $\sup \int f_n \rho_M d\mu = \sup \int f_n d\mu < \infty$. Hence $f\rho_M$ is integrable. Consequently f is integrable. This type of proofs of reducing μ-ae to everywhere will be skipped starting from next lemma. □

20-3.3. **Lemma** Let f_n, g_n, f, g be upper functions. Suppose that for all n, $f_n \le g_n$, $f_n \to f$ and $g_n \to g$, μ-ae. If $\int g_n d\mu \to \int g d\mu < \infty$, then all f_n, f are integrable and $\int f_n d\mu \to \int f d\mu$.

Proof. Without loss of generality, we may assume that all given conditions are true pointwise everywhere. Letting $n \to \infty$ in $f_n \le g_n$, we have $f \le g$ and hence f is integrable. Since each $g_n - f_n$ is an upper function, Fatou's lemma implies

$$\int \liminf_{n\to\infty}(g_n - f_n)d\mu \le \liminf_{n\to\infty} \int (g_n - f_n)d\mu \le \limsup_{n\to\infty} \int g_n d\mu + \liminf_{n\to\infty} \int -f_n d\mu,$$

i.e.
$$\int (g-f)d\mu \le \int g d\mu - \limsup_{n\to\infty} \int f_n d\mu.$$

Since $0 \le g - f \le g$, we have $\int (g-f)d\mu \le \int g d\mu < \infty$. Hence we obtain

$$\limsup_{n\to\infty} \int f_n d\mu \le \int g d\mu - \int (g-f) d\mu \le \int f d\mu.$$

Fatou's lemma applied to $\{f_n\}$ gives $\int f d\mu = \int \liminf_{n\to\infty} f_n d\mu \le \liminf_{n\to\infty} \int f_n d\mu$. Therefore $\int f d\mu = \liminf_{n\to\infty} \int f_n d\mu = \limsup_{n\to\infty} \int f_n d\mu$. □

20-3.4. **Positive Dominated Convergence Theorem** Let f_n, f, g be upper functions such that $f_n \to f$ and $f_n \le g$, μ-ae for all n. If g is integrable, then all f_n, f are integrable, $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$ and $\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$.

Proof. Letting all $g_n = g$, we have the first equality. From $0 \le |f_n - f| \le 2g$, μ-ae; we obtain $\int |f_n - f| d\mu \to \int 0 d\mu = 0$. □

20-3.5. Let μ be a positive measure on a semiring $\mathcal{S}$ over X. The extensions to decent sets and measurable sets successively are also denoted by μ. A measurable set M is said to be ***integrable*** if $\mu M < \infty$. Since every decent set is covered by a finite union of sets in $\mathcal{S}$, all decent sets are integrable.

20-3.6. **Theorem** Let M be an integrable set. For every $\varepsilon > 0$, there is a finite disjoint union B of sets in $\mathcal{S}$ such that $\mu(M \triangle B) \le \varepsilon$.

Proof. By §20-2.2a, choose $D \in \mathbb{D}$ such that $D \subset N$ and $\mu(M \setminus D) \le \frac{1}{2}\varepsilon$. By §18-3.4, there is a finite disjoint union B of sets in $\mathcal{S}$ such that $\mu(D \triangle B) \le \frac{1}{2}\varepsilon$. Then we have $\mu(M \triangle B) = \int |\rho_M - \rho_B| d\mu \le \int |\rho_M - \rho_D| d\mu + \int |\rho_D - \rho_B| d\mu \le \mu(M \setminus D) + \mu(D \triangle B) \le \varepsilon$. □

20-3.7. **Theorem** Let $\mathbb{D}$ be the δ-ring generated by a family $\mathcal{K}$ of subsets of X. Then a measurable set M is null if $\mu(B \cap M) = 0$ for each $B \in \mathcal{K}$.

Proof. Let $A \in \mathbb{D}$ be a decent set. There are $B_1, B_2, \cdots, B_n \in \mathcal{K}$ such that $A \subset \bigcup_{j=1}^n B_j$. Hence $0 \le \mu(A \cap M) \le \sum_{j=1}^n \mu(B_j \cap M) = 0$. Therefore M is a null set. □

20-4 Sigma Finiteness

20-4.1. Let μ be a *positive* measure on a δ-space $(X, \mathbb{D})$. A subset M of X is said to be *σ-finite* for μ, or μ-σ-finite if there are measurable sets H_n such that $M = \bigcup_{n=1}^\infty H_n$ and $\mu H_n < \infty$ for all n. A measure μ is σ-finite if X is σ-finite.

20-4.2. **Example** The Lebesgue measure on $\mathbb{R}$ is σ-finite but the counting measure on $\mathbb{R}$ is *not*.

20-4.3. **Theorem** Let f be an integrable upper function.
(a) The set $A = \{x \in X : f(x) = \infty\}$ is a null set. As a result, every integrable upper function is finite-valued almost everywhere.
(b) The set $B = \{x \in X : f(x) > 0\}$ is σ-finite.

Proof. Since f is an upper function, ρ_A is a simple function. For each $n \geq 1$, integrating $\rho_A \leq (1/n)f$, we have $0 \leq \mu A \leq \int \rho_A d\mu \leq (1/n)\int f d\mu \to 0$ as $n \to \infty$ because f is integrable. Therefore $\mu A = 0$ as required. Next since the set $B_n = \{x \in X : f(x) > 1/n\}$ is measurable, integrating $\rho_{B_n} \leq nf$ gives $\mu B_n = \int \rho_{B_n} d\mu \leq n \int f d\mu < \infty$. Consequently $B = \bigcup_{n=1}^{\infty} B_n$ is σ-finite. □

20-4.4. **Exercise** Prove that for measurable sets M_n, if $\sum_{n=1}^{\infty} \mu M_n < \infty$, then the set of points belonging to infinitely many M_n is null.

20-4.5. **Lemma** If M is a null set and f an upper function, then $\int f \rho_M d\mu = 0$.

Proof. Let f_n be simple functions satisfying $0 \leq f_n \uparrow f$. Then we have $0 \leq f_n \rho_M \uparrow f \rho_M$ and all $f_n \rho_M$ are simple functions. Write $f_n = \sum_{i=1}^{m} \alpha_i \rho_{H_i}$ where $\alpha_i \geq 0$ and all H_i are measurable. Thus

$$0 \leq \int f_n \rho_M d\mu = \sum_{i=1}^{m} \alpha_i \int \rho_{H_i \cap M} d\mu = \sum_{i=1}^{m} \alpha_i \mu(H_i \cap M) = 0.$$

Therefore $\int f \rho_M d\mu = \lim \int f_n \rho_M d\mu = 0$. □

20-4.6. **Theorem** For upper functions f, g; if $f = g$, μ-ae; then $\int f d\mu = \int g d\mu$.

Proof. Let N be a null set such that $f(x) = g(x)$ for each $x \in X \setminus N$. Then $f \rho_{X \setminus N} = g \rho_{X \setminus N}$ everywhere. Consequently,

$$\begin{aligned} \int f d\mu &= \int f \rho_{X\setminus N} d\mu + \int f \rho_N d\mu = \int f \rho_{X \setminus N} d\mu \\ &= \int g \rho_{X \setminus N} d\mu = \int g \rho_{X\setminus N} d\mu + \int g \rho_N d\mu = \int g d\mu. \end{aligned}$$ □

20-4.7. **Corollary** (a) If $f \leq g$, μ-ae; then $\int f d\mu \leq \int g d\mu$. Furthermore if g is integrable, then so is f.
(b) If $f_n \uparrow g$, μ-ae; then $\int f_n d\mu \uparrow \int g d\mu$.

20-4.8. **Lemma** For every σ-finite set H there exist decent sets $A_m \subset A_{m+1}$ and a null set N such that $H = \bigcup_{m=1}^{\infty} A_m \cup N$.

Proof. Firstly assume $\mu H < \infty$. Since $\mu H = \sup\{\mu A : H \supset A \in \mathbb{D}\}$, for each i there is $A_i \in \mathbb{D}$ such that $A_i \subset H$ and $\mu H \leq \mu A_i + 1/i$. Replacing A_i by $A_1 \cup A_2 \cup \cdots \cup A_i$, we may assume $A_i \subset A_{i+1}$. Let $M = \bigcup_{i=1}^{\infty} A_i$ and $N = H \setminus M$. Then $H = M \cup N$. Since $A_i \uparrow M$ we obtain $\mu H - 1/i \leq \mu A_i \uparrow \mu M \leq \mu H$. Letting $i \to \infty$, we have $\mu M = \mu H$. Hence from $\mu H = \mu M + \mu N$, we get

$\mu N = 0$. Therefore the lemma is true for a measurable set with finite measure. In general, write $H = \bigcup_{i=1}^{\infty} H_i$ where H_i is measurable satisfying $\mu H_i < \infty$. There exist $A_{ij} \in \mathbb{D}$ and null set N_i such that $H_i = \bigcup_{j=1}^{\infty} A_{ij} \cup N_i$. Clearly $A_m = \bigcup_{i,j=1}^{m} A_{ij}$ and $N = \bigcup_{i=1}^{\infty} N_i$ are the required sets. □

20-4.9. **Theorem** For every σ-finite measurable set M, there are *disjoint* decent sets $H_1, H_2, \cdots$ and a μ-null set H_0 such that $M = \bigcup_{n=0}^{\infty} H_n$.

Proof. Write $M = \bigcup_{m=1}^{\infty} A_m \cup N$ where $A_m \subset A_{m+1}$ are decent sets and N a μ-null set. Let $H_1 = A_1$ and $H_n = A_n \backslash A_{n-1}, \forall\, n > 1$. Define $H_0 = N \backslash \bigcup_{n=1}^{\infty} H_n$. It is easy to verify that these are the required sets. □

20-4.10. **Lemma** For every upper function f and every σ-finite set H, there are decent functions f_n such that $0 \le f_n \le f_{n+1} \le f\rho_H$; $f_n \uparrow f\rho_H$, μ-ae and $\int f_n d\mu \uparrow \int f\rho_H d\mu$.

Proof. Let g_n be simple functions satisfying $0 \le g_n \uparrow f$. There are decent sets A_n and a null set N such that $H = \bigcup_{n=1}^{\infty} A_n \cup N$ and $A_n \subset A_{n+1}$. Define $f_n = g_n \rho_{A_n}$. Then it is easy to verify all required conditions. □

20-4.11. **Theorem** For every integrable upper function f, there are decent functions f_n such that $0 \le f_n \le f_{n+1} \le f$; $f_n \uparrow f$, μ-ae and $\int f_n d\mu \uparrow \int f d\mu$.

Proof. Apply last lemma to the σ-finite set $H = \{x \in X : f(x) > 0\}$. □

20-4.12. **Exercise** Let μ be the point measure on the semiring of singletons of $\mathbb{R}$. Show that $\mathbb{R}$ is σ-finite but it is not a countable union of decent sets. Also show that the constant function $f = 1$ is an integrable upper function but there is no sequence of decent functions f_n such that $0 \le f_n \uparrow f$ pointwise.

20-5 Comparison of Two Positive Measures

20-5.1. This section prepares ground to deal with the sum of two measures in later chapters. Let μ, ν be *positive* measures on a δ-space $(X, \mathbb{D})$.

20-5.2. **Theorem** Suppose that $\mu \ge \nu \ge 0$.
(a) For every upper function f, we have $\int f d\mu \ge \int f d\nu$.
(b) If f is a μ-integrable upper function, then f is also ν-integrable.
(c) Every μ-null set is ν-null.
(d) Every μ-σ-finite set is ν-σ-finite.

Proof. Let f_n be simple functions such that $0 \le f_n \uparrow f$. Write $f_n = \sum_{i=1}^{m} \alpha_i \rho_{H_i}$ where $\alpha_i \ge 0$ and all H_i are measurable. Then for each decent set A, we have $\mu(H_i \cap A) \geq \nu(H_i \cap A)$, i.e. $I_\mu(f_n\rho_A) \geq I_\nu(f_n\rho_A)$, or $J_\mu(f_n) \geq J_\nu(f_n)$. Letting $n \to \infty$, we have part (a). The others are trivial. □

20-5.3. **Theorem** Let μ, ν be arbitrary positive measures. Then for every upper function f, we have $\int f d(\mu + \nu) = \int f d\mu + \int f d\nu$. Furthermore f is integrable for both μ and ν iff f is integrable for $\mu + \nu$.

Proof. Let f_n be simple functions such that $0 \leq f_n \uparrow f$. Write $f_n = \sum_{i=1}^m \alpha_i \rho_{H_i}$ where $\alpha_i \geq 0$ and all H_i are measurable. Then for each decent set A, we have

$$I_{\mu+\nu}(f_n\rho_A) = \sum_{i=1}^m \alpha_i(\mu+\nu)(A \cap H_i) = \sum_{i=1}^m \alpha_i\mu(A \cap H_i) + \sum_{i=1}^m \alpha_i\nu(A \cap H_i)$$
$$= I_\mu(f_n\rho_A) + I_\nu(f_n\rho_A) \leq \int f d\mu + \int f d\nu.$$

Taking supremum over n and $A \in \mathbb{D}$, we get $\int f d(\mu + \nu) \leq \int f d\mu + \int f d\nu$. If the left hand side is ∞, then equality holds and the result is proved. Assume $\int f d(\mu + \nu) < \infty$. Since $0 \leq \mu \leq \mu + \nu$, we have $\int f d\mu < \infty$. For every $\varepsilon > 0$, there exist an integer m and $A \in \mathbb{D}$ such that $\int f d\mu \leq I_\mu(f_m\rho_A) + \varepsilon$. Similarly there exist n and $B \in \mathbb{D}$ such that $\int f d\nu \leq I_\nu(f_n\rho_B) + \varepsilon$. Let $k = m + n$ and $D = A \cup B$. Then $D \in \mathbb{D}$ and

$$\int f d\mu + \int f d\nu \leq I_\mu(f_m\rho_A) + I_\nu(f_n\rho_B) + 2\varepsilon \leq I_\mu(f_k\rho_D) + I_\nu(f_k\rho_D) + 2\varepsilon$$
$$= I_{\mu+\nu}(f_k\rho_D) + 2\varepsilon \leq \int f d(\mu + \nu) + 2\varepsilon.$$

by the first part of this proof. Letting $\varepsilon \downarrow 0$, we have the required equality. Clearly $\int f d(\mu+\nu)$ is finite iff both $\int f d\mu$ and $\int f d\nu$ are finite. This completes the proof. □

20-5.4. **Exercise** Let μ, ν be *positive* measures and H a measurable set. Prove that $(\mu + \nu)(H) = \mu H + \nu H$. Also show that if $\mu \leq \nu$, then $\mu H \leq \nu H$.

20-5.5. **Exercise** Let $\mathcal{S}$ be a semiring over a set X and μ, ν positive measures on $\mathcal{S}$. Their extensions to decent sets are also denoted by μ, ν respectively. Prove that if $\mu A \leq \nu A$ for every $A \in \mathcal{S}$, then for every upper function f we have $\int f d\mu \leq \int f d\nu$.

20-5.6. We conclude this chapter with another reason why we prefer to work with δ-rings. Let μ be a vector measure on a δ-space $(X, \mathbb{D})$ into a Banach space E. Then μ is said to be *bounded* if $\sup_{A\in\mathbb{D}} |\mu|(A) < \infty$. By §20-2.2a, it is equivalent to $|\mu|(X) < \infty$. A ring $\mathcal{R}$ over X is called a *σ-ring* if it is closed under countable unions. Clearly, every σ-ring is a δ-ring.

20-5.7. **Theorem** Every vector measure μ on a σ-ring $\mathcal{R}$ is bounded. It is this result that we decided to abandon the framework of σ-rings and σ-algebras.

Proof. Suppose to the contrary that for each integer n there is $A_n \in \mathcal{R}$ such that $|\mu|(A_n) > n$. Because $\mathcal{R}$ is a σ-ring, $B = \bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{R}$. Since the variation $|\mu|$ is finite valued, we have $|\mu|(A_n) \leq |\mu|(B) < \infty$. Letting $n \to \infty$, we obtain a contradiction. $\square$

20-99. **References and Further Readings** : Chae, Yannelis, Rao-72, Albeverio, Antoine, Manoukain, Muldowney, Nelson and Lee.

Chapter 21

Vector Integrals

21-1 Extension to Integrable Sets

21-1.1. Let $\mathcal{S}$ be a semiring over a set X, E a Banach space and $\mu : \mathcal{S} \to E$ a vector measure, that is a countably additive charge of finite variation. Then its variation $|\mu| : \mathcal{S} \to \mathbb{R}$ is a positive measure which has an extension to the δ-ring of decent sets generated by $\mathcal{S}$. The integral $\int f d|\mu|$ is defined for every upper function f although not yet for any negative function. In particular the *positive* measure $|\mu|(M)$ *has been* defined for every *measurable* set M. A measurable set M is said to be *μ-integrable* if $|\mu|(M) < \infty$. At this moment, μ has been extended only to the ring $\mathcal{R}$ generated by $\mathcal{S}$ through simple algebraic method. In this section, we shall extend it to the δ-ring $\mathcal{K}$ of μ-integrable sets.

21-1.2. We need the following trivial side track. Let H be a vector space. A function $f \to \|f\|$ from H into $\mathbb{R}$ is called a *seminorm* on H if for all $f, g \in E$, we have $\|f\| \geq 0$, $\|0\| = 0$, $\|f + g\| \leq \|f\| + \|g\|$ and $\|\beta f\| = |\beta| \, \|f\|$ for every $\beta \in \mathbb{K}$. A seminorm with the non-degenerate condition is a norm. A vector space together with a given seminorm is called a *seminormed space*.

21-1.3. **Exercise** Let H be a seminormed space. Prove that the set N of vectors $h \in H$ with $\|h\| = 0$ is a vector subspace. Let $q : H \to H/N$ be the quotient map and $\|q(h)\| = \inf\{\|h + k\| : k \in N\}$ the quotient norm. Prove that $\|q(h)\| = \|h\|$ is a norm on the quotient space. Topological properties of H is referred to the quotient space H/N without explicit specification. For example, a sequence $\{f_n\}$ in H is said to *converge* to f if $q(f_n) \to q(f)$ in H/N. Prove that if $f_n \to f$ and $f_n \to h$, then $q(f) = q(h)$.

21-1.4. **Exercise** Let G be a dense subspace of a seminorm space H and E a Banach space. A linear map φ from G into E is compatible with N if $q(f) = q(h)$ implies $\varphi(f) = \varphi(h)$. Prove that every compatible continuous linear map $\varphi : G \to E$ has a unique continuous linear extension ξ over H compatible with N. Furthermore we have $\|\xi\| = \|\varphi\|$.

21-1.5. In measure and integration, the vector subspace N normally consists of measurable functions which are zero almost everywhere. Also $q(f) = q(h)$ if $f = h$ almost everywhere.

21-1.6. **Theorem** Every vector measure $\mu : \mathcal{S} \to E$ has an extension over the δ-ring $\mathcal{K}$ of integrable sets. This new measure is also denoted by μ for convenience. In particular, μD is defined for every decent set D.

Proof. Note that μ has a unique extension to $\mathcal{R}$ which is our starting point. Let $\mathcal{F}(\mathcal{K})$ denote the vector space of all integrable simple functions and $\mathcal{K}(\mathcal{R})$ all step functions. For every $f \in \mathcal{F}(\mathcal{K})$, $\|f\|_1 = \int |f|\, d|\mu|$ is well-defined. It is easy to show that $f \to \|f\|_1$ is a seminorm on $\mathcal{F}(\mathcal{K})$. We claim that $\mathcal{F}(\mathcal{R})$ is a *dense* subspace of $\mathcal{F}(\mathcal{K})$. Let $f \in \mathcal{F}(\mathcal{K})$ and $\varepsilon > 0$ be given. Write $f = \sum_{j=1}^n \alpha_j \rho_{D_j}$ with $\alpha_j \in \mathbb{K}$ and $D_j \in \mathcal{K}$. Select $0 < \delta < \varepsilon/(1 + \sum_{j=1}^n |\alpha_j|)$. Choose $A_j \in \mathcal{R}$ such that $|\mu|(D_j \triangle A_j) \le \delta$. Define $g = \sum_{j=1}^n \alpha_j \rho_{A_j} \in \mathcal{F}(\mathcal{R})$. Then

$$\begin{aligned}\|f - g\|_1 &\le \sum_{j=1}^n |\alpha_j| \int |\rho_{D_j} - \rho_{A_j}|\, d|\mu| \\ &\le \sum_{j=1}^n |\alpha_j|\; |\mu|(D_j \triangle A_j) \le \sum_{j=1}^n |\alpha_j| \delta \le \varepsilon.\end{aligned}$$

Therefore $\mathcal{F}(\mathcal{R})$ is dense in $\mathcal{F}(\mathcal{K})$. Now suppose $f = \sum_{j=1}^n \alpha_j \rho_{A_j}$ with $\alpha_j \in \mathbb{K}$ and $A_j \in \mathcal{S}$. Then $I_\mu(f) = \sum_{j=1}^n \alpha_j \mu A_j$ is well-defined. It follows from $\|I_\mu(f)\| \le I_{|\mu|}(|f|) = \int |f|\, d|\mu| = \|f\|_1$ that the map $I_\mu : \mathcal{F}(\mathcal{R}) = \mathcal{F}(\mathcal{S}) \to E$ is continuous linear. It has a continuous linear extension $f \to \int f d\mu$ over $\mathcal{F}(\mathcal{K})$. Clearly $D \in \mathcal{K} \to \int \rho_D d\mu$ is a required extension. □

21-1.7. **Theorem** The extension of variation is the variation of extension.

Proof. Let μ be an E-measure on semiring $\mathcal{S}$ and ν its extension to the δ-ring $\mathbb{D}$ generated by $\mathcal{S}$. Suppose that $|\mu|$ denotes the variation of μ on $\mathcal{S}$ and $|\nu|$ the variation of ν on $\mathbb{D}$. Assume that π is the extension of $|\mu|$ over $\mathbb{D}$. We want to prove that $\pi = |\nu|$ on $\mathbb{D}$. By uniqueness of extension, it suffices to show that $\pi = |\nu|$ on $\mathcal{S}$. Take any $A \in \mathcal{S}$. Let $\mathcal{S}(A), \mathbb{D}(A)$ denote the families of partitions of A by sets in $\mathcal{S}, \mathbb{D}$ respectively. Since $\mathcal{S}(A) \subset \mathbb{D}(A)$, it is obvious that $|\mu|A \le |\nu|A$. Next, let $\varepsilon > 0$ be given. Choose $D(A) = \{B_1, \cdots, B_n\}$ in $\mathbb{D}(A)$ such that $|\nu|A - \varepsilon \le \sum_{j=1}^n \|\nu B_j\|$. Working with ν, for each j there are disjoint $C_{ij} \in \mathcal{S}$ such that $|\nu|(B_j \triangle C_j) \le \varepsilon/n$ where $C_j = \bigcup_{i=1}^{k(j)} C_{ij}$. From

$$\begin{aligned}\left\|\nu B_j - \sum_i \nu C_{ij}\right\| &\le \left\|\int (\rho_{B_j} - \rho_{C_j}) d\nu\right\| \\ &\le \int |\rho_{B_j} - \rho_{C_j}|\, d|\nu| = |\nu|(B_j \triangle C_j) \le \varepsilon/n\end{aligned}$$

we have

$$\sum_{j=1}^{n} \|\nu B_j\| \leq \sum_{j=1}^{n} \left\|\sum_{i=1}^{k(j)} \nu C_{ij}\right\| + \varepsilon$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{k(j)} \|\mu C_{ij}\| + \varepsilon \leq \sum_{B \in S(A)} \|\mu B_j\| + \varepsilon \leq |\mu| A + \varepsilon$$

where $S(A) = \{C_{ij} : 1 \leq j \leq n,\ 1 \leq i \leq k_j\}$ is obviously a partition of A by sets in $\mathcal{S}$. Thus $|\nu| A \leq |\mu| A + 2\varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $|\nu| A \leq |\mu| A$. Hence $|\nu| = |\mu| = \pi$ on $\mathcal{S}$. Therefore $|\nu| = \pi$ on $\mathbb{D}$. As a result, we may identify $|\mu|$, $|\nu|$ and π. □

21-1.8. **Exercise** Show that integrable sets are σ-finite.

21-2 Integrals of Vector Maps

21-2.1. We shall develop a theory of vector integration based on the expectation of Dominated Convergence Theorem. Let E, F, FE be Banach spaces with an admissible bilinear map $F \times E \to FE$; $1 \leq p < \infty$ and q its conjugate index given by $\frac{1}{p} + \frac{1}{q} = 1$.

21-2.2. Let μ be a vector measure on a δ-space $(X, \mathbb{D})$ into E. Then the variation $|\mu|$ is a positive measure on $\mathbb{D}$. The vector measure μ has also been extended to the δ-ring $\mathcal{K}$ of all μ-integrable sets. A $\mathcal{K}$-step F-map $f : X \to F$ is of the form $f = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ where $\alpha_j \in F$ and $A_j \in \mathcal{K}$. The integral $\int f d\mu = \sum_{j=1}^{n} \alpha_j \mu A_j \in FE$ has been defined in §17-2.10. Clearly we have $\|\int f d\mu\| \leq \int |f|\, d|\mu|$ as in §17-4.10.

21-2.3. For every measurable map $f : X \to F$, $|f|^p$ is an upper function. Hence the *L_p-norm* $\|f\|_p = \left(\int |f|^p d|\mu|\right)^{1/p}$ is well-defined. It is finite iff $|f|^p$ is $|\mu|$-integrable. The *$F - L_p$-space* for μ is the set $L_p(X, \mathbb{D}, E, \mu, F)$ of all measurable maps $f : X \to F$ such that $|f|^p$ is $|\mu|$-integrable. Drop F if it is $\mathbb{K}$ and also drop $X, \mathbb{D}, E, \mu$ if there is no ambiguity. Write $L_p^r(\mu)$ for $F = \mathbb{R}$ and $L_p^+(\mu) = \{f \in L_p^r(\mu) : f \geq 0\}$. Members of $L_1(\mu, F)$ are called *μ-integrable* or simply *integrable* maps. Clearly, a measurable map $f : X \to E$ is integrable iff the upper function $|f|$ is $|\mu|$-integrable. In this case, we have $\|f\|_1 = \int |f|\, d|\mu| < \infty$. A measurable set M is said to be *μ-null* if it is $|\mu|$-null, i.e. $|\mu|(M) = 0$. A property $p(x)$ is true *μ-almost-everywhere* or μ-ae if it is true $|\mu|$-ae. A set is *μ-σ-finite* if it is $|\mu|$-σ-finite. Practically everything is defined in terms of the variation of a measure.

21-2.4. **Lemma** A simple map f belongs to $L_p(\mu, F)$ iff it is a $\mathcal{K}$-step map.

Proof. Let $f = \sum_{j=1}^{n} \alpha_j \rho_{A_j} \in L_p(\mu, F)$ where α_j are nonzero vectors in F and A_j are disjoint measurable subsets of X. Then we have

$$\|\alpha_k\|^p |\mu|(A_k) \leq \int \sum_{j=1}^{n} \|\alpha_j\|^p \rho_{A_j} d|\mu| = \int |f|^p d|\mu| = \|f\|_p^p < \infty.$$

Thus $A_k \in \mathcal{K}$. Therefore f is a $\mathcal{K}$-step map. The converse is obvious. □

21-2.5. **Lemma** For every $f \in L_p(\mu, F)$, there are decent maps f_n such that both $|f_n| \uparrow |f|$ and $f_n \to f$ converge μ-ae.

Proof. Since the upper function $|f|^p$ is $|\mu|$-integrable, $M = X \setminus f^{-1}(0)$ is $|\mu|$-σ-finite. There are disjoint decent sets A_n and a $|\mu|$-null set N such that $M = N \cup \bigcup_{j=1}^{\infty} A_j$. The measurable map f is simply approximable on each decent set A_j. For each j, there are simple maps $g_{jk} \to f\rho_{A_j}$ on A_j as $k \to \infty$ and $|g_{jk}| \uparrow_k |f\rho_{A_j}|$ on the decent set A_j. All simple maps g_{jk} are decent maps with $g_{jk}(X \setminus A_j) = 0$. Hence $f_n = \sum_{k=1}^{n} g_{nk}$ are also decent maps. Take any $x \in A_j$. For all $n \geq j$, we have $f_n(x) = g_{jk}(x) \to f(x)$ as $k \to \infty$. For $x \notin M$, we have $f_n(x) = f(x) = 0$. Hence $f_n \to f$ on $X \setminus N$, that is μ-ae. It is an exercise to verify $|f_n| \uparrow |f|$, μ-ae. □

21-2.6. **Exercise** Let $X = \mathbb{R}$ be equipped with the counting measure. Show that the identity map $f : X \to \mathbb{R}$ is measurable but does not belong to any $L_p(\mu)$ for $1 \leq p < \infty$. Is it possible to find decent maps $f_n \to f$ almost everywhere?

21-2.7. **Lemma** Let $f : X \to F$ be an integrable map. If f_n are simple maps such that $|f_n| \leq |f|$ and $f_n \to f$, μ-ae; then every f_n is integrable; the limit $\lim_{n\to\infty} \int f_n d\mu$ exists in FE and it is independent of the choice of f_n.

Proof. Since f is μ-integrable, $|f|$ is $|\mu|$-integrable. Thus every $|f_n|$ is $|\mu|$-integrable and hence every simple map f_n is μ-integrable and $\int f_n d\mu$ is well-defined. By $|f_n - f| \leq 2|f|$, the Positive Dominated Convergence Theorem gives $\int |f_n - f|\, d|\mu| \to 0$ as $n \to \infty$. Observe that

$$\left\| \int f_m d\mu - \int f_n d\mu \right\| = \left\| \int (f_m - f_n) d\mu \right\| \leq \int |f_m - f_n|\, d|\mu|$$

$$\leq \int |f_m - f|\, d|\mu| + \int |f - f_n|\, d|\mu| \to 0 \text{ as } m, n \to \infty.$$

Therefore the Cauchy sequence $\{\int f_n d\mu : n \geq 1\}$ in the Banach space FE is convergent. Next suppose that g_n are simple maps such that $|g_n| \leq |f|$ and $g_n \to f$, μ-ae. Let $r = \lim \int f_n d\mu$ and $s = \lim \int g_n d\mu$. From $|f_n - g_n| \leq 2|f|$, Positive Dominated Convergence Theorem ensures that

$$\|r - s\| \le \left\| r - \int f_n d\mu \right\| + \int |f_n - g_n| \, d|\mu| + \left\| \int g_n d\mu - s \right\| \to 0$$

as $n \to \infty$, or $r = s$. This completes the proof. □

21-2.8. For every integrable map $f : X \to F$, there are simple maps $f_n \to f$ with $|f_n| \uparrow |f|$, μ-ae. The new integral $\displaystyle\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$ is well-defined in FE by last lemma and it is an extension of integrals of integrable simple maps. So, it also agrees with the integrals of upper functions. Obviously $\int f d\mu$ is linear in f. Note that the integral of a negative integrable function on $\mathbb{R}$ is not defined until now.

21-2.9. **Theorem** Let f, g be measurable maps such that $f = g$, μ-ae. If f is integrable, then so is g. Furthermore we have $\int f d\mu = \int g d\mu$.

Proof. Since $|f| = |g|$, $|\mu|$-ae; we have $\int |g| \, d|\mu| = \int |f| \, d|\mu| < \infty$. Hence g is integrable. To prove the equality, we may assume $g = 0$ by considering $\int (f - g) d\mu$. Since $|f| = 0$, μ-ae; we obtain $\int |f| \, d|\mu| = 0$. For any simple maps $f_n \to f$ satisfying $|f_n| \le |f|$, we have $\| \int f_n d\mu \| \le \int |f_n| \, d|\mu| \le \int |f| \, d|\mu| = 0$, that is $\int f d\mu = \lim \int f_n d\mu = 0$. □

21-2.10. **Theorem** For every $f \in L_p(\mu, F)$, there is an integrable map $g = f$, μ-ae; and decent maps $g_n \to g$ with $|g_n| \uparrow |g|$ everywhere.

Proof. Choose decent maps $f_n \to f$ with $|f_n| \uparrow |f|$, μ-ae. There is a null set N such that $f_n \to f$ with $|f_n| \uparrow |f|$ on $M = X \setminus N$. Clearly $g_n = f_n \rho_M$ and $g = f \rho_M$ are the required maps. □

21-2.11. **Theorem** For every integrable map f, we have $\| \int f d\mu \| \le \int |f| \, d|\mu|$.

Proof. Choose simple maps $f_n \to f$ with $|f_n| \uparrow |f|$, μ-ae. For all f_n are integrable simple maps, we have $\| \int f_n d\mu \| \le \int |f_n| \, d|\mu|$. Since $|f_n - f| \le 2|f|$, the Positive Dominated Convergence Theorem implies that

$$\left| \int |f_n| \, d|\mu| - \int |f| \, d|\mu| \right| \le \int |f_n - f| \, d|\mu| \to 0,$$

or

$$\int |f_n| \, d|\mu| \to \int |f| \, d|\mu|.$$

Therefore $\displaystyle\left\| \int f d\mu \right\| = \lim \left\| \int f_n d\mu \right\| \le \lim \int |f_n| \, d|\mu| = \int |f| \, d|\mu|.$ □

21-2.12. **Exercise** Let f, g be real μ-integrable functions. Prove that if $f \le g$, μ-ae; then $\int f d|\mu| \le \int g d|\mu|$.

21-2.13. **Exercise** Let f be a measurable map and g a measurable function. Prove that if $|f| \leq g$, μ-ae and if $g \in L_p(\mu)$, then $f \in L_p(\mu, F)$.

21-2.14. **Chebyshev's Inequality** If $f \in L_p(\mu, F)$ where $1 \leq p < \infty$, then for every $\varepsilon > 0$, we have $|\mu|\{x \in X : \|f(x)\| \geq \varepsilon\} \leq (\|f\|_p/\varepsilon)^p$.

Proof. For $M = \{x \in X : \|f(x)\| \geq \varepsilon\}$, the result follows from

$$\|f\|_p^p = \int |f|^p d|\mu| \geq \int \varepsilon^p \rho_M d|\mu| = \varepsilon^p |\mu|(M).$$ □

21-3 L_p-Spaces for $1 \leq p < \infty$

21-3.1. Let E, F, FE be Banach spaces with an admissible bilinear map $\varphi : F \times E \to FE$; F' the dual space of continuous linear forms on F; and μ a vector measure on a δ-space $(X, \mathbb{D})$ into E. Similar to §§3-3.5,6; we develop the basic properties of $L_p(\mu, F)$. Only scalar version of Holder's inequality is required to prove Minkowski's inequality but the general version will be needed for duality theory in later chapter.

21-3.2. **Holder's Inequality** Let $p, q > 1$ be conjugate indices. If $h \in L_q(\mu, F')$ and $f \in L_p(\mu, F)$, then $hf \in L_1(\mu)$ and $\|\int hf d\mu\| \leq \|h\|_q \|f\|_p$.

Proof. If $\|f\|_p = 0$; then $f = 0$, μ-ae; or $hf = 0$, μ-ae; thus hf is integrable and $\|\int hf d\mu\| = \|h\|_q \|f\|_p = 0$. Without loss of generality, we may assume that $\|h\|_q \neq 0$ and $\|f\|_p \neq 0$. From

$$\left|\frac{h}{\|h\|_q} \frac{f}{\|f\|_p}\right| \leq \left|\frac{h}{\|h\|_q}\right| \left|\frac{f}{\|f\|_p}\right| \leq \frac{1}{q}\left|\frac{h}{\|h\|_q}\right|^q + \frac{1}{p}\left|\frac{f}{\|f\|_p}\right|^p,$$

hf is integrable. Integrating with respect to $|\mu|$, we have

$$\frac{\|\int hf d\mu\|}{\|h\|_q \|f\|_p} \leq \int \left|\frac{h}{\|h\|_q} \frac{f}{\|f\|_p}\right| d|\mu| \leq \frac{1}{q}\int \left|\frac{h}{\|h\|_q}\right|^q d|\mu| + \frac{1}{p}\int \left|\frac{f}{\|f\|_p}\right|^p d|\mu| = 1$$

from which the required inequality follows. □

21-3.3. **Minkowski's Inequality** For $1 \leq p < \infty$, if f, g are in $L_p(\mu, F)$ then so is $f + g$. Furthermore, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. Clearly $f + g$ and $|f + g|^p$ are measurable. Observe that

$$|f + g|^p \leq (|f| + |g|)^p \leq (2|f|)^p + (2|g|)^p.$$

Since the right hand side is integrable, we have $f + g \in L_p(\mu, F)$. For $p = 1$,

$$\|f + g\|_1 = \int |f + g| \, d|\mu| \leq \int (|f| + |g|) \, d|\mu| = \int |f| d\mu + \int |g| d\mu = \|f\|_1 + \|g\|_1.$$

Next assume $1 < p < \infty$. Suppose $\|f + g\|_p \neq 0$, otherwise the required inequality holds already. Observe that

$$\|f+g\|_p^p = \int |f+g|^p d|\mu| = \int |f+g|\ |f+g|^{p-1} d|\mu|$$
$$\le \int |f|\ |f+g|^{p-1} d|\mu| + \int |g|\ |f+g|^{p-1} d|\mu|.$$

Since $f+g \in L_p(\mu, F)$, $|f+g|^{(p-1)q} = |f+g|^p$ is integrable, i.e. $|f+g|^{p-1} \in L_q$. Furthermore, its L_q-norm is given by

$$\left(\int |f+g|^{(p-1)q} d|\mu|\right)^{1/q} = \left(\int |f+q|^p d|\mu|\right)^{1/q} = \|f+g\|_p^{p/q} = \|f+g\|_p^{p-1}.$$

Hence the Holder's Inequality gives

$$\|f+g\|_p^p \le \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1} = (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}.$$

The result follows after dividing by $\|f+g\|_p^{p-1}$. □

21-3.4. **Exercise** Prove that $L_p^r(\mu)$ is a vector lattice. Show that if $f, g \in L_p^r(\mu)$, then so are $f \vee g, f \wedge g, f_+$ and f_-.

21-3.5. **Theorem** For $1 \le p < \infty$, $L_p(\mu, F)$ is a Banach space when two maps f, g are identified by $f = g$, μ-ae. Furthermore, the integration is a continuous linear map from $L_1(\mu, F)$ into FE.

Proof. Let $f, g \in L_p(\mu, F)$. By §21-2.10 without loss of generality, there are simple maps f_n, g_n such that $f_n \to f$, $g_n \to g$, $|f_n| \le |f|$ and $|g_n| \le |g|$ everywhere. Since $f_n + g_n \to f+g$ in F, the sum $f+g$ is measurable. From $|f+g|^p \le (2|f|)^p + (2|g|)^p$, the upper function $|f+g|^p$ is $|\mu|$-integrable, that is $f+g \in L_p(\mu, F)$. In particular, for $p = 1$ we have

$$\int (f+g) d\mu = \lim_{n\to\infty} \int (f_n + g_n) d\mu = \lim_{n\to\infty} \left(\int f_n d\mu + \int g_n d\mu\right) = \int f d\mu + \int g d\mu.$$

Similarly, for every $t \in \mathbb{K}$ we get $tf \in L_p(\mu, F)$ and for $p = 1$ we obtain $\int tf d\mu = t \int f d\mu$. Therefore $L_p(\mu, F)$ is a vector space. The continuity of integration on $L_1(\mu, F)$ follows from $\|\int f d\mu\| \le \int |f|\ d|\mu| = \|f\|_1$.

To prove the completeness, it suffices to show the convergence of a special type of Cauchy sequences. Let $\{g_n\}$ be a sequence in $L_p(\mu, F)$ such that $\|g_{n+1} - g_n\|_p \le 2^{-n}$ for all $n \ge 1$. Define $h_n = |g_1| + \sum_{k=1}^n |g_{k+1} - g_k|$ and $h = |g_1| + \sum_{k=1}^\infty |g_{k+1} - g_k|$. Since $L_p(\mu, F)$ is a normed space, we have

$$\|h_n\|_p \le \|g_1\|_p + \sum_{k=1}^n \|g_{k+1} - g_k\|_p \le \|g_1\|_p + 1,$$

that is,
$$\int h_n^p d|\mu| = \|h_n\|_p^p \le (\|g_1\|_p + 1)^p < \infty.$$

Since h_n^p are positive measurable functions and since $h_n^p \uparrow h^p$, Positive Monotone Convergence Theorem ensures that h^p is $|\mu|$-integrable. Thus h^p and consequently h itself are finite-valued μ-ae. The set $N = \{x \in X : h(x) = \infty\}$ is a null set. Let $M = X \setminus N$. Then the map $f : X \to F$ defined by the infinite series $f(x) = g_1(x)\rho_M(x) + \sum_{k=1}^{\infty} [g_{k+1}(x) - g_k(x)]\rho_M(x)$ is absolutely convergent on M and is zero on N. Hence $g_n\rho_M \to f$ on X. Therefore f is measurable. The absolutely convergent series $-g_n\rho_M = \sum_{k=n}^{\infty}(g_{k+1} - g_k)\rho_M$ gives $|g_n\rho_M| \le \sum_{k=n}^{\infty} |g_{k+1} - g_k| \le h$, that is $|g_n\rho_M|^p \le h^p$. Letting $n \to \infty$, we have $|f|^p \le h^p$. Since h^p is integrable, so is $|f|^p$, i.e. $f \in L_p(\mu, F)$. Because $|g_n\rho_M - f|^p \to 0$ and $|g_n\rho_M - f|^p \le (2h)^p$, it follows from Positive Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int |g_n - f|^p d|\mu| = \lim_{n\to\infty} \int |g_n\rho_M - f|^p d|\mu| = \int \lim_{n\to\infty} |g_n\rho_M - f|^p d|\mu| = 0,$$

that is $\|g_n - f\|_p \to 0$. This completes the proof. □

21-3.6. **Corollary** For $E = F = \mathbb{K}$ and $\mu \ge 0$, L_2 is a Hilbert space under the inner product given by the following expression: $< f, g >= \int f(x)\overline{g(x)}d\mu(x)$.

21-3.7. Let X be a set equipped with the counting measure μ on the semiring of all singletons and let $E = F = \mathbb{K}$. Then $\ell_p(X) = L_p$ is a Banach space. Also $\ell_2(X)$ is a Hilbert space under the inner product given by $< f, g >= \sum_{x\in X} f(x)g^-(x)$ where $g^-(x)$ denotes the complex conjugate of $g(x)$. It can be proved that every Hilbert space H is of the form $\ell_2(X)$ where X is any orthonormal basis of H.

21-3.8. **Example** Let μ be a scalar measure on a δ-space $(X, \mathbb{D})$, $L_1(\mu)$ the Banach space of μ-integrable functions and $T : L_1(\mu) \to E$ a continuous linear map. For every decent set A, let $\nu A = T\rho_A$. Then $\nu : \mathbb{D} \to E$ is a measure with $|\nu| \le \|T\|\ |\mu|$.

Proof. Clearly ν is finitely additive on $\mathbb{D}$. Next, assume that $A_n \downarrow \emptyset$ in $\mathbb{D}$. Then $|\mu|A_n \downarrow 0$. Thus $\|\nu A_n\| = \|T\rho_{A_n}\| \le \|T\|\ \|\rho_{A_n}\|_1 = \|T\|\ |\mu|A_n \to 0$. Hence ν is countably additive. Finally suppose that $A = \bigcup_{j=1}^{n} B_j$ be a disjoint union where $A, B_j \in \mathbb{D}$. Then $\sum_{j=1}^{n} \|\nu B_j\| \le \sum_{j=1}^{n} \|T\|\ |\mu|B_j = \|T\|\ |\mu|A$. Thus we obtain $|\nu|(A) = \sup_{P(A)} \sum_{D\in P(A)} \|\nu D\| \le \|T\|\ |\mu|A < \infty$. Therefore ν is of finite variation. Consequently, ν is a measure on $(X, \mathbb{D})$. □

21-3.9. **Exercise** Let μ be a positive measure on a δ-space $(X, \mathbb{D})$. For every $A \in \mathbb{D}$, let $\nu A = \rho_A \in E = L_1(\mu)$. Prove that ν is an E-measure with $|\nu| = \mu$.

21-3.10. **Exercise** Let λ be the Lebesgue measure on $\mathbb{R}$, $1 < p < \infty$ and $E = L_p(\lambda)$. For every decent set of $\mathbb{R}$, let $\mu A = \rho_A \in E$. Show that μ is countably additive but it is *not* of finite variation.

21-4 Mean Convergence

21-4.1. Convergence in the $L_p(\mu, F)$-space for $1 \le p < \infty$ is also called the *L_p-convergence* or *p*-th *mean convergence.* It is the results in this section that Lebesgue integral distinguishes itself from Riemann integral. Let E, F, FE be Banach spaces with an admissible bilinear map $F \times E \to FE$ and μ a vector measure on a δ-space $(X, \mathbb{D})$ into E.

21-4.2. **Dominated Convergence Theorem** Let $f_n, f : X \to F$ be measurable maps. Suppose $f_n \to f$, μ-ae. If there is $g \in L_p^r(\mu)$ with $|f_n| \le g$, μ-ae for all n; then all $f_n, f \in L_p(\mu, F)$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

Proof. Letting $n \to \infty$ in $|f_n| \le g$, we have $|f| \le g$, μ-ae. Since $|f|^p \le g^p$, μ-ae and g^p is integrable, so is $|f|^p$, i.e. $f \in L_p(\mu, F)$. Similarly, all $f_n \in L_p(\mu, F)$. Next observe that $|f_n - f|^p \le (|f_n| + |f|)^p \le (2|g|)^p$. It follows from Positive Dominated Convergence Theorem that

$$\lim_{n\to\infty} \|f_n - f\|_p^p = \lim_{n\to\infty} \int |f_n - f|^p d\mu = \int \lim_{n\to\infty} |f_n - f|^p d\mu = 0. \qquad \square$$

21-4.3. **Exercise** Show that the functions $n\rho_{(0,1/n]}, \rho_{(n,n+1]}$ and $(1/n)\rho_{(0,n]}$ converge to zero but their integrals are equal to one. Explain why Dominated Convergence Theorem is not applicable.

21-4.4. **Monotone Convergence Theorem** Let $f_n, f : X \to F$ be measurable maps. Suppose $f_n \to f$, μ-ae and all $f_n \in L_p(\mu, F)$. If $|f_n| \uparrow |f|$, μ-ae. and $\sup \|f_n\|_p < \infty$, then $f \in L_p(\mu, F)$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

Proof. Clearly $|f_n|^p \uparrow |f|^p$, μ-ae and $\sup \int |f_n|^p d|\mu| \le \sup \|f_n\|_p^p < \infty$. By Positive Monotone Convergence Theorem, we have

$$\int |f|^p d\mu = \lim_{n\to\infty} \int |f_n|^p d\mu \le \sup \|f_n\|_p^p < \infty.$$

Hence the upper function $|f|^p$ is integrable, i.e. $f \in L_p(\mu, F)$. Now all $|f_n| \le |f|$ and $f_n \to f$, μ-ae. Last theorem gives $\|f_n - f\|_p \to 0$ as $n \to \infty$. $\square$

21-4.5. **Theorem** Let $f_n, f \in L_p(\mu, F)$ where $1 \le p < \infty$. If $f_n \to f$, μ-ae; and if $\|f_n\|_p \to \|f\|_p$ then $\|f_n - f\|_p \to 0$.

Proof. Without loss of generality, we may assume $f_n \to f$ on X. Let $g_n = (2|f_n|)^p + (2|f|)^p$ and $g = 2^{p+1}|f|^p$. Then the given condition ensures that $\int g_n d\mu \to \int g d\mu$. Since $|f_n - f|^p \leq (|f_n| + |f|)^p \leq g_n$ and $\int g d|\mu| < \infty$, by §20-3.3 we obtain $\int |f_n - f|^p d\mu \to \int 0 d\mu = 0$ which gives $\|f_n - f\|_p \to 0$. □

21-4.6. **Density Theorem** The set of all decent maps is dense in $L_p(\mu, F)$ for $1 \leq p < \infty$. More precisely, for every $f \in L_p(\mu, F)$ there are decent maps $f_n \to f$ with $|f_n| \uparrow |f|$, μ-ae and $\|f_n - f\|_p \to 0$. Furthermore if $f \geq 0$, we may choose all $f_n \geq 0$.

Proof. By §21-2.10, there are f_n satisfying all except the last condition which also follows from last theorem. □

21-4.7. **Corollary** If $\mathbb{D}$ is generated by a semiring $\mathcal{S}$, then the set of $\mathcal{S}$-step F-map is dense in $L_p(\mu, F)$ as a result of §20-3.6.

21-4.8. **Corollary** If $\mathbb{D}$ is generated by a countable family $\mathcal{K}$ of subsets of X and if F is separable, then $L_p(\mu, F)$ is also separable. In particular, the function space $L_p(\mu)$ is separable.

Proof. Let $\mathcal{K}_1 = \mathcal{K}$; $\mathcal{K}_n$ the family consisting of sets $A \cup B$ and $A \setminus B$ for all $A, B \in \mathcal{K}_{n-1}$ and $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. Clearly $\mathcal{R}$ is the ring generated by $\mathcal{K}$ and $\mathcal{R}$ is countable. Let H be a countable dense set in F. Then $\mathcal{F}_n = \{\sum_{j=1}^{n} \alpha_j \rho_{A_j} : \alpha_j \in H, A_j \in \mathcal{R}\}$ is countable and so is $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$. It is an exercise to prove from last corollary that $\mathcal{F}$ is dense in $L_p(\mu, F)$. The last statement follows because $\mathbb{K}$ is separable. □

21-4.9. **Integration Term by Term** Let $\{f_n\}$ be a sequence of measurable maps. If the upper function $\sum_{n=1}^{\infty} |f_n|$ is integrable, then all f_n and $\sum_{n=1}^{\infty} f_n$ are integrable and $\displaystyle\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proof. Let $g = \sum_{j=1}^{\infty} |f_j|$, $h_n = \sum_{j=1}^{n} f_j$ and $h = \sum_{j=1}^{\infty} f_j$. Since g is integrable, it is finite value almost everywhere. Without loss of generality, assume that g is finite value everywhere. Because $g(x) = \sum_{j=1}^{\infty} \|f_j(x)\|$ is convergent, h is well-defined. The limit h of measurable maps h_n is measurable. Since g is integrable and $|f_n| \leq g$, all f_n is integrable. From $|h_n| \leq \sum_{j=1}^{n} |f_j| \leq g$, Dominated Convergence Theorem implies $\int \sum_{j=1}^{\infty} f_j d\mu = \int h d\mu = \int \lim h_n d\mu = \lim \int h_n d\mu = \lim \sum_{j=1}^{n} \int f_j d\mu = \sum_{j=1}^{\infty} \int f_j d\mu$. □

21-4.10. **Exercise** Let f_n, g be integrable maps. Prove that $f_n \to g$, μ-ae if $\sum_{n=1}^{\infty} \|f_n - g\|_1 < \infty$.

21-4.11. **Exercise** Let f be a map from a metric space T into a metric space and let $a \in T$. Then $\lim f(t) = c$ as $t \to a$ iff for every sequence $t_n \to a$ in T with all $t_n \neq a$ we have $f(t_n) \to c$.

21-4.12. **Continuously Dominated Convergence Theorem** Let T be a metric space and μ an E-measure on a δ-space. Let $f : X \times T \to F$ be a map such that for each $t \in T$, $f(x,t)$ is μ-integrable in x. Suppose that $a \in T$ and for each $x \in X$, we have $\lim_{t\to a} f(x,t) = f(x,a)$. If there is a μ-integrable function $g : X \to \mathbb{R}$ satisfying the dominating condition: $|f(x,t)| \leq g(x)$ for all $(x,t) \in X \times T$, then we have $\displaystyle\lim_{t\to a} \int f(x,t)d\mu(x) = \int f(x,a)d\mu(x)$.

Proof. For any sequence $t_n \to a$ in T, define $h_n(x) = f(x,t_n)$ for all n. Then we have $\displaystyle\lim_{n\to\infty} \int f(x,t_n)d\mu(x) = \int f(x,a)d\mu(x)$ by applying the discrete version to $\{h_n\}$. The result follows from last exercise. □

21-4.13. **Differentiation under Integral Sign** Let $f : X \times (a,b) \to \mathbb{R}$ be a real function such that for each $t \in (a,b)$, $f(x,t)$ is μ-integrable in x. Suppose for each $x \in X$, $\partial f/\partial t$ exists. If there is a μ-integrable function $g : X \to \mathbb{R}$ satisfying the dominating condition: $\left|\dfrac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for all $(x,t) \in X \times (a,b)$, then we have $\displaystyle\frac{d}{dt}\int f(x,t)d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)d\mu(x)$ on $X \times (a,b)$.

Proof. Let $\varphi(t) = \int f(x,t)d\mu(x)$ and $a < t_0 < b$. Take any $t_n \to t_0$ in (a,b) with all $t_n \neq t_0$. Define $h_n(x) = \dfrac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$. Then each h_n is a measurable function on X and its pointwise limit $\dfrac{\partial f}{\partial t}(x,t_0) = \displaystyle\lim_{n\to\infty} h_n(x)$ is also measurable. By Mean-Value Theorem, we have $|h_n(x)| = \left|\dfrac{\partial f}{\partial t}(x,\theta)\right| \leq g(x), \forall\, x \in X$ where θ is between t_n and t_0. The Dominated Convergence Theorem ensures that $(\partial f/\partial t)(x,t_0)$ is integrable so that the right hand side of the required identity is well-defined. Furthermore,

$$\lim_{n\to\infty} \frac{\varphi(t_n) - \varphi(t_0)}{t_n - t_0} = \lim_{n\to\infty} \int h_n(x)d\mu(x) = \int \frac{\partial f}{\partial t}(x,t_0)d\mu(x).$$

Consequently, φ is differentiable at t_0 and the required identity holds. □

21-4.14. **Exercise** State and prove a continuous version of L_p-Dominated Convergence Theorem.

21-5 L_∞-Spaces

21-5.1. Let $\mathcal{K}$ be a family of measurable sets in a δ-space $(X, \mathbb{D})$. Assume that $\mathcal{K}$ satisfies the following two conditions:
(a) If $A_n \in \mathcal{K}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{K}$.
(b) If A is a measurable set with $A \subset B \in \mathcal{K}$, then $A \in \mathcal{K}$.
A proposition $p(x)$ about $x \in X$ is ***true*** $\mathcal{K}$-ae if there is $N \in \mathcal{K}$ such that $p(x)$ is true for all $x \in X \setminus N$. The $\mathcal{K}$-***essential sup-norm*** of a measurable map f from X into a Banach space F is defined by $\|f\|_\infty = \inf_{N \in \mathcal{K}} \sup_{x \in X \setminus N} \|f(x)\|$. Also f is $\mathcal{K}$-***essentially bounded*** if $\|f\|_\infty$ is finite. The set of all $\mathcal{K}$-essentially bounded measurable maps into F is denoted by $L_\infty(\mathcal{K}, F)$. The generality is interesting in itself and will be used later in spectral measures §26-4. If μ is a vector measure on X into a Banach space E, the family of all μ-null sets is really what we want to study. The set of all μ-essentially bounded measurable maps from X into F is denoted by $L_\infty(X, \mathbb{D}, E, \mu, F)$. Drop some symbols if there is no ambiguity.

21-5.2. **Lemma** For every sequence of measurable maps f_k, there is $N \in \mathcal{K}$ such that $\|f_k\|_\infty = \sup_{x \in X \setminus N} \|f_k(x)\|$ for all k. As a result, if $\|f_n - g\|_\infty \to 0$ in $L_\infty(\mathcal{K}, F)$, then there is $N \in \mathcal{K}$ such that $f_n \to g$ uniformly on $X \setminus N$.

Proof. Suppose that $\|f_k\|_\infty < \infty$. Choose $t_{jk} \downarrow \|f_k\|_\infty$ as $j \to \infty$. Since $\|f_k\|_\infty < t_{jk}$, there is $N_{jk} \in \mathcal{K}$ such that $\sup_{x \in X \setminus N_{jk}} \|f_k(x)\| \le t_{jk}$. On the other hand, if $\|f_k\|_\infty = \infty$, let $N_{jk} = \emptyset$ and $t_{jk} = \infty$ so that $\sup_{x \in X \setminus N_{jk}} \|f_k(x)\| \le t_{jk}$ and $t_{jk} \to \|f_k\|_\infty$ as $j \to \infty$. Clearly, $N = \bigcup_{j,k=1}^{\infty} N_{jk} \in \mathcal{K}$. If $x \notin N$, then $x \notin N_{jk}$, or $\|f_k(x)\| \le t_{jk}$ for all j, that is $\|f_k(x)\| \le \|f_k\|_\infty$ by letting $j \to \infty$. Therefore, $\|f_k\|_\infty = \inf_M \sup_{x \in X \setminus M} \|f_k(x)\| \le \sup_{x \in X \setminus N} \|f_k(x)\| \le \|f_k\|_\infty$. Consequently, $\|f_k\|_\infty = \sup_{x \in X \setminus N} \|f_k(x)\|$. For the last statement, choose $N \in \mathcal{K}$ such that $\|f_n - g\|_\infty = \sup_{x \in X \setminus N} \|f_n(x) - g(x)\|$ for all n. Then $f_n \to g$ uniformly on $X \setminus N$. □

21-5.3. **Theorem** Let f, g be measurable maps.
(a) If $|f| \le t < \infty$, $\mathcal{K}$-ae; then $f \in L_\infty(\mathcal{K}, F)$ and $\|f\|_\infty \le t$.
(b) $|f| \le \|f\|_\infty$, $\mathcal{K}$-ae; equivalently $Q = \{x \in X : |f(x)| > \|f\|_\infty\} \in \mathcal{K}$.
(c) $\|f\|_\infty = \inf\{t \in [0, \infty] : |f| \le t, \ \mathcal{K}\text{-ae}\} = \sup_{x \in X \setminus Q} |f(x)|$.

(d) $f = g$, $\mathcal{K}$-ae iff $\|f - g\|_\infty = 0$.

Proof. (a) Choose $K \in \mathcal{K}$ such that $\|f(x)\| \le t < \infty$ for all $x \in X \setminus K$. Thus $\|f\|_\infty = \inf_N \sup_{x\in X\setminus N} \|f(x)\| \le \sup_{x\in X\setminus K} \|f(x)\| \le t < \infty$.
(b) Choose $N \in \mathcal{K}$ with $\|f\|_\infty = \sup_{x\in X\setminus N} \|f(x)\|$. Then $\|f(x)\| \le \|f\|_\infty$ for all $x \in X \setminus N$. Thus $|f| \le \|f\|_\infty$, $\mathcal{K}$-ae. Since $Q \subset N$, we have $Q \in \mathcal{K}$. Conversely if $Q \in \mathcal{K}$, then $\|f(x)\| \le \|f\|_\infty$ for all $x \in X \setminus Q$. Hence $|f| \le \|f\|_\infty$, $\mathcal{K}$-ae.
(c) Let $\alpha = \inf\{t \in [0,\infty] : |f| \le t,\ \mathcal{K}\text{-ae}\}$ and $\beta = \sup_{x\in X\setminus Q} |f(x)|$. Since $|f(x)| \le \beta$ for all $x \in X \setminus Q$ and $Q \in \mathcal{K}$, we have $\alpha \le \beta$. Also $\beta \le \|f\|_\infty$ by definition of Q. If $\alpha = \infty$, then $\alpha = \beta = \|f\|_\infty$. Assume that $\alpha < \infty$. For every n, there is $t_n \ge |f|$, $\mathcal{K}$-ae with $t_n \le \alpha + \frac{1}{n}$. Choose $M_n \in \mathcal{K}$ such that $|f(x)| \le t_n$ for all $x \in X \setminus M_n$. Then $M = \bigcup_{n=1}^\infty M_n \in \mathcal{K}$. For every $x \in X \setminus M$, we have $|f(x)| \le t_n \le \alpha + \frac{1}{n}$ for all n, i.e. $|f(x)| \le \alpha$. Therefore $\|f\|_\infty \le \sup_{x\in M} |f(x)| \le \alpha$. Consequently we also have $\alpha = \beta = \|f\|_\infty$.
(d) is left as an exercise. □

21-5.4. **Theorem** The space $L_\infty(\mathcal{K}, F)$ is a Banach space when two maps f, g are identified by $f = g$, $\mathcal{K}$-ae.

Proof. Let $f, g \in L_\infty(\mathcal{K}, F)$. Then $|f + g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$, $\mathcal{K}$-ae. Hence $f+g \in L_\infty(\mathcal{K}, F)$ and $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. We leave it as an exercise to show that $L_\infty(\mathcal{K}, F)$ is a normed space. To prove that $L_\infty(\mathcal{K}, F)$ is complete, let $\{f_n\}$ be a Cauchy sequence in $L_\infty(\mathcal{K}, F)$. Then for every $\varepsilon > 0$ there is k such that $\|f_m - f_n\|_\infty \le \varepsilon$ for all $m, n \ge k$. Since $|f_m - f_n| \le \|f_m - f_n\|_\infty$ and $|f_n| \le \|f_n\|_\infty$, $\mathcal{K}$-ae; there is $N \in \mathcal{K}$ such that for every $x \in M = X \setminus N$ we have $\|f_m(x) - f_n(x)\| \le \|f_m - f_n\|_\infty$ and $\|f_n(x)\| \le \|f_n\|_\infty$. Hence for every $x \in X$, $\{f_n(x)\rho_M(x) : n \ge 1\}$ is a Cauchy sequence in F. Define $f(x) = \lim_{n\to\infty} f_n(x)\rho_M(x)$. Then f is a measurable map. Because

$$\|f_m(x)\rho_M(x)\| \le \|f_m(x)\rho_M(x) - f_k(x)\rho_M(x)\| + \|f_k(x)\rho_M(x)\| \le \varepsilon + \|f_k\|_\infty,$$

setting $m \to \infty$ we have $\|f(x)\| \le \varepsilon + \|f_k\|_\infty$ for all $x \in X$. Therefore $f \in L_\infty(\mathcal{K}, F)$. Since $\|f_m(x)\rho_M(x) - f_n(x)\rho_M(x)\| \le \|f_m - f_n\|_\infty \le \varepsilon$, letting $m \to \infty$ we obtain $\|f(x) - f_n(x)\rho_M(x)\| \le \varepsilon$ for all $n \ge k$. Thus,

$$\|f - f_n\|_\infty \le \|f - f_n\rho_M\|_\infty + \|f_n - f_n\rho_M\|_\infty \le \varepsilon,\ \forall\, n \ge k.$$

This proves $f_n \to f$ in $L_\infty(\mathcal{K}, F)$. □

21-5.5. Before spectral measures §26-4, we work with $L_\infty(X, \mathbb{D}, E, \mu, F)$ only.

21-5.6. **Exercise** Show that in general, decent functions are *not* dense in L_∞.

21-5.7. **Exercise** Let $\mathcal{F}(\mathbb{D})$ denote the set of all decent functions on X. Then it is a vector subspace of L_∞. Let $c_0(X)$ denote the closure of $\mathcal{F}(\mathbb{D})$ in L_∞. Then $c_0(X)$ is a Banach space under the essential sup-norm and $\mathcal{F}(\mathbb{D})$ is dense in $c_0(X)$. Prove that a measurable function f is in $c_0(X)$ iff for every $\varepsilon > 0$ there is a decent set D such that $\|f\rho_{(X\setminus D)}\|_\infty \le \varepsilon$.

21-5.8. **Theorem** Let F' be the dual space of F and let $1 \le p < \infty$. If $h \in L_\infty(\mu, F')$ and $f \in L_p(\mu, F)$, then $hf \in L_p$ and $\|hf\|_p \le \|h\|_\infty \|f\|_p$.

Proof. Since $(u,v) \to uv : F' \times F \to \mathbb{K}$ is continuous bilinear, the function hf is measurable. Without loss of generality, we may assume $|h| \le \|h\|_\infty$ everywhere. Thus $|hf|^p \le |h|^p |f|^p \le \|h\|_\infty^p |f|^p \in L_1^r(\mu)$. Therefore $|hf|^p$ is integrable, that is, $hf \in L_p(\mu)$. From $\int |hf|^p d|\mu| \le \|h\|_\infty^p \int |f|^p d|\mu|$, we obtain $\|hf\|_p \le \|h\|_\infty \|f\|_p$. □

21-5.9. **Exercise** Let μ be a positive measure on a δ-space X and h a measurable function on X such that $hf \in L_2(\mu)$ for every $f \in L_2(\mu)$. Clearly the multiplication operator $T : L_2(\mu) \to L_2(\mu)$ given by $T(f) = hf$ is linear. Prove that if T is continuous, then $h \in L_\infty$. Show that T is a normal operator. Also T is self-adjoint iff f is real-valued, μ-ae.

21-6 Convergence in Measure

21-6.1. Let μ be an E-measure on a δ-space X and $f_n, f : X \to F$ be measurable maps. Then $\{f_n\}$ *converges to f in measure* if

$$\lim_{n\to\infty} |\mu|\{x \in X : \|f_n(x) - f(x)\| \ge \varepsilon\} = 0$$

for every $\varepsilon > 0$. Also $\{f_n\}$ is *Cauchy in measure* if for every $\varepsilon > 0$ we have $\lim |\mu|\{x \in X : \|f_m(x) - f_n(x)\| \ge \varepsilon\} = 0$ as $m, n \to \infty$. Note that *positive* measures are defined on measurable sets by §20-2.1 with properties listed in §20-2.2, 5.2a,4 although a vector measure is defined only on its *integrable* sets.

21-6.2. **Theorem** Let μ, ν be measures on X with $|\mu| \le |\nu|$. If $f_n \to f$ in ν-measure, then $f_n \to f$ in μ-measure. The same result holds for being Cauchy in measure.

21-6.3. **Theorem** If $f_n \to f$ in measure for both measures μ, ν, then $f_n \to f$ in measure for $\alpha\mu+\beta\nu$ where $\alpha, \beta \in \mathbb{K}$. The same result holds for being Cauchy in measure.

Proof. It follows from $|\alpha\mu + \beta\nu| \le |\alpha|\ |\mu| + |\beta|\ |\nu|$. □

21-6.4. Since only the total variation $|\mu|$ is involved, we shall assume that μ is a *positive* measure throughout the rest of this section. It is easy to prove that if $\mu\{x \in X : \|f_n(x) - f(x)\| \geq 1/s\} \to 0$ for every integer $s \geq 1$, then $f_n \to f$ in measure. Similarly, $\{f_n\}$ is Cauchy in measure if for each integers $s \geq 1$ we have $\lim \mu\{x \in X : \|f_m(x) - f_n(x)\| \geq 1/s\} = 0$ as $m, n \to \infty$.

21-6.5. **Theorem** Let $1 \leq p < \infty$. Then the following statements hold.
(a) If $f_n \to f$ in $L_p(\mu, F)$, then $f_n \to f$ in measure.
(b) If $\{f_n\}$ is Cauchy in $L_p(\mu, F)$, then it is Cauchy in measure.

Proof. (a) By Chebyshev's Inequality, we have

$$\mu\{x : \|f_n(x) - f(x)\| \geq \varepsilon\} \leq (\|f_n - f\|_p/\varepsilon)^p \to 0.$$

(b) It is left as an exercise. □

21-6.6. **Exercise** Let $f_n = g_n$ and $f = g$, μ-ae. Prove that if $f_n \to f$ in measure, then $g_n \to g$ in measure.

21-6.7. **Theorem** If $f_n \to f$ and $f_n \to g$ in measure, then $f = g$, μ-ae.

Proof. For each integer $s \geq 1$, let $A_n = \{x \in X : \|f_n(x) - f(x)\| \geq 1/(2s)\}$, $B_n = \{x \in X : \|f_n(x) - g(x)\| \geq 1/(2s)\}$ and $H_s = \{x \in X : \|f(x) - g(x)\| \geq 1/s\}$. Since $f_n \to f$ and $f_n \to g$ in measure, we have $\mu A_n \to 0$ and $\mu B_n \to 0$ as $n \to \infty$. Now because $H_s \subset A_n \cup B_n$, we have $\mu H_s \leq \mu A_n + \mu B_n \to 0$. Letting $n \to \infty$, we get $\mu H_s = 0$. Therefore $\bigcup_{s=1}^{\infty} H_s = \{x \in X : f(x) \neq g(x)\}$ is a null set, i.e. $f = g$, a.e. □

21-6.8. **Theorem** If $\{f_n\}$ is Cauchy in measure and if $\{f_n\}$ has a subsequence which is convergent to f in measure, then $f_n \to f$ in measure.

Proof. For each $\alpha > 0$, let $A_n(\alpha) = \{x \in X : \|f(x) - f_n(x)\| \geq \alpha\}$ and $A_{mn}(\alpha) = \{x \in X : \|f_m(x) - f_n(x)\| \geq \alpha\}$. Since f_n is Cauchy in measure for every $\varepsilon > 0$, there is $t > 0$ such that for all $m, n \geq t$, we have $\mu A_{mn}(\alpha/2) \leq \varepsilon/2$. Given that $\{f_n\}$ has a subsequence which converges to f in measure, there are integers $1 \leq m(1) < m(2) < m(3) < \cdots$ such that $\forall\ \varepsilon > 0$, $\exists$ integer j, $\forall\ i \geq j$, we have $\mu A_{m(i)}(\alpha/2) \leq \varepsilon/2$. Let $k = m(j + t)$. Take any $n \geq k$. Then $\mu A_k(\alpha/2) = \mu A_{m(j+t)}(\alpha/2) \leq \varepsilon/2$. Next, since $n \geq k \geq t$, we have $\mu A_{kn}(\alpha/2) \leq \varepsilon/2$. Clearly $A_n(\alpha) \subset A_k(\alpha/2) \cup A_{kn}(\alpha/2)$. We get $\mu A_n(\alpha) \leq \varepsilon$. Thus $\mu A_n(\alpha) \to 0$ as $n \to \infty$. Therefore $f_n \to f$ in measure. □

21-6.9. **Exercise** Let $f_n = g_n$, μ-ae. Prove that if $\{f_n\}$ is Cauchy in measure, then so is $\{g_n\}$.

21-6.10. **Exercise** Consider the Lebesgue measure on $\mathbb{R}$.
(a) Show that $n\rho_{(0,1/n]} \to 0$ pointwise and in measure but not in L_1.
(b) Show that $\rho_{(n,n+1]} \to 0$ pointwise but not in measure.
(c) Show that $(1/n)\rho_{(n,2n]} \to 0$ uniformly but not in L_1.
(d) For every integer $m > 0$, let $g_{jm} = \rho_{(\,(j-1)/2^m, j/2^m\,]}$ for all $0 < j \leq 2^m$. Fixed m, sketch g_{jm} for various j. Let $\{f_n\}$ be an enumeration of $\{g_{jm}\}$. Show that $f_n \to 0$ in measure and in L_1 but not pointwise.

21-6.11. **Exercise** Let $f_n \to f$, $g_n \to g$ in measure. Show that for all $\alpha, \beta \in \mathbb{K}$, we have $\alpha f_n + \beta g_n \to \alpha f + \beta g$ in measure. Give a counter example of functions to show that $f_n g_n \to fg$ in measure is false in general.

21-7 Almost Uniform Convergence

21-7.1. Let μ be an E-measure on a δ-space X and $f_n, f : X \to F$ be measurable maps. Then $\{f_n\}$ *converges to f almost uniformly* if for every $\varepsilon > 0$ there is a measurable set B such that $|\mu|(B) \leq \varepsilon$ and $f_n \to f$ uniformly on $X \setminus B$. Similarly, $\{f_n\}$ is said to be *Cauchy almost uniformly* if for every $\varepsilon > 0$ there is a measurable set B such that $|\mu|(B) \leq \varepsilon$ and $\{f_n\}$ is Cauchy uniformly on $X \setminus B$. Clearly if $f_n \to f$ uniformly, then $f_n \to f$ almost uniformly.

21-7.2. **Theorem** If $f_n \to f$ almost uniformly for measures μ, ν, then $f_n \to f$ almost uniformly for $\alpha\mu + \beta\nu$ where $\alpha, \beta \in \mathbb{K}$.

Proof. Choose measurable sets A, B such that $|\mu|A \leq \varepsilon$, $|\nu|B \leq \varepsilon$ and $f_n \to f$ uniformly on $X \setminus A$ and $X \setminus B$. Then $f_n \to f$ uniformly on $X \setminus (A \cap B)$ and $|\alpha\mu + \beta\nu|(A \cap B) \leq |\alpha|\ |\mu|A + |\beta|\ |\nu|B \leq (|\alpha| + |\beta|)\varepsilon$. □

21-7.3. Since only the total variation $|\mu|$ is involved, we shall assume that μ is a *positive* measure throughout the rest of this section unless it is otherwise specified.

21-7.4. **Theorem** If $f_n \to f$ almost uniformly, then $f_n \to f$, μ-ae.

Proof. Let $f_n \to f$ almost uniformly. For every integer $s \geq 1$, there is a measurable B_s such that $\mu B_s \leq 1/s$ and $f_n \to f$ uniformly on $X \setminus B_s$. Then $B = \bigcap_{s=1}^{\infty} B_s$ is measurable and $\mu B \leq \mu B_s \leq 1/s$ for all s, i.e. $\mu B = 0$. Next choose any $x \in X \setminus B$. Then $x \in X \setminus B_s$ for some s. Since $f_n \to f$ uniformly on $X \setminus B_s$, we get $f_n(x) \to f(x)$. Hence $f_n \to f$ on $X \setminus B$. Therefore $f_n \to f$, μ-ae. □

21-7.5. **Theorem** If $\{f_n\}$ is Cauchy almost uniformly, then $\{f_n\}$ is convergent almost uniformly.

Proof. For each $s \geq 1$, there is a measurable set B_s such that $\mu B_s \leq 1/s$ and $\{f_n\}$ is Cauchy uniformly on $X \setminus B_s$. Then $B = \bigcap_{s=1}^{\infty} B_s$ is a null set. Now choose any $x \notin B$. Then $x \in X \setminus B_s$ for some s. Since $\{f_n(x)\}$ is Cauchy in F, $f(x) = \lim_{n\to\infty} f_n(x)$ exists. For $x \in B$, let $f(x) = 0$. Thus f is defined on X and $f_n \rho_B \to f$. Finally observe that for each $s \geq 1$, $\{f_n\}$ is uniformly Cauchy on $X \setminus B_s$. Hence $\forall\, \varepsilon > 0$, $\exists\, k$, $\forall\, m, n \geq k$, $\forall\, x \in X \setminus B_s$, $\|f_m(x) - f_n(x)\| \leq \varepsilon$. Letting $m \to \infty$ we have $\|f(x) - f_n(x)\| \leq \varepsilon$, $\forall\, n \geq k$ and $x \in X \setminus B_s$. Therefore $f_n \to f$ uniformly on $X \setminus B_s$ and $\mu B_s \leq 1/s$, i.e. $f_n \to f$ almost uniformly. Since all f_n are measurable, so are $f_n \rho_B$ and their pointwise limit f. □

21-7.6. **Theorem** If $f_n \to f$ almost uniformly, then $f_n \to f$ in measure.

Proof. For every $\varepsilon > 0$, let $A_n = \{x \in X : \|f_n(x) - f(x)\| \geq \varepsilon\}$. Since $f_n \to f$ almost uniformly, there is a measurable set B such that $\mu B \leq \varepsilon$ and $f_n \to f$ uniformly on $X \setminus B$. Thus $\exists\, k$, $\forall\, n \geq k$, $\forall\, x \in X \setminus B$, $\|f_n(x) - f(x)\| < \varepsilon$. Hence $\forall\, n \geq k$, $A_n \subset B$, that is $\mu A_n \leq \mu B \leq \varepsilon$. Therefore $\lim_{n\to\infty} \mu A_n = 0$, or $f_n \to f$ in measure. □

21-7.7. **Theorem** If $\{f_n\}$ is Cauchy in measure, then it has a subsequence which is convergent almost uniformly.

Proof. For every $\alpha > 0$, let $A_{mn}(\alpha) = \{x : \|f_m(x) - f_n(x)\| \geq \alpha\}$. Given that $\{f_n\}$ is Cauchy in measure, we have $\lim \mu A_{mn}(\alpha) = 0$ as $m, n \to \infty$. Inductively, there are integers $1 < n(1) < n(2), \cdots$, such that for all $m, n \geq n(i)$, we have $\mu A_{mn}(2^{-i}) \leq 2^{-i}$. Define $g_i = f_{n(i)}$ for each $i \geq 1$. We claim that the subsequence $\{g_i\}$ is Cauchy almost uniformly. In fact, let

$$B_i = \{x : \|g_{i+1}(x) - g_i(x)\| \geq 2^{-i}\}.$$

Then
$$\mu B_i = \mu A_{n(i+1)n(i)}(2^{-i}) \leq 2^{-i}.$$

Hence
$$\mu\left(\bigcup_{i=k}^{\infty} B_i\right) \leq \sum_{i=k}^{\infty} \mu B_i \leq 2^{-k+1}.$$

On the other hand, take any $x \in X \setminus \bigcup_{i=k}^{\infty} B_i$, we have $\forall\, i \geq k, x \notin B_i$, that is $\|g_{i+1}(x) - g_i(x)\| < 2^{-i}$. Hence for all $j > i \geq k$, we obtain

$$\|g_j(x) - g_i(x)\| < 2^{-i+1} \leq 2^{-k+1}.$$

Therefore $\{g_i : i \geq 1\}$ is Cauchy uniformly on $X \setminus \bigcup_{i=k}^{\infty} B_i$. Consequently, $\{g_i\}$ is Cauchy almost uniformly and hence $\{g_i\}$ converges almost uniformly. □

21-7.8. **Corollary** If $\{f_n\}$ is Cauchy in measure then it is convergent in measure.

21-7.9. **Dominated Convergence Theorem in Measure** Let $f, f_n \in L_p(\mu, F)$ where $1 \leq p < \infty$. Suppose that $f_n \to f$ in measure. If there is $g \in L_p(\mu)$ with $|f_n| \leq g$, μ-ae for all n; then $f_n \to f$ in $L_p(\mu, F)$.

Proof. Suppose to the contrary that $f_n \to f$ in $L_p(\mu, F)$ is false. Then for some $\varepsilon > 0$, there is a subsequence $\{g_n\}$ of $\{f_n\}$ with $\|g_n - f\|_p \geq \varepsilon$ for all n. Now $\{g_n\}$ has a subsequence $\{h_n\}$ convergent almost uniformly to some measurable map h. Since $\{h_n\}$ converges in measure to both f, h; we have $f = h$, μ-ae. From $h_n \to f$ and $|h_n| \leq g \in L_p(\mu)$, μ-ae; Dominated Convergence Theorem ensures $\|h_n - f\|_p \to 0$ contrary to $\|g_n - f\|_p \geq \varepsilon$. □

21-7.10. **Egorov's Theorem** Let μ be an E-measure on a δ-space X and $f_n, f : X \to F$ be measurable maps. Suppose that either $|\mu|(X) < \infty$, or there is an integrable function g such that $|f_n| \leq g$, μ-ae for all n. If $f_n \to f$, μ-ae; then $f_n \to f$ almost uniformly.

Proof. Without loss of generality we may assume that $\mu \geq 0$, $f_n \to f$ pointwise and in second case $|f_n| \leq g$ everywhere. To show $f_n \to f$ almost uniformly, for every $\varepsilon > 0$ we have to find a measurable set B such that $\mu B \leq \varepsilon$ and $f_n \to f$ uniformly on $X \setminus B$, i.e. $\forall\ \delta > 0$, we have to find k such that $\forall\ n \geq k$, $\forall\ x \in X \setminus B$, $\|f_n(x) - f(x)\| < \delta$. Now for all integers $m, n \geq 1$ define $A_n(m) = \{x \in X \setminus B : \|f_n(x) - f(x)\| \geq 1/m\}$ and $B_k(m) = \bigcup_{n=k}^{\infty} A_n(m)$. We claim $\bigcap_{k=1}^{\infty} B_k(m) = \emptyset$. Suppose to the contrary that there is some $x \in \bigcap_{k=1}^{\infty} B_k(m)$, that is $x \in B_k(m)$ for every k. Choose some $n \geq k$ satisfying $x \in A_n(m)$, i.e. $\|f_n(x) - f(x)\| \geq 1/m$. Thus $f_n \to f$ pointwise is false. This contradiction establishes what we claimed. Clearly $B_{k+1}(m) \subset B_k(m)$. Therefore $B_k(m) \downarrow \emptyset$ as $k \to \infty$. Suppose $\mu X < \infty$. Then for every k we have $\mu B_k(m) < \infty$ by monotone property of positive measures. Next, suppose $\|f_n(x)\| \leq g(x)$ for all x. Then

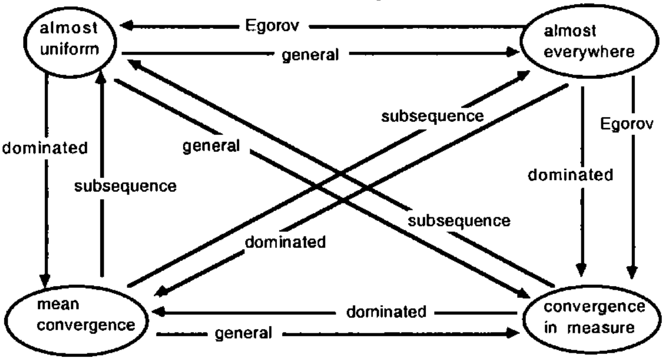
$$\|f_n(x) - f(x)\| \leq \|f_n(x)\| + \|f(x)\| \leq 2g(x), \forall\ x \in X$$

or
$$B_k(m) \subset \bigcup_{n=1}^{\infty} A_n(m) \subset \{x \in X : g(x) \geq 1/(2m)\}.$$

Since g is integrable, we have $\mu\{x \in X : g(x) \geq \frac{1}{2m}\} < \infty$. Again we have $\mu B_k(m) < \infty$ for all k. Thus we obtain $\mu B_k(m) \downarrow 0$ when $k \to \infty$. Hence for any $\varepsilon > 0$ and m, there is $k(m)$ satisfying $\mu B_{k(m)}(m) \leq \varepsilon/2^m$. Let $B = \bigcup_{m=1}^{\infty} B_{k(m)}(m)$. Then B is measurable. By countable subadditivity, we get $\mu B \leq \sum_{m=1}^{\infty} \mu B_{k(m)}(m) \leq \sum_{m=1}^{\infty} \varepsilon/2^m = \varepsilon$. For every $\delta > 0$, let $m > 1/\delta$. Now take any $n \geq k(m)$, and $x \in X \backslash B$. We have $x \notin B_{k(m)}(m)$, i.e. $x \notin A_n(m)$, or $\|f_n(x) - f(x)\| < \frac{1}{m} < \delta$. This proves $f_n \to f$ uniformly on $X \setminus B$. □

21-7.11. **Corollary** If $f_n \to f$, μ-ae on a σ-finite set M; then there are disjoint decent sets A_j and a null set N such that $M = \bigcup_{j=1}^{\infty} A_j \cup N$ and that $f_n \to f$ uniformly on each A_j.

Proof. Write $M = \bigcup_{j=1}^{\infty} D_j \cup D$ where D_j are disjoint decent sets and D is null. Since $f_n \to f$, μ-ae on D_j; there is a measurable subset B_j^1 of D_j such that $|\mu| B_j^1 \leq 1$ and $f_n \to f$ uniformly on $A_j^1 = D_j \setminus B_j^1$. Inductively there is a measurable subset B_j^k of B_j^{k-1} such that $|\mu| B_j^k \leq 1/k$ and $f_n \to f$ uniformly on $A_j^k = B_j^{k-1} \setminus B_j^k$. Because $|\mu| N_j \leq |\mu| B_{jk} \leq 1/k$ for all k, $N_j = \bigcap_{k=1}^{\infty} B_j^k$ is null and $N = \bigcup_{j=1}^{\infty} N_j \cup D$ is also null. Since A_j^k, B_j^k are measurable subset of the decent set D_j, every A_j^k is a decent set. By construction, $f_n \to f$ uniformly on each A_j^k. From $M = \bigcup_{j=1}^{\infty} D_j \cup D = \bigcup_{j=1}^{\infty} (\bigcup_{k=1}^{\infty} A_j^k \cup N_j) \cup D = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_j^k \cup N$, the result follows by an enumeration of $\{A_j^k : j, k \geq 1\}$. □



21-7.12. **Exercise** Show that as $n \to \infty$ the functions $\rho_{(n,\infty)} \to 0$ converges pointwise but not almost uniformly on $\mathbb{R}$. Show that x^n converges almost uniformly but not uniformly on $[0, 1]$. Are these two sequences of functions convergent in measure ?

21-7.13. **Theorem** Let $|\mu|(X) < \infty$. If $f_n \to f$ uniformly and if all $f_n \in L_p(\mu, F)$, then $f \in L_p(\mu, F)$ and $\|f_n - f\|_p \to 0$.

Proof. Since $f_n \to f$ uniformly, $\{f_n\}$ is Cauchy uniformly. For $\varepsilon = 1$, $\exists\, k$, $\forall\, n \geq k$, $\sup_{x \in X} \|f_n(x) - f_k(x)\| \leq 1$, i.e. $|f_n| \leq |f_k| + 1$. Because $|\mu| X < \infty$ we have $1 \in L_p^r$. Consequently $|f_k| + 1 \in L_p^r$. Now the result follows immediately from L_p-Dominated Convergence Theorem. □

21-7.14. **Exercise** Let $f_n = g_n$ and $f = g$, μ-ae. Prove that if $f_n \to f$ almost uniformly, then $g_n \to g$ almost uniformly.

21-7.15. **Exercise** Which of the functions in §23-6.10 converge almost uniformly?

21-8 More Than One Measure

21-8.1. Let μ, ν be vector measures on a δ-space $(X, \mathbb{D})$ into a Banach space E where $\mathbb{D}$ is generated by a semiring $\mathcal{S}$. If $|\mu| \le |\nu|$ on $\mathcal{S}$ then by §20-5.2, we have $|\mu| \le |\nu|$ on measurable sets; hence for every ν-integrable map $f : X \to F$, the upper function $|f|$ is $|\nu|$-integrable; $|\mu|$-integrable and finally μ-integrable.

21-8.2. **Theorem** If $f : X \to F$ is integrable for both μ and ν, then f is integrable for $\mu + \nu$ and $\alpha\mu$ for every $\alpha \in \mathbb{K}$. Furthermore we have

$$\int f d(\mu + \nu) = \int f d\mu + \int f d\nu \qquad \text{and} \qquad \int f d(\alpha\mu) = \alpha \int f d\mu.$$

Proof. Since f is μ-integrable, the upper function $|f|$ is integrable for $|\mu|$. Similarly $|f|$ is integrable for $|\nu|$. From

$$\int |f| \; d|\mu + \nu| \le \int |f| \; d(|\mu| + |\nu|) = \int |f| \; d|\mu| + \int |f| \; d|\nu| < \infty,$$

$|f|$ is integrable for $|\mu + \nu|$. Therefore f is integrable for $\mu + \nu$. Next choose simple maps f_n such that $|f_n| \le |f|$ and $f_n \to f$. Then all f_n are integrable simple maps for $\mu, \nu, \mu + \nu$. Hence

$$\int f d(\mu + \nu) = \lim \int f_n d(\mu + \nu) = \lim \left(\int f_n d\mu + \int f_n d\nu \right)$$
$$= \lim \int f_n d\mu + \lim \int f_n d\nu = \int f d\mu + \int f d\nu.$$

The second equality is left as an exercise. □

21-8.3. **Theorem** The set of all σ-finite E-measures on a δ-space $(X, \mathbb{D})$ forms a vector space.

Proof. Let μ, ν be two σ-finite vector measures on X. There are disjoint decent sets $A_1, A_2, \cdots$ and a μ-null set M such that $X = M \cup \bigcup_{i=1}^{\infty} A_i$. Since M is a measurable subset of the ν-σ-finite set X, M is itself ν-σ-finite. There are disjoint decent sets $B_1, B_2, \cdots$ and a ν-null set N such that $M = N \cup (\bigcup_{j=1}^{\infty} B_j)$.

So we obtain $X = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{j=1}^{\infty} B_j\right) \cup N$. Since N is null for both μ, ν, we have $|\mu|(N \cap D) = |\nu|(N \cap D) = 0$ and hence

$$|\mu + \nu|(N \cap D) \leq |\mu|(N \cap D) + |\nu|(N \cap D) = 0, \qquad \forall\ D \in \mathbb{D}.$$

Therefore N is null for $\mu + \nu$. Consequently $\mu + \nu$ is σ-finite. Similarly $\alpha\mu$ is σ-finite for each $\alpha \in \mathbb{K}$. This completes the proof. □

21-9 Integration on Subspaces

21-9.1. Let $(X, \mathbb{D})$ be a δ-space and $\mathbb{M}(X)$ the family of all measurable sets of X. We shall study the induced structures on a measurable subset Q of X. Let F, E, FE be Banach spaces with an admissible bilinear map $F \times E \to FE$.

21-9.2. **Theorem** (a) The family $\mathbb{D}(Q) = \{A \cap Q : A \in \mathbb{D}\}$ is a δ-ring over Q. Furthermore, we have $\mathbb{D}(Q) \subset \mathbb{D}$. It is natural to call Q together with $\mathbb{D}(Q)$, a *measurable subspace* of X.
(b) The family $\mathbb{M}(Q) = \{\ H \cap Q : H \in \mathbb{M}(X)\ \}$ consists of precisely all measurable sets of the measurable subspace Q.

Proof. (a) Since the inverse images of the inclusion map $f(x) = x$ from Q into X preserves unions, intersections and relative complements, $\mathbb{D}(Q)$ is a δ-ring. Because Q is measurable, we have $\mathbb{D}(Q) \subset \mathbb{D}$.
(b) Let $\mathbb{N}(Q)$ denote the family of all measurable sets of the measurable subspace Q. Take any $H \in \mathbb{M}(X)$ and $B \in \mathbb{D}(Q)$. Then $B = A \cap Q$ for some $A \in \mathbb{D}(X)$. Thus $H \cap A \in \mathbb{D}$ and $(H \cap A) \cap Q \in \mathbb{D}(Q)$, i.e. $(H \cap Q) \cap B \in \mathbb{D}(Q)$. Hence $\mathbb{M}(Q) \subset \mathbb{N}(Q)$. Conversely, take any $H \in \mathbb{N}(Q)$ and $A \in \mathbb{D}$. Then we have $A \cap Q \in \mathbb{D}(Q)$ and $H \cap A = (H \cap Q) \cap A = H \cap (A \cap Q) \in \mathbb{D}(Q) \subset \mathbb{D}$. Thus $H \in \mathbb{M}(X)$, or $H = H \cap Q \in \mathbb{M}(Q)$. Therefore $\mathbb{N}(Q) \subset \mathbb{M}(Q)$. □

21-9.3. **Theorem** The restriction of a decent map $f : X \to F$ to Q is a decent map on the measurable subspace $(Q, \mathbb{D}(Q))$ which will be denoted by Q for simplicity. The same is true for simple maps, measurable maps and upper functions.

Proof. Let $g : X \to F$ be a map of the form $g = \sum_{j=1}^{n} \alpha_j \rho_{A_j}$ where $\alpha_j \in F$ and $A_j \subset X$. Then the restriction $h : Q \to F$ is given by $h = \sum_{j=1}^{n} \alpha_j \rho_{A_j \cap Q}$. If all $A_j \in \mathbb{D}$, then $A_j \cap Q \in \mathbb{D}(Q)$. Hence if g is a decent map on X, then h is a decent map on Q. The same result holds for simple maps. Suppose that f is a measurable map on X. Take any decent subset A of X. Choose simple maps $g_n \to f$ on A. Then all $g_n|Q$ are simple maps on Q and $g_n|Q \to f|Q$ on

$A \cap Q$. Therefore $f|Q$ is measurable. We leave the case of upper functions as an exercise. □

21-9.4. For every map $g : Q \to F$, define $g\rho_Q : X \to F$ by $(g\rho_Q)(x) = g(x)$ if $x \in Q$ and $(g\rho_Q)(x) = 0$, otherwise. It is easy to prove that g is measurable on Q iff $g\rho_Q$ is measurable on X. Similar criterion holds for an extended function $h : Q \to [0, \infty]$ to be upper functions.

21-9.5. **Exercise** Let $Q = \bigcup_n Q_n$ be a countable union of measurable sets Q_n of X. Prove that a map $f : Q \to F$ is measurable on Q iff f is measurable on each Q_n. Also prove the case for upper functions.

21-9.6. Let μ be an E-measure on a δ-space $(X, \mathbb{D})$ and Q a measurable set. Because of $\mathbb{D}(Q) \subset \mathbb{D}$, the restriction of μ to $\mathbb{D}(Q)$ is a well-defined measure. For simplicity, this restriction of μ to $\mathbb{D}(Q)$ is again denoted by μ. A map $f : X \to F$ is said to be *integrable on* Q if $f\rho_Q$ is integrable on X. In this case, write $\int_Q f d\mu = \int f\rho_Q d\mu$. A map $g : Q \to F$ is said to be *integrable on* Q if $g\rho_Q$ is integrable on X. The same convention for upper functions and their integrals on Q are defined.

21-9.7. **Exercise** Prove that if Q is a null set of X, then every measurable map f on X is integrable on Q. Furthermore, $\int_Q f d\mu = 0$.

21-9.8. **Exercise** Let $M \subset N$ be measurable sets. Prove that if $f : N \to F$ is integrable on N, then it is integrable on M.

21-9.9. **Theorem** Let $Q = \bigcup_n Q_n$ be a countable union of *disjoint* measurable sets of X. If f is integrable on each Q_n such that $\sum_n \int_{Q_n} |f|\, d|\mu| < \infty$, then f is integrable on Q. Furthermore, we have $\int_Q f d\mu = \sum_n \int_{Q_n} f d\mu$. Similar statement holds for upper functions.

Proof. The integral of the upper function $|f\rho_Q|$ is given by

$$\int |f\rho_Q|\, d|\mu| = \sum\nolimits_{n=1}^{\infty} \int |f\rho_{Q_n}|\, d|\mu| < \infty.$$

Therefore $f\rho_Q$ is $|\mu|$-integrable on X, or equivalently f is μ-integrable on Q. Finally we shall prove the equality for infinite sequence $\{Q_n\}$ while the finite case is left as an exercise. Since $\{Q_n\}$ are disjoint, $|f\rho_Q| = \sum_{n=1}^{\infty} |f\rho_{Q_n}|$ is integrable. Integration term by term gives

$$\int_Q f d\mu = \int f\rho_Q d\mu = \int \sum_{n=1}^{\infty} f\rho_{Q_n} d\mu = \sum_{n=1}^{\infty} \int f\rho_{Q_n} d\mu = \sum_{n=1}^{\infty} \int_{Q_n} f d\mu. \square$$

21-9.10. **Exercise** Let f be a map on $[a, b]$ into F. Prove that if f is (Lebesgue) integrable on one of the following intervals $(a, b], [a, b], (a, b], [a, b)$, then it is integrable on all of them. Furthermore, we have

$$\int_{(a,b]} f(x)dx = \int_{[a,b]} f(x)dx = \int_{(a,b)} f(x)dx = \int_{[a,b)} f(x)dx.$$

21-9.11. **Exercise** Let f be an integrable map on an interval $[a, b]$. Define $\int_a^b f(x)dx = \int_{[a,b]} f(x)dx$. Prove that if $a \le \alpha < \beta \le b$, then f is integrable on $[\alpha, \beta]$. In this case, write $\int_\beta^\alpha f(x)dx = -\int_\alpha^\beta f(x)dx$. Prove that for all $\alpha, \beta, \gamma \in [a, b]$, we have $\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx$.

21-9.12. **Exercise** Let f be an integrable map on $[a, b]$ into F. Prove that if f is continuous at $c \in (a, b)$, then the map g on $[a, b]$ given by $g(x) = \int_a^x f(t)dt$ is differentiable at c. Furthermore, we have $g'(c) = f(c)$.

21-9.13. **Exercise** Let f, g be continuous maps on $[a, b]$ into F. Suppose that for every $x \in (a, b)$, g is differentiable at x and $g'(x) = f(x)$. Prove that the Lebesgue integral can be evaluated by $\int_a^b f(x)dx = g(b) - g(a)$ as in elementary calculus. Evaluate the Lebesgue integral $\int_0^\pi x \sin x dx$.

21-9.14. **Exercise** Note that if f is integrable on $(0, 1]$, then $f\rho_{(0,1]}$ is Lebesgue integrable but it has nothing to do with the *improper integral*. Let f be defined on $(0, 1]$ by $f(x) = n(-1)^n$ whenever $\frac{1}{n+1} < x \le \frac{1}{n}$. Show that the improper integral $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 f(x)dx$ exist but f is not integrable on $(0, 1]$.

21-9.15. **Exercise** Note that if f is integrable on $[1, \infty)$, then $f\rho_{[1,\infty)}$ is Lebesgue integrable but it has nothing to do with the *improper integral*. Let f be defined on $[1, \infty)$ by $f(x) = \dfrac{(-1)^n}{n}$ whenever $n \le x < n + 1$. Show that the improper integral $\displaystyle\lim_{b\to\infty} \int_1^b f(x)dx$ exists but f is not integrable on $[1, \infty)$.

21-9.16. **Exercise** Show that if $f \in L_1^r(\mathbb{R})$, then so is $\dfrac{\sin x}{x} f(x)$.

21-99. **References and Further Readings** : Dinculeanu-67, Diestel-77, Duchon, Taylor-65, Okikiolu, Figiel, Mikusinski, Rao-91,93, Byers, Dorbrakov and Pallu.

Chapter 22

Finite Products of Measures

22-1 Product Measurable Spaces

22-1.1. Semirings, decent sets and Unique Extension Theorem provide the natural and elegant tools for the study of the product structures. In this section, we only deal with geometric aspects. In this chapter, integrands are restricted to scalar functions. Further investigation on vector integrands is required. Let $\mathcal{S}, \mathcal{T}$ be semirings over sets X, Y respectively.

22-1.2. **Theorem** The product family $\mathcal{S} \times \mathcal{T} = \{A \times B : A \in \mathcal{S},\ B \in \mathcal{T}\}$ is a semiring over the product set $X \times Y$.

Proof. Let $A \times B$, $S \times T$ be in $\mathcal{S} \times \mathcal{T}$. Write $A \setminus S = \bigcup_{i=1}^m A_i$ and $B \setminus T = \bigcup_{j=1}^n B_j$. Then the definition of semirings can be verified easily by the identities $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$ and

$$\begin{aligned}(A \times B) \setminus (S \times T) &= [(A \setminus S) \times B] \cup [(A \cap S) \times (B \setminus T)] \\ &= \left[\textstyle\bigcup_{i=1}^m (A_i \times B)\right] \cup \left[\textstyle\bigcup_{j=1}^n (A \cap S) \times B_j\right]. \end{aligned}$$ □

22-1.3. Let $\mathcal{R}(\mathcal{S})$, $\mathcal{R}(\mathcal{T})$, $\mathcal{R}(\mathcal{S} \times \mathcal{T})$ be the rings generated by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{S} \times \mathcal{T}$ respectively. Let $\mathcal{A} = \mathbb{D}(\mathcal{S})$, $\mathcal{B} = \mathbb{D}(\mathcal{T})$, $\mathcal{A} \otimes \mathcal{B} = \mathbb{D}(\mathcal{S} \times \mathcal{T})$ denote the δ-rings generated by $\mathcal{S}$, $\mathcal{T}$, $\mathcal{S} \times \mathcal{T}$ respectively. Sets in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ are called ***decent sets*** of X, Y, $X \times Y$ respectively. The families of measurable sets of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ are denoted by $\mathbb{M}(\mathcal{A})$, $\mathbb{M}(\mathcal{B})$, $\mathbb{M}(\mathcal{A} \otimes \mathcal{B})$ respectively.

22-1.4. **Exercise** (a) Prove that $\mathcal{R}(\mathcal{S} \times \mathcal{T}) = \mathcal{R}[\mathcal{R}(\mathcal{S}) \times \mathcal{R}(\mathcal{T})]$.
(b) Prove that if f, g are step functions on X, Y respectively; then so is the function on $X \times Y$ defined by $(f \otimes g)(x, y) = f(x)g(y)$.

22-1.5. **Theorem** Products of decent sets are decent sets.

Proof. To prove $\mathcal{A} \times \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B}$, take any $B \in \mathcal{T}$. Define

$$\mathcal{E} = \{P \in \mathcal{A} : P \times B \in \mathcal{A} \otimes \mathcal{B}\}.$$

Then the following identities

$$(A \cup P) \times B = (A \times B) \cup (P \times B)$$

$$(\bigcap_n A_n) \times B = \bigcap_n (A_n \times B)$$

and
$$(A \setminus P) \times B = (A \times B) \setminus [(A \cap P) \times B]$$

show that $\mathcal{E}$ is a δ-ring. Since $\mathcal{S} \times \{B\} \subset \mathbb{D}(\mathcal{S} \times \mathcal{T}) = \mathcal{A} \otimes \mathcal{B}$, we have $\mathcal{S} \subset \mathcal{E}$ and consequently $\mathcal{A} = \mathbb{D}(\mathcal{S}) \subset \mathcal{E}$, i.e. $\mathcal{A} \times \{B\} \subset \mathcal{A} \otimes \mathcal{B}$. Similarly take any $A \in \mathcal{A}$ and define $\mathcal{F} = \{T \in \mathcal{B} : A \times T \in \mathcal{A} \otimes \mathcal{B}\}$. Then $\mathcal{F}$ is a δ-ring containing $\mathcal{T}$ and hence $\mathcal{B} = \mathbb{D}(\mathcal{T}) \subset \mathcal{F}$ which is equivalent to the required statement. □

22-1.6. **Exercise** Prove that if f, g are decent functions; then so is $f \otimes g$.

22-1.7. **Corollary** $\mathcal{A} \otimes \mathcal{B}$ is the δ-ring generated by the semiring $\mathcal{A} \times \mathcal{B}$.

Proof. It follows from last theorem that $\mathbb{D}(\mathcal{A} \times \mathcal{B}) \subset \mathcal{A} \otimes \mathcal{B}$. On the other hand, since $\mathcal{S} \times \mathcal{T} \subset \mathcal{A} \times \mathcal{B}$ we have $\mathcal{A} \otimes \mathcal{B} = \mathbb{D}(\mathcal{S} \times \mathcal{T}) \subset \mathbb{D}(\mathcal{A} \times \mathcal{B})$. □

22-1.8. **Theorem** Products of measurable sets are measurable.

Proof. To prove $\mathbb{M}(\mathcal{A}) \times \mathbb{M}(\mathcal{B}) \subset \mathbb{M}(\mathcal{A} \otimes \mathcal{B})$, let $M \in \mathbb{M}(\mathcal{A})$, $N \in \mathbb{M}(\mathcal{B})$. Take any decent sets $A \in \mathcal{A}$, $B \in \mathcal{B}$. Then

$$(M \times N) \cap (A \times B) = (M \cap A) \times (N \cap B) \in \mathcal{A} \times \mathcal{B}$$

is a decent set of $X \times Y$. Since $\mathcal{A} \times \mathcal{B}$ generates $\mathcal{A} \otimes \mathcal{B}$, it follows that $M \times N$ is a measurable set of $X \times Y$. □

22-1.9. **Exercise** Prove that if f, g are simple functions; then so is $f \otimes g$.

22-1.10. **Exercise** Prove that if f, g are measurable functions; then so is $f \otimes g$.

22-1.11. Let P be a subset of $X \times Y$ and (x, y) a point in $X \times Y$. Then the *x-sections* and *y-sections* of P are defined by $P_x = \{y \in Y : (x, y) \in P\}$ and $P_y = \{x \in X : (x, y) \in P\}$ respectively.

22-1.12. **Theorem** If P is a decent subset of $X \times Y$, then all x-sections P_x and all y-sections P_y are decent subsets of Y, X respectively.

Proof. Let $x \in X$ be given and let $\mathcal{F} = \{P \in \mathcal{A} \otimes \mathcal{B} : P_x \in B\}$. The identities $(P \cup Q)_x = P_x \cup Q_x; (P \setminus Q)_x = P_x \setminus Q_x$ and $(\bigcap_n P_n)_x = \bigcap_n (P_n)_x$ show that $\mathcal{F}$ is a δ-ring. Take any $P = A \times B \in \mathcal{A} \times \mathcal{B}$. Then $P_x = \emptyset$ or B which are in $\mathcal{B}$. Hence $\mathcal{F}$ is a δ-ring containing $\mathcal{A} \times \mathcal{B}$. Since $\mathcal{A} \otimes \mathcal{B}$ is generated by $\mathcal{A} \times \mathcal{B}$ we have $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{F}$ which is equivalent to say that all P_x are decent subsets of Y for $P \in \mathcal{A} \otimes \mathcal{B}$. The other part follows by symmetry. □

22-1.13. **Theorem** Let x be a point in a decent subset A of X. If P is a measurable subset of $X \times Y$, then P_x is a measurable subset of Y.

Proof. Let B be a decent subset of Y. Since P is measurable, $(A \times B) \cap P$ is a decent set in $\mathcal{A} \otimes \mathcal{B}$. Hence $B \cap P_x = [(A \times B) \cap P]_x$ is a decent set. Consequently, P_x is measurable. □

22-1.14. **Exercise** Let $\mathcal{S}$ denote the semiring of semi-intervals of the x-axis and $\mathcal{T}$ the semiring of singletons of the y-axis. Sketch a few sets in $\mathcal{R}(\mathcal{S} \times \mathcal{T})$. Is the closed unit rectangle $[0,1]^2$ a decent set in $\mathbb{D}(\mathcal{S} \times \mathcal{T})$? Is the open unit disk a decent set in $\mathbb{D}(\mathcal{S} \times \mathcal{T})$? Is it a σ-set in the product space $\mathbb{R}^2$? Answer the same questions for the closed unit disk. Is the closed unit disk a $\sigma\delta$-set in $\mathbb{R}^2$? Show that the line segment $\{(x,x) : 0 < x \le 1\}$ is a measurable set.

22-2 Product Measures

22-2.1. In earlier chapters, we used semi-rectangles to interpret the abstract theory graphically but we have not proved whether taking areas is a measure. Now we have enough tools to prove this fact. It would be nice to have a simple direct and elementary proof for this particular case. Let E_1, E_2, E be Banach spaces with an admissible bilinear map $\varphi : E_1 \times E_2 \to E$ denoted by $\varphi(u,v) = uv$. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be δ-spaces and $\mu : \mathcal{A} \to E_1$, $\nu : Y \to E_2$ be vector measures. Strictly speaking, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ define $\mu \times_\varphi \nu = \varphi(\mu A, \nu B)$ and the measure obtained in the following theorem should be written as $\mu \otimes_\varphi \nu$. We drop φ in order to simplify notations.

22-2.2. **Theorem** There is a unique measure $\mu \otimes \nu$ on the product δ-space $(X \otimes Y, \mathcal{A} \times \mathcal{B})$ into E such that $(\mu \otimes \nu)(A \times B) = (\mu A)(\nu B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. Furthermore, we have $|\mu \otimes \nu| \le |\mu| \otimes |\nu|$ on the product δ-ring $\mathcal{A} \otimes \mathcal{B}$. Finally, if both μ, ν are positive measures; then so is $\mu \otimes \nu$. Note that μ, ν need not be σ-finite.

Proof. Define $\mu \times \nu : \mathcal{A} \times \mathcal{B} \to \mathbb{K}$ by $(\mu \times \nu)(A \times B) = (\mu A)(\nu B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. Let $A \times B = \bigcup_{n=1}^{\infty} A_n \times B_n$ be a disjoint union where $A, A_n \in \mathcal{A}$ and $B, B_n \in \mathcal{B}$. Then $\rho_{A \times B} = \sum_{n=1}^{\infty} \rho_{A_n \times B_n}$, or $\rho_A(x)\rho_B(y) = \sum_{n=1}^{\infty} \rho_{A_n}(x)\rho_{B_n}(y)$ for all $(x,y) \in X \times Y$. For each fixed y, $\rho_A(x)\rho_B(y)$ is a decent map of x and hence it is μ-integrable in x. Integrate last equation term by term with respect to μ and $|\mu|$ as follow:

$$(\mu A)\rho_B(y) = \sum_{n=1}^{\infty} (\mu A_n)\rho_{B_n}(y) \qquad \#1$$

and $$(|\mu|A)\rho_B(y) = \sum_{n=1}^{\infty}(|\mu|A_n)\rho_{B_n}(y).$$

Since the right hand side of

$$\sum_{n=1}^{\infty} \|(\mu A_n)\rho_{B_n}(y)\| \le \sum_{n=1}^{\infty}(|\mu|A_n)\rho_{B_n}(y) = (|\mu|A)\rho_B(y)$$

is the decent map $(|\mu|A)\rho_B(y)$ in y, integrate the vector map in #1 term by term with respect to ν to get $(\mu A)(\nu B) = \sum_{n=1}^{\infty}(\mu A_n)(\nu B_n)$, that is $(\mu \times \nu)(A \times B) = \sum_{n=1}^{\infty}(\mu \times \nu)(A_n \times B_n)$. Therefore $\mu \times \nu$ is countably additive on the semiring $\mathcal{A} \times \mathcal{B}$. Replacing μ, ν by $|\mu|, |\nu|$ respectively; $|\mu| \times |\nu|$ is also countably additive on $\mathcal{A} \times \mathcal{B}$. Since $|\nu| \times |\mu| \ge 0$, it is a measure. To show that $\mu \times \nu$ is of finite variation, let $\bigcup_{i=1}^{n}(A_i \times B_i) = A \times B$ be a finite partition where $A, A_i \in \mathcal{A}$ and $B, B_i \in \mathcal{B}$. Then,

$$\sum_{i=1}^{n} \|(\mu \times \nu)(A_i \times B_i)\| = \sum_{i=1}^{n} \|(\mu A_i)(\nu B_i)\| \le \sum_{i=1}^{n} \|\mu A_i\| \; \|\nu B_i\|$$
$$\le \sum_{i=1}^{n}(|\mu|A_i)(|\nu|B_i) = \sum_{i=1}^{n}(|\mu| \times |\nu|)(A_i \times B_i) \le (|\mu| \times |\nu|)(A \times B) < \infty.$$

Taking supremum, we have $|\mu \times \nu|(A \times B) \le (|\mu| \times |\nu|)(A \otimes B)$. Consequently $\mu \times \nu$ is of finite variation and we have $|\mu \times \nu| \le |\mu| \times |\nu|$ on $\mathcal{A} \times \mathcal{B}$. We conclude that $\mu \times \nu$ is a measure on the semiring $\mathcal{A} \times \mathcal{B}$. By Unique Extension Theorem, $\mu \times \nu$ can be extended uniquely to a measure on the δ-ring $\mathcal{A} \otimes \mathcal{B}$. This proves the first part. Since $\mathcal{A} \otimes \mathcal{B}$ is the δ-ring generated by $\mathcal{A} \times \mathcal{B}$, we also get $|\mu \otimes \nu| \le |\mu| \otimes |\nu|$ on $\mathcal{A} \otimes \mathcal{B}$. Finally if $\mu, \nu \ge 0$; then $(\mu \times \nu)(A \times B) = (\mu A)(\nu B) \ge 0$ and consequently $(\mu \otimes \nu)(D) \ge 0$ for every decent set D of $X \times Y$, that is $\mu \otimes \nu \ge 0$. □

22-2.3. A bilinear map $E_1 \times E_2 \to E$ is *scalar* if $\|e_1 e_2\| = \|e_1\| \; \|e_2\|$ for all $e_1 \in E_1$, $e_2 \in E_2$. This is the case if E is the tensor product of E_1, E_2; in particular, if E_1 or E_2 is the scalar field $\mathbb{K}$ and the bilinear map is the scalar multiplication. Clearly every scalar bilinear map is admissible.

22-2.4. **Theorem** $|\mu \otimes \nu| = |\mu| \otimes |\nu|$ on the product δ-ring $\mathcal{A} \otimes \mathcal{B}$ if the bilinear map $E_1 \times E_2 \to E$ is scalar.

Proof. Let $A = \bigcup_{i=1}^{m} A_i$ and $B = \bigcup_{j=1}^{n} B_j$ be disjoint unions where $A, A_i \in \mathcal{A}$ and $B, B_j \in \mathcal{B}$. Then $A \times B = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} A_i \times B_j$ is a disjoint union. Observe

$$\left(\sum_{i=1}^{m} \|\mu A_i\|\right)\left(\sum_{j=1}^{n} \|\nu B_j\|\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \|\mu A_i\| \; \|\nu B_j\|$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \|(\mu A_i) \otimes (\nu B_j)\| \qquad \text{by given condition}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} |(\mu \times \nu)(A_i \times B_j)| \le \sum_{i=1}^{m}\sum_{j=1}^{n} |\mu \times \nu|(A_i \times B_j)$$
$$= |\mu \times \nu|(A \times B).$$

Taking supremum over all finite partitions of A, B by A_i, B_j respectively; we have $(|\mu| \times |\nu|)(A \times B) = |\mu|(A)\ |\nu|(B) \le |\mu \times \nu|(A \times B)$. Therefore we have $|\mu| \times |\nu| \le |\mu \times \nu|$ on $\mathcal{A} \times \mathcal{B}$. Consequently, $|\mu \otimes \nu| = |\mu| \otimes |\nu|$ on $\mathcal{A} \otimes \mathcal{B}$. □

22-2.5. **Exercise** Let $E_1 = E_2 = E = \mathbb{R}$ and μ, ν the Lebesgue measure. Suppose that $(u, v) \to uv : E_1 \times E_2 \to E$ is defined by $uv = 0$ for all $u \in E_1$ and $v \in E_2$. For $A = B = (1, 3]$, compare $|\mu \times \nu|(A \times B)$ and $(|\mu|A)(|\nu|B)$. □

22-2.6. **Theorem** Let M, N be measurable subsets of X, Y respectively. If either M or N is null, then $M \times N$ is null.

Proof. Take any $A \times B \in \mathcal{A} \times \mathcal{B}$. Then we have

$$\begin{aligned} &|\mu \otimes \nu|\ [(A \times B) \cap (M \times N)] \\ \le\ &|\mu| \otimes |\nu|\ [(A \cap M) \times (B \cap N)] = |\mu|(A \cap M)\ |\nu|(B \cap N) = 0. \end{aligned}$$

It follows from §20-3.7 that $M \times N$ is $\mu \otimes \nu$-null. □

22-2.7. **Theorem** If both M, N are σ-finite sets of X, Y respectively; then $M \times N$ is σ-finite.

Proof. Write $M = \bigcup_{i=0}^{\infty} M_i$ and $N = \bigcup_{j=0}^{\infty} N_j$ where M_0, N_0 are null sets for μ, ν and M_i, N_i for $i \ge 1$ are decent subsets of X, Y respectively. Then

$$M \times N = \left[\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (M_i \times N_j)\right] \cup (M \times N_0) \cup (M_0 \times N).$$

The first term is a countable union of decent sets in $X \times Y$ and the last two are null set by last theorem. Therefore $M \times N$ is σ-finite. □

22-2.8. **Exercise** Prove that $\mu \otimes \nu$ is linear in μ, ν separately.

22-2.9. **Exercise** Let μ, ν be the counting measure on the set $\mathbb{N}$ of natural numbers $1, 2, 3, \cdots$ weighted by complex functions f, g respectively. Describe the product measure $\mu \otimes \nu$ on the product space $\mathbb{N}^2$ in terms of $f \otimes g$.

22-3 Repeated Integrals

22-3.1. Let E_1, E_2, E be Banach spaces with an admissible bilinear map $E_1 \times E_2 \to E$. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be δ-spaces and let $\mu : \mathcal{A} \to E_1$, $\nu : Y \to E_2$ be vector measures. For convenience, decent sets in $X \times Y$ which are countable unions of sets in $\mathcal{A} \times \mathcal{B}$ are called *σ-product sets*. Countable intersections of σ-product sets are called *σδ-product sets*. For convenience, we shall restrict ourselves to scalar functions in this section.

22-3.2. **Lemma** Let D be a decent subset of $M \times N \in \mathcal{A} \times \mathcal{B}$. Then the measure μD_y of y-section D_y is a ν-integrable map of $y \in Y$. Furthermore,

$$(\mu \otimes \nu)(D) = \int_Y \mu D_y d\nu(y).$$

Proof. Case 1: Assume $D = A \times B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $y \in B$ then $D_y = A$ and $\rho_B(y) = 1$. If $y \notin B$ then $D_y = \emptyset$ and $\mu D_y = 0 = \rho_B(y)$. In both cases, we have $\mu D_y = (\mu A)\rho_B(y)$ which is a decent map on Y. Hence it is ν-integrable on Y and

$$(\mu \otimes \nu)(D) = (\mu A)(\nu B) = \int_Y (\mu A)\rho_B(y) d\nu(y) = \int_Y \mu D_y d\nu(y).$$

Case 2: Assume that $D = \bigcup_{n=1}^{\infty} D_n$ is a σ-product set where each $D_n \in \mathcal{A} \times \mathcal{B}$. Since $\mathcal{A} \times \mathcal{B}$ is a semiring, we may assume $\{D_n\}$ to be *disjoint*. Then $D_y = \bigcup_{n=1}^{\infty} D_{ny}$ is a disjoint union of decent sets of Y. Applying the measures $|\mu|$ and μ, we have

$$|\mu| D_y = \sum_{n=1}^{\infty} |\mu| D_{ny} \qquad \text{and} \qquad \mu D_y = \sum_{n=1}^{\infty} \mu D_{ny}. \qquad \#1$$

By Case 1, both $|\mu| D_{ny}$ and μD_{ny} are integrable in y for $|\nu|$ and ν respectively. Observe

$$\int_Y \sum_{n=1}^{\infty} \|\mu D_{ny}\| \, d|\nu|(y) \leq \int_Y \sum_{n=1}^{\infty} |\mu| D_{ny} d|\mu|(y)$$

$$= \sum_{n=1}^{\infty} \int_Y |\mu| D_{ny} d|\nu|(y) = \sum_{n=1}^{\infty} (|\mu| \times |\nu|) D_n = (|\mu| \times |\nu|) D < \infty.$$

It follows that #1 is a ν-integrable map in y and

$$\int_Y \nu D_y d\nu(y) = \sum_{n=1}^{\infty} \int_Y \mu D_{ny} d\nu(y) = \sum_{n=1}^{\infty} (\mu \otimes \nu) D_n = (\mu \otimes \nu) D.$$

Case 3: Assume that D is a $\sigma\delta$-product set. Then there are σ-product set $D_n \downarrow D$. Clearly $Q_n = D_n \cap (M \times N)$ is again a σ-product set and $Q_n \downarrow D$. Since $Q_n \subset M \times N$, every Q_n is a decent subset of $X \times Y$ so that Case 2 is applicable to Q_n. Hence, $\|\mu Q_{ny}\| \leq |\mu| Q_{ny} \leq |\mu|(M \times N)_y = (|\mu| M)\rho_N(y)$. By Dominated Convergence Theorem, $\mu D_y = \lim_{n\to\infty} \mu Q_{ny}$ is ν-integrable in y. Furthermore, we have

$$(\mu \otimes \nu) D = \lim_{n\to\infty} (\mu \otimes \nu) Q_n = \lim_{n\to\infty} \int_Y \mu Q_{ny} d\nu(y) = \int_Y \mu D_y d\nu(y).$$

Case 4: Assume $|\mu \otimes \nu|(D) = 0$. Then for each $n \geq 1$ there is $A_j^n \times B_j^n \in \mathcal{A} \times \mathcal{B}$ such that $D \subset \bigcup_{j=1}^{\infty} (A_j^n \times B_j^n)$ and $\sum_{j=1}^{\infty} |\mu \otimes \nu|(A_j^n \times B_j^n) \leq \frac{1}{n}$. Let

$$H_n = \bigcup_{j=1}^{\infty} (A_j^n \times B_j^n) \cap (M \times N)$$

and $H = \bigcap_{n=1}^{\infty} H_n$. Then $D \subset H \subset H_n \subset M \times N$. Observe that

$$|\mu \otimes \nu|(H) \le |\mu \otimes \nu|(H_n) \le \sum\nolimits_{j=1}^{\infty} |\mu \otimes \nu|(A_j^n \times B_j^n) \le \frac{1}{n},$$

i.e. $|\mu \otimes \nu|(H) = 0$. Since H is a $\sigma\delta$-product subset of $M \times N$, applying Case 3 to $|\mu| \otimes |\nu|$ we have

$$\int_Y |\mu| H_y d|\nu|(y) = (|\mu| \otimes |\nu|)(H) = |\mu \otimes \nu|(H) = 0,$$

i.e. $$|\mu D_y| \le |\mu|(D_y) \le |\mu|(H_y) = 0,\ \nu\text{-ae},$$

or $\mu D_y = 0$, ν-ae. Consequently, $\int_Y \mu D_y d\nu(y) = 0 = (\mu \otimes \nu)(D)$.

Case 5: In general, there is a $\sigma\delta$-product set H and a $\mu \otimes \nu$-null set N such that $D \cup N = H \subset M \times N$, $D \cap N = \emptyset$ and $|\mu \otimes \nu|(N) = 0$. Then

$$\mu D_y = \mu(H \setminus N)_y = \mu(H_y \setminus N_y) = \mu H_y - \mu N_y$$

is ν-integrable in y. Furthermore,

$$\begin{aligned}\int_Y \nu D_y d\nu(y) &= \int_Y \nu H_y d\nu(y) - \int_Y \mu N_y d\nu(y) \\ &= (\mu \otimes \nu)(H) - (\mu \otimes \nu)(N) = (\mu \otimes \nu)(D).\end{aligned}$$ □

22-3.3. **Lemma** Let $f : X \times Y \to \mathbb{K}$ be a decent function.

(a) For each $y \in Y$, $f(x, y)$ is μ-integrable in x on X.

(b) The map $\int_X f(x, y) d\mu(x)$ is ν-integrable in y on Y.

(c) $$\int f d(\mu \otimes \nu) = \int_Y \left[\int_X f d\mu\right] d\nu.$$

Proof. Take any decent subset D of $X \times Y$. There are *disjoint* sets $H_i = M_i \times N_i \in \mathcal{A} \times \mathcal{B}$ such that $D \subset \bigcup_{i=1}^n H_i$. Then $D = \bigcup_{i=1}^n (D \cap H_i)$ is a disjoint union of decent sets. By last lemma, $\rho_{(D \cap H_i)_y}(x) = \rho_{D \cap H_i}(x, y)$ is μ-integrable in x on X and hence so is $\rho_D(x, y) = \sum_{i=1}^n \rho_{D \cap H_i}(x, y)$. The map

$$\begin{aligned}\int_X \rho_D(x, y) d\mu(x) &= \sum\nolimits_{i=1}^n \int_X \rho_{D \cap H_i}(x, y) d\mu(x) \\ &= \sum\nolimits_{i=1}^n \int_X \rho_{(D \cap H_i)y}(x) d\mu(x) = \sum\nolimits_{i=1}^n \mu(D \cap H_i)_y\end{aligned}$$

is a ν-integrable in y on Y. Furthermore

$$\begin{aligned}\int \rho_D d\mu \otimes \nu &= \int \sum\nolimits_{i=1}^n \rho_{D \cap H_i} d\mu \otimes \nu = \sum\nolimits_{i=1}^n \int \rho_{D \cap H_i} d\mu \otimes \nu \\ &= \sum\nolimits_{i=1}^n (\mu \otimes \nu)(D \cap H_i) = \sum\nolimits_{i=1}^n \int_Y \mu(D \cap H_i)_y d\nu(y) \\ &= \int_Y \sum\nolimits_{i=1}^n \mu(D \cap H_i)_y d\nu(y) = \int_Y \left[\int_X \rho_D(x, y) d\mu(x)\right] d\nu(y).\end{aligned}$$

The proof is completed by linearity in f. □

22-3.4. In order to express a double integrals in terms of repeated integrals, we have to deal with maps which are not defined everywhere but merely defined almost everywhere. Let μ be a measure on a δ-space $(X, \mathcal{A})$. Let f be a map defined only on a subset of X with values in F. A measurable set M is called a ***μ-domain*** of f if $X \setminus M$ is a μ-null set and for each $x \in M$, $f(x) \in F$ is well-defined. The map f is called a ***μ-map***, or equivalently f is said to be ***defined*** μ-ae if f has at least one μ-domain. For every μ-domain M of a μ-map, define $f\rho_M(x) = f(x)$ if $x \in M$ and $f\rho_M(x) = 0$, otherwise. Then a μ-map f is said to be ***μ-measurable*** if there is a μ-domain M such that $f\rho_M$ is measurable and ***μ-integrable*** if $f\rho_M$ is μ-integrable. It can be proved that if f is μ-integrable (respectively μ-measurable), then so is $f\rho_N$ for every μ-domain N of f. Furthermore, $\int f\rho_N d\mu$ is independent of the choice of N and hence the integral $\int f d\mu = \int f\rho_N d\mu$ is well-defined. Properties of measurable and μ-integrable maps can be trivially extended to μ-maps. For example, $f = 0$, μ-ae means $f\rho_M = 0$, μ-ae. Clearly, every sequence of μ-maps has a common μ-domain which facilitates the algebraic and limiting operations.

22-3.5. Let $f : X \times Y \to \mathbb{K}$ be a function satisfying the following conditions.
(a) For ν-almost all $y \in Y$, $f(x, y)$ is an upper function of $x \in X$ if $\mu \geq 0$; or a μ-integrable function of x for arbitrary μ.
(b) The integral $\int f(x, y) d\mu(x)$ is an upper function of $y \in Y$ if $\nu \geq 0$; or a ν-integrable map of y for arbitrary ν. In this case, the ***value of repeated integral in this order*** is denoted by $\int \int f(x, y) d\mu(x) d\nu(y)$, or $\int \int f d\mu d\nu$.

22-3.6. A function f on $X \times Y$ is said to be ***μ-σ-finite*** on X if f is ***concentrated*** on some μ-σ-finite subset M of X, i.e. for every $x \in X \setminus M$ and for every $y \in Y$, $f(x, y) = 0$. Clearly if X is μ-σ-finite, then every function is μ-σ-finite on X. Similarly ν-σ-finiteness on Y is also defined.

22-3.7. **Exercise** In $\mathbb{R}^2$, let x-axis be equipped with the counting measure μ on the semiring $\mathcal{A}$ of singletons and y-axis with the Lebesgue measure ν on the semiring $\mathcal{B}$ of semi-intervals. Sketch a few sets in $\mathcal{A} \times \mathcal{B}$ and find their product measure. Show that $N = \{(x, y) \in \mathbb{R}^2 : x = y,\ 0 < x \leq 1\}$ is a measurable set. Verify that $\int \rho_N d(\mu \otimes \nu) = (\mu \otimes \nu)N = 0$ but $\int \int \rho_N d\mu d\nu = 1$. Is ρ_N μ-σ-finite on the x-axis? What is the outer measure $(\mu \times \nu)^*(N)$?

22-3.8. **Lemma** Let f be a measurable function on $X \times Y$. If f is σ-finite on X and Y, then there are null subsets H of X, K of Y and a sequence of decent functions f_n on $X \times Y$ such that $|f_n| \leq |f_{n+1}| \leq |f|$ on $X \times Y$ and $f_n \to f$ outside the set $(H \times Y) \cup (X \times K)$.

Proof. Let $g_n : X \times Y \to F$ be simple functions such that $|g_n| \leq |g_{n+1}| \leq |f|$ and $g_n \to f$ on $X \times Y$. For σ-finiteness of f on X and Y, there are decent subsets $A_n \subset A_{n+1}$ of X, $B_n \subset B_{n+1}$ of Y and null subsets H of X, K of Y respectively such that f is concentrated on $M = \bigcup_{n=1}^{\infty} A_n \cup H$ and $N = \bigcup_{n=1}^{\infty} B_n \cup K$. Then $f_n = g_n \rho_{A_n \times B_n}$ are decent functions on $X \times Y$ satisfying $|f_n| \leq |f_{n+1}| \leq |f|$ on $X \times Y$. Clearly if $x \in X \setminus M$ or if $y \in Y \setminus N$ then $f(x,y) = 0$ and thus $f_n(x,y) = 0$. Also if $x \in M \setminus H$ and $y \in N \setminus K$ then $(x,y) \in A_j \times B_j$ for some j, hence $f_n(x,y) = g_n(x,y) \to f(x,y)$ as $j \leq n \to \infty$. Therefore $f_n \to f$ outside $(H \times Y) \cup (X \times K)$. □

22-3.9. **Fubini's Theorem** If f is a $(|\mu| \otimes |\nu|)$-integrable function on $X \times Y$ and if f is σ-finite on X and Y, then the repeated integral exists. Furthermore we have $\displaystyle\int f d(\mu \otimes \nu) = \int\int f d\mu d\nu$.

Proof. Let f_n, H, K be constructed by last lemma. It follows from that both $g_n(y) = \int f_n(x,y) d\mu(x)$ and $h_n(y) = \int |f_n(x,y)|\ d|\mu|(x)$ are ν-integrable on Y. Since $|f_n| \leq |f_{n+1}|$, $h = \sup_{n \geq 1} h_n$ is an upper function on X. Observe

$$\int h_n d|\nu| = \int d|\nu| \int |f_n|\ d|\mu| = \int |f_n|\ d|\mu| \otimes |\nu| \leq \int |f|\ d|\mu| \otimes |\nu| < \infty.$$

By Monotone Convergence Theorem, h is ν-integrable. There is a null subset L of Y such that $0 \leq h(y) < \infty$ for every $y \in Y \setminus L$. Without loss of generality, assume $K \subset L$. Fix any $y \in Y \setminus L$ and let x vary. Then for each $y \in Y \setminus K$, $|f_n(x,y)| \uparrow |f(x,y)|$ as $n \to \infty$. Hence $\int |f_n(x,y)|\ d|\mu|(x) = h_n(y) \leq h(y) < \infty$. By Monotone Convergence Theorem, $|f(x,y)|$ is $|\mu|$-integrable in x, i.e. $f(x,y)$ is μ-integrable in x. Thus, $g(y) = \int f(x,y) d\mu(x)$ exists for each $y \in Y \setminus L$. By Dominated Convergence Theorem,

$$g_n(y) = \int f_n(x,y) d\mu(x) \to \int f(x,y) d\mu(x) = g(y).$$

Hence g is a ν-function and $g_n \to g$, ν-ae as $n \to \infty$. Since

$$|g_n(y)| = \left| \int f_n(x,y) d\mu(x) \right| \leq \int |f_n(x,y)|\ d|\mu|(x) = h_n(y) \leq h(y),$$

Dominated Convergence Theorem ensures that g is ν-integrable. Furthermore, because $|f_n| \leq |f|$, $f_n \to f$ and f is $\mu \otimes \nu$-integrable, we have

$$\int\int f d\mu d\nu = \int g(y) d\nu(y) = \lim_{n \to \infty} \int g_n(y) d\nu(y)$$

$$= \lim_{n\to\infty} \int d\nu(y) \int f_n(x,y)d\mu(x) = \lim_{n\to\infty} \int f_n(x,y)d(\mu\otimes\nu)(x,y)$$
$$= \int f(x,y)d(\mu\otimes\nu)(x,y) = \int fd(\mu\otimes\nu). \qquad \square$$

22-3.10. **Tonelli's Theorem** Let f be an upper function on $X \times Y$ and let μ, ν be *positive* measures. If f is σ-finite on X and Y, then the repeated integral exists. Furthermore, we have $\int fd(\mu\otimes\nu) = \int\int fd\mu d\nu$.

Proof. After trivial modification of last theorem, there are null sets H of X, K of Y and a sequence of decent functions f_n on $X \times Y$ such that $0 \le f_n \le f_{n+1} \le f$ and $f_n \uparrow f$ outside the null set $(H \times Y) \cup (X \times K)$. The following is a simplified version of last theorem. Now $g_n(y) = \int f_n(x,y)d\mu(x)$ is ν-integrable on X and clearly $0 \le g_n \le g_{n+1}$. Fix any $y \in Y \setminus K$ and let x vary. Then for each $x \in X \setminus H$, $f_n(x,y) \uparrow f(x,y)$ as $n \to \infty$. Hence $g(y) = \int f(x,y)d\mu(x)$ exists in $[0,\infty]$ and $0 \le g_n \uparrow g$. Finally,

$$\int\int fd\mu d\nu = \int gd\nu = \lim_{n\to\infty} \int g_n d\nu$$
$$= \lim_{n\to\infty} \int d\nu \int f_n d\mu = \lim_{n\to\infty} \int f_n d\mu\otimes\nu = \int fd\mu\otimes\nu. \qquad \square$$

22-3.11. Usually we work with σ-finite measures μ, ν. In this case, all functions on $X \times Y$ are σ-finite on both X, Y.

22-3.12. **Integration in Reversed Order** With more precise notations of §§17-2.10, 22-2.1, the Fubini's and Tonelli's theorems, give the following identity

$$\int fd(\mu\otimes_\varphi \nu) = \int_\varphi \left(\int fd\mu, d\nu\right) = \int_\varphi \left(d\mu, \int fd\nu\right).$$

22-3.13. **Exercise** Show that the characteristic function f of xy-axes is an upper function on $\mathbb{R}^2$. Find its repeated integrals.

22-3.14. **Exercise** Evaluate the repeated integrals of $f(x,y) = (x-y)/(x+y)^3$ for $0 \le x, y \le 1$.

22-3.15. **Exercise** Prove that the repeated integral $\int_0^1 \int_0^\infty e^{-x} \cos txdxdt$ exists. Show that $\int_0^\infty e^{-x} \frac{\sin x}{x}dx = \frac{1}{4}\pi$ by changing the order of integral.

22-3.16. **Exercise** Let μ, ν be vector measures on X, Y respectively and f a measurable function on $X \times Y$. Show that if $\int\int |f|\, d|\mu| \otimes |\nu| < \infty$

or $\displaystyle\int d|\mu| \int |f|\, d|\nu| < \infty$, then f is integrable on $X \times Y$.

22-3.17. **Exercise** Show that if $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} |\alpha_{mn}|) < \infty$ where $\alpha_{mn} \in \mathbb{C}$, then both sums $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} \alpha_{mn})$ and $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} \alpha_{mn})$ exist and they are equal.

22-3.18. **Theorem** If μ, ν are vector measures on δ-spaces X, Y respectively and if f, g are integrable functions on X, Y into $\mathbb{K}$ respectively; then $f \otimes g$ is integrable on $X \times Y$. Furthermore we have $\int f \otimes g d\mu \otimes \nu = \left(\int f d\mu\right)\left(\int g d\nu\right)$.

Proof. Since f is μ-integrable, there are decent functions f_n on X such that $|f_n| \le |f_{n+1}| \le |f|$ and $f_n \to f$, μ-ae. There is a μ-null set $M \subset X$ satisfying $f_n(x) \to f(x)$ for all $x \in X \setminus M$. Similarly, there are decent functions g_n on Y and a ν-null set $N \subset Y$ such that $|g_n| \le |g_{n+1}| \le |g|$ and $g_n(y) \to g(y)$ for all $y \in Y \setminus N$. Then $H = (X \times N) \cup (M \times Y)$ is $\mu \otimes \nu$-null. Clearly, $f_n \otimes g_n \to f \otimes g$ on $X \times Y \setminus H$. Observe that

$$|(f_n \otimes g_n)(x,y)| = |f_n(x) g_n(y)| = |f_n(x)|\ |g_n(y)| \le |f(x)|\ |g(y)| = |(f \otimes g)(x,y)|.$$

With trivial modification, we obtain $|f_n \otimes g_n| \le |f_{n+1} \otimes g_{n+1}|$. Therefore $|f_n \otimes g_n| \uparrow |f \otimes g|$, $\mu \otimes \nu$-ae. Now calculate,

$$\int |f_n \otimes g_n|\, d|\mu \otimes \nu| \le \int |f_n|\ |g_n|\, d|\mu| \otimes |\nu| = \int \left(\int |f_n|\ |g_n|\, d|\mu|\right) d|\nu|$$
$$= \left(\int |f_n|\, d|\mu|\right)\left(\int |g_n|\, d|\nu|\right) \le \left(\int |f|\, d|\mu|\right)\left(\int |g|\, d|\nu|\right) < \infty.$$

Hence $|f \otimes g|$ is $|\mu \otimes \nu|$-integrable, or $f \otimes g$ is $\mu \otimes \nu$-integrable. Since f, g are σ-finite on X, Y respectively; $f \otimes g$ is σ-finite on X and Y. Finally we have

$$\int f \otimes g d\mu \otimes \nu = \int \int f(x) g(y) d\mu(x) d\nu(y)$$
$$= \int g(y) \int f(x) d\mu(x) d\nu(y) = \left(\int f d\mu\right)\left(\int g d\nu\right). \qquad \square$$

22-3.19. **Theorem** Let μ, ν be vector measures on δ-spaces X, Y respectively. Then for every $1 \le p < \infty$, every function f in $L_p(X \times Y, \mu \otimes \nu, F)$ can be approximated by finite sums of functions of the form $\alpha \rho_A \rho_B$ where $\alpha \in \mathbb{K}$ and A, B are decent subsets of X, Y respectively. Furthermore, if f and μ, ν are positive (or real); then all α can be chosen as positive (or real respectively).

Proof. Consider a special case first. Let $f = \rho_D$ where D is a decent subset of $X \times Y$. Then for every $\varepsilon > 0$, there are finite number of decent sets $A_j \subset X$ and $B_j \subset Y$ such that $\{A_j \times B_j\}$ are disjoint and $(|\mu| \otimes |\nu|)(D \triangle E) \le \varepsilon$ where

$E = \cup_{j=1}^n A_j \times B_j$. Then $g_j = \rho_{A_j}$ and $h_j = \rho_{B_j}$ are decent functions on X, Y respectively. The following calculation proves the special case:

$$\left\|f - \sum_{j=1}^n g_j \otimes h_j\right\|_p^p = \int \left|f - \sum_{j=1}^n g_j \otimes h_j\right|^p d|\mu \otimes \nu|$$

$$= \int \left(\rho_{D \setminus \bigcup_{j=1}^n A_j \times B_j} + \rho_{\bigcup_{j=1}^n A_j \times B_j \setminus D}\right) d|\mu \otimes \nu|$$

$$= \int \rho_{D \triangle E} d|\mu \otimes \nu| \leq \int \rho_{D \triangle E} d(|\mu| \otimes |\nu|) = (|\mu| \otimes |\nu|)(D \triangle E) \leq \varepsilon.$$

The general case follows form Density Theorem. The special positive and real cases are left as exercises. □

22-3.20. **Exercise** Let μ, ν be vector measures on δ-spaces X, Y respectively. Prove that if $\mathcal{F}, \mathcal{G}$ are dense subsets of $L_p(X), L_p(Y)$ respectively; then the set of all finite sums of functions of the form $f \otimes g$ where $f \in \mathcal{F}$, $g \in \mathcal{G}$, is dense in $L_p(X \times Y)$.

22-3.21. **Exercise** Let μ, ν be vector measures on the δ-rings $\mathbb{D}_X, \mathbb{D}_Y$ generated by semirings $\mathcal{S}_X, \mathcal{S}_Y$ over X, Y respectively. Prove that the set of all finite sums of functions of the form $\alpha\rho_A\rho_B$ where $\alpha \in \mathbb{K}$, $A \in \mathcal{S}_X$ and $B \in \mathcal{S}_Y$ is dense in $L_p(X \times Y, \mu \otimes \nu, F)$ for all $1 \leq p < \infty$. Furthermore, if f and μ, ν are positive (or real); then all α can be chosen as positive (or real respectively).

22-3.81. **Exercise** Investigate the case when f is a vector map but E_1 or E_2 is the scalar field $\mathbb{K}$. Finally investigate the general case when all E_1, E_2, F are Banach spaces.

22-3.82. **Exercise** Let E_1, E_2, E, G be Banach spaces; $\varphi : E_1 \times E_2 \to E$ an admissible bilinear map; $\xi : E \to G$ a continuous linear map; μ, ν measures on X, Y into E_1, E_2 respectively. Investigate the integral of functions on $X \times Y$ with respect to the composite map $\xi\varphi$.

22-99. **References and Further Readings** : Bartle, Bledsoe, Dudley-71,72, Elliott, Fernandres, Godfrey, Masani, Swarts, Jefferies, Kawabe, Garcia and Fernandez.

Chapter 23

Measures on Finite Dimensional Spaces

23-1 Decent Sets of $\mathbb{R}^n$

23-1.1. The general measure theory will be applied to the space $\mathbb{R}^n$. In this section, we shall study certain related topological properties of $\mathbb{R}^n$.

23-1.2. Points of $\mathbb{R}^n$ are n-tuples $a = (a_1, a_2, \cdots, a_n)$ where $a_j \in \mathbb{R}$. Write $a < b$ if $a_j < b_j$ for each $1 \le j \le n$. Similarly $a \le b$ if $a_j \le b_j$ for all j. Define

semi-intervals by $\quad (a, b] = \{x \in \mathbb{R}^n : a < x \le b\}$
open-intervals by $\quad (a, b) = \{x \in \mathbb{R}^n : a < x < b\}$
and *closed-intervals* by $\quad [a, b] = \{x \in \mathbb{R}^n : a \le x \le b\}$.

Clearly $(a, b] = \prod_{j=1}^n (a_j, b_j]$. As a product of semirings, the family $\mathbb{S}$ of all semi-intervals forms a semiring over $\mathbb{R}^n$. Sets of the δ-ring $\mathbb{D}$ generated by $\mathbb{S}$ are decent subsets of $\mathbb{R}^n$. Since each decent set can be covered by a semi-interval, it must be a bounded set. The σ-sets are countable unions of semi-intervals and the $\sigma\delta$-sets are countable intersections of σ-sets. Measurable sets of $\mathbb{R}^n$ were defined in terms of semi-intervals. It has been proved that the family of all measurable sets is the σ-algebra generated by semi-intervals. Every measure on $\mathbb{R}^n$ can be initiated from semi-intervals and uniquely extended to all decent sets by Unique Extension Theorem. The topology of $\mathbb{R}^n$ is given by the norm $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Let us recall the notations of:

an open ball $\quad \mathbb{B}(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$
and a closed ball $\quad \overline{\mathbb{B}}(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \le r\}$.

23-1.3. **Theorem** Every non-empty open set is a σ-set.

Proof. Let $\mathbb{F}$ denote the set of all semi-intervals $(a, b] \subset V$ such that all coordinates a_i, b_i are rational. Enumerate $\mathbb{F}$ by $A_1, A_2, \cdots$. Then $V = \bigcup_{j \ge 1} A_j$ is a σ-set. □

23-1.4. **Exercise** Prove that open sets are measurable sets and continuous functions on $\mathbb{R}^n$ are measurable functions.

23-1.5. **Exercise** Show that bounded measurable sets are decent sets. Prove that open sets and closed sets are measurable. Conclude that compact sets are decent sets.

23-1.6. **Theorem** For every non-empty open set V, there is a sequence of open sets W_m such that all closures $\overline{W}_m$ are compact subsets of V, $W_m \uparrow V$ and $\overline{W}_m \uparrow V$.

Proof. Let $B_1, B_2, \cdots$ be an enumeration of all open balls $\mathbb{B}(a,r)$ such that all coordinates a_i and the radii r are rational and that $\mathbb{B}(a,2r) \subset V$. Then $W_m = \bigcup_{j=1}^m B_j$ is a required sequence. □

23-1.7. **Theorem** For every compact set K, there is a sequence of bounded open sets $V_m \downarrow K$.

Proof. For each $m \geq 1$, the compact set K is covered by the family of open balls $\{\mathbb{B}(x, 1/m) : x \in K\}$. There is a finite subset J_m of K such that $\{\mathbb{B}(x,1/m) : x \in J_m\}$ covers K. Let $W_m = \cup\{\mathbb{B}(x,1/m) : x \in J_m\}$ and $V_m = \bigcap_{k=1}^m W_k$. Then $\{V_m\}$ forms a required sequence of open sets. □

23-1.8. **Lemma** Let $e = (1, 1, \cdots, 1) \in \mathbb{R}^n$. Then for all $a < b$ in $\mathbb{R}^n$ we have $(a, b + \frac{1}{m}e) \downarrow (a, b]$, $(a - \frac{1}{m}e, b] \downarrow [a, b]$ and $[a + \frac{1}{m}e, b] \uparrow (a, b]$ as $m \to \infty$.

23-1.9. **Exercise** Prove that every open interval of $\mathbb{R}^n$ is an open set and every closed interval is a compact set.

23-1.10. **Exercise** Prove that the measurable sets are the σ-algebra generated by open intervals, or closed intervals, or bounded open sets, or compact sets.

23-2 Regularity

23-2.1. Let E, F, FE be Banach spaces with an admissible bilinear map $F \times E \to FE$ and μ a vector measure on $\mathbb{R}^n$ into E. We shall show that μ is completely determined by its values on bounded open sets, or compact sets. We shall also study the role of continuous maps. This section motivates the development of measures on locally compact spaces.

23-2.2. **Outer Regularity Theorem** For every decent set D in $\mathbb{R}^n$, there are bounded open sets $V_m \downarrow D$ such that $\lim_{m\to\infty} \mu V_m = \mu D$.

Proof. Since D is bounded, it is covered by some bounded open set V_0. Inductively, suppose that V_{m-1} has been found. There is a sequence $\{A_j\}$ of disjoint semi-intervals such that $D \subset \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} |\mu| A_j \leq |\mu| D + 1/m$. There are open intervals B_j such that $A_j \subset B_j$ and $|\mu| B_j \leq |\mu| A_j + 1/(m2^j)$. Hence $V_m = V_{m-1} \cap \bigcup_{j=1}^{\infty} B_j$ is a bounded open set with $D \subset V_m \subset V_{m-1}$. Furthermore, $|\mu| V_m \leq \sum_{j=1}^{\infty} |\mu| B_j \leq \sum_{j=1}^{\infty} \left(|\mu| A_j + \frac{1}{m2^j}\right) \leq |\mu| D + \frac{2}{m}$. Finally, $\|\mu V_m - \mu D\| = \|\mu(V_m \setminus D)\| \leq |\mu|(V_m \setminus D) = |\mu| V_m - |\mu| D \leq \frac{2}{m} \to 0$ completes the proof. □

23-2.3. For every bounded open set V, there are compact sets $K_m \uparrow V$. As a result, we have the inner regularity property: $\lim_{m\to\infty} \mu K_m = \mu V$. Therefore a measure μ is completely determined by compact sets or by inner regularity.

23-2.4. **Theorem** Every continuous map $f : \mathbb{R}^n \to F$ is locally μ-integrable. Every continuous map g with compact support belongs to $L_p(\mu, F)$ for all $1 \leq p \leq \infty$.

Proof. Since the image $f(A)$ of a decent set A is contained in the compact set $f(\overline{A})$, it is separable. For every open set B of F, the inverse $f^{-1}(B)$ is open in $\mathbb{R}^n$ and hence measurable. By §19-5.7, f is measurable. It is an exercise to complete the proof. See §27-1.6. □

23-2.5. **Continuous Approximation Theorem** Let $1 \leq p < \infty$. If $f \in L_p(\mu, F)$, then for every $\varepsilon > 0$, there is a continuous map $g : \mathbb{R}^n \to F$ with compact support such that $\|f - g\|_p \leq \varepsilon$. Furthermore, if $f \geq 0$, then we can choose $g \geq 0$.

Proof. Consider $\rho_{(a,b]}$ where $a < b$. Let $m > 0$. For each $1 \leq j \leq n$, it is obvious by drawing pictures that there is a continuous function $g_j : \mathbb{R} \to \mathbb{R}$ such that $0 \leq g_j \leq 1$ and $\rho_{(a_j+\frac{1}{m},b_j]} \leq g_j \leq \rho_{(a_j,b_j+\frac{1}{m}]}$. Hence $g^m(x) = g_1(x_1)g_2(x_2)\cdots g_n(x_n)$ is a continuous function of $x = (x_1, x_2, \cdots, x_n)$ on $\mathbb{R}^n$. It has compact support contained in $[a, b + \frac{1}{m}e]$ where $e = (1, 1, ..., 1)$. Since $\rho_{(a+\frac{1}{m}e,b]} \leq g^m \leq \rho_{(a,b+\frac{1}{m}e]}$, we obtain

$$0 \leq g^m - \rho_{(a+\frac{1}{m}e,b\,]} \leq \rho_{(a,b+\frac{1}{m}e]} - \rho_{(a+\frac{1}{m}e,b]}.$$

It follows from Dominated Convergence Theorem that

$$\lim_{m\to\infty} \int |g^m - \rho_{(a,b]}|^p d|\mu| \leq \lim_{m\to\infty} \int \left|\rho_{(a,b+\frac{1}{m}e]} - \rho_{(a+\frac{1}{m}e,b]}\right|^p d|\mu| \to 0,$$

i.e. $\|g^m - \rho_{(a,b]}\|_p \to 0$. Hence the characteristic function of a semi-interval can be approximated by continuous functions with compact support. The general case follows from §21-4.7. Also see §27-1.11 below. □

23-2.6. **Corollary** If μ, ν are vector measures on $\mathbb{R}^n$ into E such that $\int g d\mu = \int g d\nu$ for all continuous functions g with compact support, then we have $\mu = \nu$.

Proof. From last proof, we have $\|g^m - \rho_{(a,b]}\|_1 \to 0$, that is $\mu(a,b] = \lim \int g^m d\mu = \lim \int g^m d\nu = \nu(a,b]$. Therefore $\mu = \nu$. □

23-2.7. **Exercise** Consider the point measure δ_n at $\frac{1}{n}$ on the semiring $\mathcal{S}$ of semi-intervals of $\mathbb{R}$. Let $\mu A = \sum_{n=1}^{\infty} \frac{\delta_n A}{n(n+1)}, \forall A \in \mathcal{S}$. Show that μ is a positive measure on $\mathcal{S}$. Show that $Q = \{\frac{1}{2n} : n = 1, 2, 3, \cdots\}$ is a decent set of $\mathbb{R}$. Find an open set W such that $\mu Q = \mu W$. For the open set $V = (0, 1)$, find a compact subset K of V such that $\mu V \leq \mu K + \frac{1}{10}$. Show that the constant function $f(x) = 1$ is μ-integrable. Find a continuous function g with compact support such that $\|f - g\|_1 \leq \frac{1}{10}$.

23-3 Translation Invariance

23-3.1. Translation invariance of measure and integrals will be studied in detail in this section. Semi-intervals clearly play a crucial role in this and the next sections. Hence it justifies the approach of starting with semirings. Only elementary techniques are used. It serves as a motivation for more general treatment such as change-variables of next section, or Harr measures on locally compact groups which is beyond the scope of this book.

23-3.2. **Theorem** If Q is a decent set (respectively a measurable set), then so is the translate $a + Q$ by any $a \in \mathbb{R}^n$.

Proof. Let $\mathbb{D}$ denote the family of all decent sets of $\mathbb{R}^n$. It is routine to verify that $\mathbb{F} = \{Q \subset \mathbb{R}^n : a + Q \in \mathbb{D}\}$ is a δ-ring containing all semi-intervals and hence we have $\mathbb{D} \subset \mathbb{F}$ which means translates of decent sets are decent sets. For measurable sets, apply similar argument to the σ-algebra generated by semi-intervals. □

23-3.3. **Corollary** If f is a decent map (respectively, simple, or upper, or measurable map) on $\mathbb{R}^n$ into a Banach space F, then so is the *translate* $T_a f(x) = f(x + a)$ by any $a \in \mathbb{R}^n$.

Proof. For decent or simple maps, it follows from the identity $T_a \rho_Q = \rho_{Q-a}$. For upper functions, consider the pointwise limit of sequences of simple functions on X and for measurable maps on decent sets only. □

23-3.4. For every semi-interval $(a,b] = \prod_{j=1}^{n}(a_j,b_j]$ with $a < b$, let $\lambda(a,b] = \prod_{j=1}^{n}(b_j - a_j)$. Obviously, define $\lambda\emptyset = 0$. Being product measure, λ is a positive measure on the semiring of all semi-intervals. It is called the *Lebesgue measure* on $\mathbb{R}^n$. We do *not* deal with Lebesgue measurable sets. Our measurable sets are independent of any measure.

23-3.5. A measure μ on $\mathbb{R}^n$ is said to be *translation invariant* if $\mu(a+Q) = \mu Q$, for all decent sets Q and all $a \in \mathbb{R}^n$. If a measure μ on $\mathbb{R}^n$ is translation invariant on all semi-intervals, then it is also translation invariant on all decent sets because the family $\mathbb{F} = \{Q \in \mathbb{D} : \mu Q = \mu(a+Q)\}$ is a δ-ring containing all semi-intervals. In particular, the Lebesgue measure λ is translation invariant.

23-3.6. **Uniqueness Theorem** If μ is a translation invariant measure on $\mathbb{R}^n$ into a Banach space E and if $\mu(0,1]^n = t \in E$, then $\mu = t\lambda$ on all decent sets. Therefore we work with Lebesgue measure λ for the rest of this section.

Proof. Consider $n = 1$ first. Let $k, p > 0$ be integers. Then by translation invariance $\mu(0,k] = \mu(0,1] + \mu(1,2] + \cdots + \mu(k-1,k] = k\mu(0,1] = kt$ and

$$\mu(0,k] = \mu\left(0, \frac{k}{p}\right] + \mu\left(\frac{k}{p}, \frac{2k}{p}\right] + \cdots + \mu\left(\frac{(p-1)k}{p}, k\right] = p\mu\left(0, \frac{k}{p}\right].$$

Hence $\mu(0, \frac{k}{p}] = kt/p$. Now for all $a < b$, let r_j be rational numbers such that $r_j \downarrow (b-a)$ as $j \to \infty$. Then,

$$\mu(a,b] = \mu(0,b-a] = \lim_{j\to\infty} \mu(0,r_j] = \lim_{j\to\infty} r_j t = t\lambda(a,b].$$

Hence $\mu = t\lambda$ on all semi-intervals on $\mathbb{R}$. Next, assume $n > 1$. For all $a_1 < b_1$, define $\nu(a_1,b_1] = \mu\{(a_1,b_1] \times (0,1]^{n-1}\}$. It is easy to verify that ν is a translation invariant measure on the semi-intervals of $\mathbb{R}$. Hence by first case, we have $\nu(a_1,b_1] = \nu(0,1]\lambda(a_1,b_1] = t\lambda(a_1,b_1]$. Now, let $a_1 < b_1$ be fixed. For all $a_j < b_j$ where $2 \le j \le n$, define $\varphi\prod_{j=2}^{n}(a_j,b_j] = \mu(a,b]$. Since μ is translation invariant, so is φ on the semi-intervals of $\mathbb{R}^{n-1}$. Hence by induction,

$$\begin{aligned}\mu(a,b] &= \varphi\prod_{j=2}^{n}(a_j,b_j] = \{\varphi(0,1]^{n-1}\}\prod_{j=2}^{n}(b_j - a_j)\\ &= \mu\{(a_1,b_1] \times (0,1]^{n-1}\}\prod_{j=2}^{n}(b_j - a_j)\\ &= \nu(a_1,b_1]\prod_{j=2}^{n}(b_j - a_j) = t(b_1 - a_1)\prod_{j=2}^{n}(b_j - a_j) = t\lambda(a,b].\end{aligned}$$

Therefore $\mu = t\lambda$ on all semi-intervals of $\mathbb{R}^n$. By Unique Extension Theorem, $\mu = t\lambda$ on all decent sets of $\mathbb{R}^n$. □

23-3.7. **Lemma** For any measurable set M, we have $\lambda(a+M) = \lambda(M)$. In particular, translates of null sets are null.

Proof. $\lambda(a+M) = \sup\{\lambda(D) : D \in \mathbb{D}, D \subset M+a\}$
$= \sup\{\lambda(B+a) : B+a \in \mathbb{D}, B+a \subset M+a\}$, for $D = B+a$
$= \sup\{\lambda(B) : B \in \mathbb{D}, B \subset M\} = \lambda(M)$. □

23-3.8. **Theorem** If $f(x)$ is an integrable map into some Banach space F or an upper function on $\mathbb{R}^n$, then so is every translate $f(a+x)$ where $a \in \mathbb{R}^n$. Furthermore, the integral is also translation invariant, that is

$$\int f(a+x)d\lambda(x) = \int f(x)d\lambda(x).$$

Proof. Case 1: Suppose that $f = \sum_{j=1}^k \alpha_j \rho_{Q_j}$ is a decent map into F where $\alpha_j \in F$ and $Q_j \in \mathbb{D}$. Then

$$\int f(a+x)d\lambda(x) = \sum_{j=1}^k \alpha_j \int \rho_{Q_j - a}(x)d\lambda(x)$$
$$= \sum_{j=1}^k \alpha_j \lambda(Q_j - a) = \sum_{j=1}^k \alpha_j \lambda Q_j = \int f(x)d\lambda(x).$$

Case 2: For an upper function f on $\mathbb{R}^n$, choose simple functions $0 \le f_m \uparrow f$. Clearly $T_a f_m$ are simple functions and $0 \le T_a f_m \uparrow T_a f$. Hence $T_a f$ is also an upper function. By last lemma, we have

$$\int f(a+x)d\lambda(x) = \lim_{m\to\infty} \int f_m(a+x)d\lambda(x) = \lim_{m\to\infty} \int f_m(x)d\lambda(x) = \int f(x)d\lambda(x).$$

Case 3: Let $f : \mathbb{R}^n \to F$ be a λ-integrable map. Its translate is measurable. By Density Theorem, there are decent maps f_m such that $|f_m| \le |f|$ on $\mathbb{R}^n$ and $f_m \to f$, λ-ae. Clearly $|T_a f_m| \le |T_a f|$ on $\mathbb{R}^n$ and $T_a f_m \to T_a f$, λ-ae. Since $\int |T_a f| d\lambda = \int |f| d\lambda < \infty$, $T_a f$ is integrable. Finally Dominated Convergence Theorem completes the proof as in Case 2. □

23-3.9. **Theorem** Let $f \in L_p(\lambda, F)$ and $1 \le p < \infty$. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $\|a\| \le \delta$ in $\mathbb{R}^n$, we have $\|T_a f - f\|_p \le \varepsilon$.

Proof. Suppose $\varepsilon > 0$ is given. Let $g : \mathbb{R}^n \to F$ be a continuous map with compact support such that $\|f-g\|_p \le \varepsilon$. There is $t > 0$ such that for all $\|a\| \le 1$ in $\mathbb{R}^n$ we have $a+\text{supp}(g) \subset Q = (-t,t]^n$. Since g is uniformly continuous, there is $0 < \delta < 1$ such that for all $\|x-y\| \le \delta$, we have $\|g(x) - g(y)\| \le \varepsilon/(2t)^{n/p}$. Take any $\|a\| \le \delta$. Since both supp(f) and a + supp(f) are subsets of Q, we obtain

$$\left[\int_{\mathbb{R}^n} \|g(a+x) - g(x)\|^p d\lambda(x)\right]^{1/p} \le \left[\int_Q \left\{\frac{\varepsilon}{(2t)^{n/p}}\right\}^p d\lambda(x)\right]^{1/p} \le \varepsilon.$$

Therefore, $\|T_a f - f\|_p \le \|T_a f - T_a g\|_p + \|T_a g - g\|_p + \|g - f\|_p \le 3\varepsilon$. □

23-3.10. **Exercise** Show that the function $f(x) = \dfrac{1}{1+x^2}$ is integrable on $\mathbb{R}$. Find a continuous function g with compact support such that $\|f-g\|_1 \le \frac{1}{10}$.

23-3.11. **Exercise** Prove that the graph of a continuous function from $\mathbb{R}^n$ into $\mathbb{R}$ is a null set in $\mathbb{R}^{n+1}$.

23-4 Relation to Outer Measures

23-4.1. Occasionally we want to estimate the size of a subset of $\mathbb{R}^n$ but we do not know whether it is measurable. Outer measure is handy for this situation. Furthermore a set cannot have any interior point if its outer measure is zero.

23-4.2. Let λ be a *finite-valued positive* measure on a δ-space $\mathbb{D}$. The family of all $\mathbb{D}$-measurable sets is defined by localization §19-1.1 and λ is extended to all $\mathbb{D}$-measurable sets by inner regularity §20-2.2. This extension is also denoted by λ. On the other hand, suppose that $\mathbb{D}$ is generated by a semi-ring $\mathcal{S}$ and the restriction of λ to $\mathcal{S}$ is denoted by μ. An outer measure μ^* is constructed by sequential covers §18-2.3 and the family of all μ^*-measurable sets is defined by nice cuts §18-2.5. A λ-null set N §20-2.5 is a $\mathbb{D}$-measurable set with $\lambda N = 0$ and a μ^*-null set Q is any subset of X with $\mu^* Q = 0$. Since both families of $\mathbb{D}$-measurable and μ^*-measurable sets are δ-rings containing $\mathcal{S}$, we have $\lambda A = \mu^* A < \infty$ for all $A \in \mathbb{D}$ by uniqueness of extension. Suppose that $X = \bigcup_{n=1}^\infty X_n$ for some *disjoint* decent sets $X_n \in \mathbb{D}$.

23-4.3. **Theorem** Every $\mathbb{D}$-measurable set M is μ^*-measurable with $\lambda M = \mu^* M$. In particular, every λ-null set is a μ^*-null set.

Proof. Since each $M \cap X_n \in \mathbb{D}$ is μ^*-measurable, $M = \bigcup_{n=1}^\infty (M \cap X_n)$ is also μ^*-measurable. Furthermore we have

$$\lambda M = \sum_{n=1}^\infty \lambda(M \cap X_n) = \sum_{n=1}^\infty \mu^*(M \cap X_n) = \mu^* M. \qquad \square$$

23-4.4. **Lemma** Let $\mu^* H < \infty$.

(a) There is a $\mathbb{D}$-measurable set M such that $H \subset M$ and $\mu^* H = \lambda M$.

(b) There exist a $\mathbb{D}$-measurable set A and a subset B of some λ-null set such that $H = A \cup B$, $A \cap B = \emptyset$ and $\lambda^* H = \lambda A$.

Proof. (a) For every k, there are $A_{jk} \in \mathcal{S}$ such that $H \subset \bigcup_{j=1}^\infty A_{jk}$ and $\sum_{j=1}^\infty \mu A_{jk} \le \mu^* H + 1/k$. Then $M = \bigcap_{k=1}^\infty \bigcup_{j=1}^\infty A_{jk}$ is $\mathbb{D}$-measurable. Clearly we have $H \subset M$. Observe that

$$\lambda M \le \lambda\left(\bigcup_{j=1}^\infty A_{jk}\right) \le \sum_{j=1}^\infty \lambda A_{jk} = \sum_{j=1}^\infty \mu A_{jk} \le \mu^* H + 1/k$$

for all k. Hence $\lambda M \le \mu^* H \le \mu^* M = \lambda M$, that is $\lambda M = \mu^* H$.

(b) Since $\mu^*(M \setminus H) = \mu^* M - \mu^* H = \lambda^* M - \mu^* H = 0$, there is a $\mathbb{D}$-measurable set N such that $M \setminus H \subset N$ and $\lambda N = \mu^*(M \setminus H) = 0$. Hence N is a

λ-null set. Now $A = M \setminus N$ is a $\mathbb{D}$-measurable set and $B = N \cap H$ is a subset of the λ-null set N. Since $A \cap B \subset (M \setminus N) \cap N = \emptyset$, we get $A \cap B = \emptyset$. From $M \setminus H \subset N$, we have $A = M \setminus N \subset H$. Also $B \subset H$ by definition. Hence $A \cup B \subset H$. Using the notation $Q' = X \setminus Q$ for all $Q \subset X$. Then $H \setminus A = H \cap (M \cap N')' = H \cap (M' \cup N) = (H \cap M') \cup (H \cap N) = B$ because $H \cap M' = \emptyset$ for $H \subset M$. Therefore $H = A \cup B$. □

23-4.5. **Theorem** (a) Every μ^*-null set is a subset of some λ-null set.
(b) Every μ^*-measurable H is a disjoint union of a $\mathbb{D}$-measurable set A and a subset B of a λ-null set. Furthermore we have $\mu^* H = \lambda A$.

Proof. (a) follows immediately from last lemma. To prove (b), since $X_n \in \mathbb{D}$, we have $\mu^*(H \cap X_n) \leq \mu^* X_n = \lambda X_n < \infty$. Write $H \cap X_n = A_n \cup B_n$ where A_n is a $\mathbb{D}$-measurable set with $\mu^*(H \cap X_n) = \lambda A_n$ and B_n is a subset of some λ-null set N_n. Now $A = \bigcup_{n=1}^\infty A_n$ is $\mathbb{D}$-measurable. Because $X = \bigcup_{n=1}^\infty X_n$ is a disjoint union, we have $\mu^* H = \sum_{n=1}^\infty \mu^*(H \cap X_n) = \sum_{n=1}^\infty \lambda A_n = \lambda A$. Clearly $N = \bigcup_{n=1}^\infty N_n$ is λ-null and $A \subset N$. □

23-4.6. We are well motivated to construct *completions* of vector measures. Because we do not need the result in this book, proofs are left as exercises.

23-4.7. Let μ be a vector measure on a δ-space $(X, \mathbb{D})$ into a Banach space. Note that a μ-null set N is a $\mathbb{D}$-measurable set with $|\mu|N = 0$. Let $\mathbb{D}_\mu$ be the family of all sets of the form $A \cup B$ where A is a decent set in $\mathbb{D}$ and B is a subset of some μ-null set. Define $\nu(A \cup B) = \mu A$.

23-4.8. **Exercise** Prove that $\mathbb{D}_\mu$ is a δ-ring generated by decent sets in $\mathbb{D}$ and subsets of μ-null sets.

23-4.9. **Exercise** Prove that ν is the unique extension of μ to a vector measure on $\mathbb{D}_\mu$ with $\nu B = 0$ for all subsets of μ-null sets. Also show that $|\nu|(A \cup B) = |\mu|(A)$.

23-4.10. **Exercise** Prove that the family of $\mathbb{D}_\mu$-measurable sets is the σ-algebra generated by $\mathbb{D}$-measurable sets and subsets of μ-null sets.

23-5 Change Variables

23-5.1. In this section, we shall prove a transformation formula to change variables starting with linear transformations. The geometrical meaning of determinants is given by §23-5.5 below. Note that the diffeomorphism *arctan* : $(-1, 1) \to \mathbb{R}$ carries the decent set $(-1, 0]$ onto the unbounded set $(-\infty, 0]$ which is not a decent set. Luckily, positive measures are defined on all measurable sets.

23-5.2. A linear transformation on $\mathbb{R}^n$ is a square matrix $T = [T_{ij}]$ of order n. With the sup-norm $\|x\| = \max\{|x_j| : 1 \le j \le n\}$, the matrix norm is given by $\|T\| = \max\left\{\sum_{j=1}^n |T_{ij}| : 1 \le i \le n\right\}$. Clearly for the identity transformation I, we have $\|I\| = 1$. Let $e = (1, 1, \cdots, 1) \in \mathbb{R}^n$. The following sets are called *cubes* with *center* a, *radius* r and *edge-length* $2r$:

a closed cube : $\{x : \|x - a\| \le r\} = [a - re, a + re]$,

an open cube : $\{x : \|x - a\| < r\} = (a - re, a + re)$,

and a semi-cube : $(a - re, a + re] = \prod_{j=1}^n (a_j - r, a_j + r]$.

Let λ denote the Lebesgue measure on $\mathbb{R}^n$. Obviously, we have

$$\lambda[a - re, a + re] = \lambda(a - re, a + re) = \lambda(a - re, a + re] = (2r)^n.$$

23-5.3. **Lemma** Every elementary row operation in Linear Algebra can be decomposed into the following two atomic operations:

(a) multiplying one row by a non-zero number α ;

(b) adding one row to another.

Proof. The following steps give the idea of how it is done:

add a multiple of row b to row a: $(a, b) \to (a, \alpha b) \to (a + \alpha b, \alpha b) \to (a + \alpha b, b)$

and interchange two rows: $(a, b) \to (a + b, b) \to (a + b, -a) \to (b, -a) \to (b, a)$. □

23-5.4. Since every matrix is of the form $T = T_1 T_2 T_3$ where T_1, T_3 are invertible and $T_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is the identity matrix of order r. Both T_1, T_3 are products of elementary matrices obtained from (a), (b) of last lemma and a projection $x \to (x_1, x_2, \cdots, x_r, 0, \cdots, 0)$. Therefore the following theorem is reduced to three cases only.

23-5.5. **Theorem** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then the image TQ of a measurable set Q is measurable and we have $\lambda(TQ) = |det\ T| \lambda Q$.

Proof. Let $Q = (a, b]$ be a semi-interval. Firstly, assume $T(x) = (\alpha x_1, x_2, \cdots, x_n)$ where $\alpha \ne 0$. If $\alpha > 0$ then $TQ = (\alpha a_1, \alpha b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]$ is decent and $\lambda(TQ) = (\alpha b_1 - \alpha a_1)(b_2 - a_2) \cdots (b_n - a_n) = \alpha \prod_{j=1}^n (b_j - a_j) = |det\ T| \lambda Q$. If $\alpha < 0$ then $TQ = [\alpha b_1, \alpha a_1) \times (a_2, b_2] \times \cdots \times (a_n, b_n]$ is decent and

$$\lambda(TQ) = (\alpha a_1 - \alpha b_1)(b_2 - a_2) \cdots (b_n - a_n) = (-\alpha) \prod_{j=1}^n (b_j - a_j) = |det\ T| \lambda Q.$$

Consequently, $\lambda(TQ) = |det\ T|\lambda Q$ for all linear transformation of the type §23-5.3a. Next assume $T(x) = (x_1 + x_2, x_2, \cdots, x_n)$. Now TQ is a decent set by drawing a picture on x_1x_2-plane or applying next theorem below. It is easy to verify that $det\ T = 1$ and $\rho_{TQ}(x) = \rho_{(a_1+x_2,b_1+x_2]}(x_1)\prod_{j=2}^n \rho_{(a_j,b_j]}(x_j)$. Hence, we have $\int \rho_{TQ}(x)dx_1 = \lambda(a_1+x_2, b_1+x_2]\prod_{j=2}^n \rho_{(a_j,b_j]}(x_j) = (b_1-a_1)\prod_{j=2}^n \rho_{(a_j,b_j]}(x_j)$. It follows from Fubini's Theorem that

$$\lambda(TQ) = \int \rho_{TQ}(x)d\lambda(x) = \int dx_n \cdots \int dx_3 \int dx_2 \int \rho_{TQ}(x)dx_1$$
$$= \int \cdots \int \int (b_1 - a_1)\prod\nolimits_{j=2}^n \rho_{(a_j,b_j]}(x_j)dx_2dx_3\cdots dx_n$$
$$= \prod\nolimits_{j=1}^n (b_j - a_j) = \lambda Q = |det\ T|\lambda Q.$$

Therefore, $\lambda(TQ) = |det\ T|\lambda Q$ for all linear transformation of the type §23-5.3b. Next, assume $T(x) = (x_1, x_2, \cdots, x_r, 0, \cdots, 0)$. If $r = n$, then T is the identity map with $det\ T = 1$ and hence $\lambda(TQ) = |det\ T|\lambda Q$. If $r < n$, then $det\ T = 0$ and $\lambda(TQ) = \lambda\big[(a_1, b_1] \times \cdots (a_r, b_r] \times \{0\} \times \cdots \times \{0\}\big] = 0$. Therefore, $\lambda(TQ) = |det\ T|\lambda Q$ for all projections T. Finally, write $T = T_1T_2\cdots T_k$ where each T_j is one of above three types. Then we have

$$\lambda(TQ) = \lambda T_1T_2\cdots T_kQ = |det\ T_1||det\ T_2|\cdots|det\ T_k|\lambda Q$$
$$= |(det\ T_1)(det\ T_2)\cdots(det\ T_k)|\lambda Q = |det\ T|\lambda Q.$$

Next, for fixed T it is obvious that both $\lambda(TQ)$ and $|det\ T|\lambda Q$ are measures in decent sets Q and they agree on semi-intervals. Therefore they agree on all decent sets by Unique Extension Theorem. Finally, since every measurable set is a countable disjoint union of decent sets in $\mathbb{R}^n$, the result follows. □

23-5.6. Let X be a non-empty open subset of $\mathbb{R}^n$ and let $f : X \to f(X) \subset \mathbb{R}^n$ be a C^1-diffeomorphism, i.e. f is a continuously differentiable bijection such that for every $a \in X$, the derivative $Df(a)$ is invertible. It follows from the Inverse Mapping Theorem that $f(X)$ is open and f^{-1} is also continuously differentiable. Decent and measurable subsets of X were defined by §21-9. For the diffeomorphism $arctan : (-1, 1) \to \mathbb{R}$, the image $(-\infty, 0]$ of the decent set $(-1, 0]$ is not bounded and hence cannot be a decent set. Therefore we have to work with measurable sets.

23-5.7. **Theorem** The image $f(M)$ of a measurable set $M \subset X$ is measurable.

Proof. Consider $Q = (a, b]$ with $K = [a, b] \subset X$ and $a < b$. The compact sets $f(K)$ and $f(B_j)$ are measurable where $B_j = [a_1, b_1] \times \cdots \times \{a_j\} \times [a_n, b_n]$ are faces of the interval K. Hence $f(Q) = f(K) \setminus \bigcup_{j=1}^n B_j$ is also measurable. Next let $\mathbb{F}$ be the family of sets $A \subset \mathbb{R}^n$ such that $f(A \cap Q)$ is measurable. Since f is bijective, $\mathbb{F}$ is a σ-algebra containing all semi-intervals. Hence $\mathbb{F}$ contains all measurable sets of $\mathbb{R}^n$. Therefore if M is a measurable subset of X, then $f(Q \cap M)$ is measurable. Finally decompose $X = \bigcup_{j=1}^{\infty} Q_j$ into disjoint union of semi-intervals Q_j. Then $f(M) = \bigcup_{j=1}^{\infty} f(Q_j \cap M)$ is measurable. □

23-5.8. **Lemma** For every closed interval $K = [a, b]$ contained in X, we have

$$\lambda f(a, b] \leq \int_{(a,b]} |det\ Df(x)| d\lambda(x).$$

Proof. Clearly $|det\ Df(x)|$ is a continuous function of $x \in X$ and it is integrable on $(a, b]$. Since Df^{-1} is continuous on the compact set $f(K)$, there is $t > 0$ such that for each $y \in f(K)$ we have $\|Df^{-1}(y)\| \leq t$. By uniform continuity on the compact set K, for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $\|x - y\| \leq \delta$ in K we have $\|Df(x) - Df(y)\| \leq \varepsilon/t$ and $|det\ Df(x) - det\ Df(y)| \leq \varepsilon$. Write $(a, b] = \bigcup_{j=1}^{\infty} A_j$ as a union of disjoint semi-cubes A_j each of which has radius $< \delta$. Take any $A_j = (c - re, c + re]$ where $0 < r < \delta$. For each $x \in X$ define $h(x) = Df(c)^{-1} f(x)$. Then h is a continuously differentiable map on X. Now for each $x \in A_j$,

$$\begin{aligned}\|Dh(x) - I\| &= \|Df(c)^{-1}Df(x) - Df(c)^{-1}Df(c)\| \\ &\leq \|Df(c)^{-1}\| \|Df(x) - Df(c)\| \leq t(\varepsilon/t) = \varepsilon,\end{aligned}$$

that is,

$$\|Dh(x)\| \leq \|I\| + \varepsilon = 1 + \varepsilon.$$

By Mean-Value Theorem, for each $x \in A_j$, we have

$$\|h(x) - h(c)\| \leq \|x - c\| \sup \|Dh(z)\| \leq r(1 + \varepsilon)$$

where z runs over the line segment between x and c. Hence we obtain

$$h(A_j) \subset [h(c) - r(1 + \varepsilon)e, h(c) + r(1 + \varepsilon)e],$$

and,

$$\lambda h(A_j) \leq \prod_{k=1}^{n} [2r(1 + \varepsilon)] = (1 + \varepsilon)^n \lambda A_j.$$

On the other hand, it follows from §23-5.5 that

$$\lambda h(A_j) = \lambda[Df(c)^{-1} f(A_j)] = |det\ Df(c)^{-1}|\ \lambda f(A_j).$$

Combining the last two expressions, we get

$$\lambda f(A_j) \leq (1 + \varepsilon)^n |det\ Df(c)| \lambda A_j \leq (1 + \varepsilon)^n \int_{A_j} [|det\ Df(x)| + \varepsilon] d\lambda(x).$$

Therefore,

$$\lambda f(a, b] = \lambda \left[\bigcup_{j=1}^{\infty} f(A_j)\right] = \sum_{j=1}^{\infty} \lambda f(A_j)$$

$$\leq (1+\varepsilon)^n \sum_{j=1}^{\infty} \int_{A_j} [\ |det\ Df(x)\ | + \varepsilon] d\lambda(x)$$
$$= (1+\varepsilon)^n \int_{(a,b]} [\ |det\ Df(x)| + \varepsilon\ ] d\lambda(x)$$
$$= (1+\varepsilon)^n \int_{(a,b]} |det\ Df(x)| d\lambda(x) + (1+\varepsilon)^n \varepsilon \lambda(a,b].$$

The result follows by letting $\varepsilon \downarrow 0$. □

23-5.9. **Lemma** For every measurable subset Q of X we have

$$\lambda f(Q) \leq \int_Q |det\ Df(x)| d\lambda(x).$$

Proof. Firstly assume that Q is an open subset of X. We can write the following disjoint union $Q = \bigcup_{j=1}^{\infty} A_j$ where each A_j is a semi-interval with closure contained in Q. By last lemma we have

$$\lambda f(Q) = \lambda\left(\bigcup_{j=1}^{\infty} f(A_j)\right) = \sum_{j=1}^{\infty} \lambda f(A_j)$$
$$\leq \sum_{j=1}^{\infty} \int_{A_j} |det\ Df(x)| d\lambda(x) = \int_Q |det\ Df(x)| d\lambda(x).$$

Next, we assume that Q is a decent set. It follows from the Outer Regularity Theorem, there is a sequence of *bounded* open sets $A_j \downarrow Q$ such that $\lambda A_j \downarrow \lambda Q$ as $j \to \infty$. Replacing A_j by $A_j \cap X$ we may assume all $A_j \subset X$. Hence,

$$\lambda f(Q) \leq \lambda f(A_j) \leq \int \rho_{A_j}(x) |det\ Df(x)| d\lambda(x).$$

By Dominated Convergence Theorem, letting $j \to \infty$,

$$\lambda f(Q) \leq \int \rho_Q(x) |det\ Df(x)| d\lambda(x) = \int_Q |det\ Df(x)| d\lambda(x).$$

Finally, let $X = \bigcup_{j=1}^{\infty} B_j$ be a disjoint union of semi-intervals B_j. Then for every measurable subset Q of X, $Q \cap B_j$ is a decent set. Furthermore we have

$$\lambda f(Q) = \sum_{j=1}^{\infty} \lambda f(Q \cap B_j) \leq \sum_{j=1}^{\infty} \int_{Q \cap B_j} |det\ Df(x)| d\lambda(x)$$
$$\leq \int_Q |det\ Df(x)| d\lambda(x).$$
□

23-5.10. **Theorem** Let g be an upper function or an integrable map on $f(X)$ into a Banach space F. Then for every measurable subset Q of X, we have

$$\int_{f(Q)} g(y) d\lambda(y) = \int_Q g \circ f(x) |det\ Df(x)| d\lambda(x).$$

Proof. Let g be an upper function on $f(X)$. Replacing g by $g\rho_Q$, we may assume $Q = X$. There are simple functions g_i on $f(X)$ such that $0 \leq g_i \uparrow g$.

Write $g_i = \sum_{j=1}^m \alpha_j \rho_{H_j}$ where $\alpha_j \geq 0$ and H_j are measurable subsets of $f(X)$. Then $K_j = f^{-1}(H_j)$ are measurable subsets of X. Hence

$$\int_{f(X)} g_i(y)d\lambda(y) = \sum\nolimits_{j=1}^m \alpha_j \lambda H_j = \sum\nolimits_{j=1}^m \alpha_j \lambda f K_j$$
$$\leq \sum\nolimits_{j=1}^m \alpha_j \int_{K_j} |det\ Df(x)|d\lambda(x) = \int_X g_i \circ f(x)|det\ Df(x)|d\lambda(x).$$

By Monotone Convergence Theorem, letting $i \to \infty$,

$$\int_{f(X)} g(y)d\lambda(y) \leq \int_X g \circ f(x)|det\ Df(x)|d\lambda(x).$$

Interchanging X and $f(X)$, replacing $g(y)$ by $g \circ f(x)|det\ Df(x)|$, we have

$$\int_X g \circ f(x)|det\ Df(x)|d\lambda(x)$$
$$\leq \int_{f(X)} g \circ f \circ f^{-1}(y)|det\ Df[f^{-1}(y)]|\ |det\ Df^{-1}(y)|d\lambda(y) = \int_{f(X)} g(y)d\lambda(y).$$

Consequently, the equality holds for all upper functions g. By linearity, it is also true for every integrable function $h = (\text{Re}\ h)_+ - (\text{Re}\ h)_- + i(\text{Im}\ h)_+ - i(\text{Im}\ h)_-$. Next for an integral map $g : f(X) \to F$, its variation $|g| : f(X) \to \mathbb{R}$ is an integrable upper function. We leave it as an exercise to show that $(g \circ f)(x)|det\ Df(x)|$ is a measurable map in $x \in X$. From

$$\int_X \|g \circ f(x)\|\ |det\ Df(x)|d\lambda(x) = \int_{f(X)} \|g(y)\|d\lambda(y) < \infty,$$

$(g \circ f)(x)|det\ Df(x)|$ is integrable in $x \in X$. Finally $vg : f(X) \to \mathbb{R}$ is an integrable function for every $v \in F'$. Hence we obtain

$$\int_{f(Q)} vg(y)d\lambda(y) = \int_Q vg \circ f(x)|det\ Df(x)|d\lambda(x),$$

that is, $$v\int_{f(Q)} g(y)d\lambda(y) = v\int_Q g \circ f(x)|det\ Df(x)|d\lambda(x).$$

The result follows because F' separates points of F. □

23-99. **References and Further Readings** : Cohn, Rao-87, Serrin, Borell and Yamasaki.

Chapter 24

Indefinite Integrals

24-1 Derivatives

24-1.1. Let E, F, FE be Banach spaces with a bilinear map $\varphi : F \times E \to FE$ which is *scalar*, that is $\|\varphi(u,v)\| = \|u\| \, \|v\|$ for all $(u,v) \in F \times E$. This is the case if E or F is the scalar field $\mathbb{K}$ and φ is the scalar multiplication. In general, this is also true if φ is the tensor product and FE takes any tensor norm. Clearly every scalar bilinear map is admissible. For convenience, write $\varphi(u,v) = uv$.

24-1.2. Let μ be an E-measure on a δ-space $(X, \mathbb{D})$. Suppose that $h : X \to F$ is a *locally μ-integrable* map, that is μ-integrable on every decent set. Since h is simply approximable on every decent set, it is measurable. Let $\nu : \mathbb{D} \to FE$ be given by $\nu(A) = \int_A h d\mu$ for all $A \in \mathbb{D}$. Then h is called the *μ-density* of ν, or the *Radon-Nikodym derivative* of ν with respect to μ (on X relative to φ). Equivalently, ν is called the *indefinite integral* of h with respect to μ. Symbolically, write $\frac{d\nu}{d\mu} = h$ or $d\nu = h d\mu$. In this section, we shall study the relationship between μ and ν.

24-1.3. **Theorem** (a) The function $|h|$ is locally $|\mu|$-integrable.
(b) ν is a measure on $(X, \mathbb{D})$.
(c) $|\nu|(A) = \displaystyle\int_A |h| \, d|\mu|$ for all $A \in \mathbb{D}$, or $|h d\mu| = |h| \, d|\mu|$.

Proof. Since h is locally μ-integrable, $h\rho_A$ is μ-integrable for every decent set A. Hence $|h|\rho_A$ is $|\mu|$-integrable, or $|h|$ is locally $|\mu|$-integrable. Clearly ν is additive on $\mathbb{D}$. Let $A_n \downarrow \emptyset$ in $\mathbb{D}$. Then $|h\rho_{A_n}| \le |h\rho_{A_1}|$, $h\rho_{A_n} \to 0$ and $|h\rho_{A_1}|$ is μ-integrable on X. Hence Dominated Convergence Theorem implies $\nu A_n = \int h\rho_{A_n} d\mu \to \int 0 d\mu = 0$. Therefore ν is countably additive on $\mathbb{D}$. Next, for any fixed decent set A, let $\{D_1, D_2, \cdots, D_k\}$ be any finite partition of A by decent sets. Then

$$\sum_{j=1}^{k} \|\nu D_j\| = \sum_{j=1}^{k} \left\| \int_{D_j} h d\mu \right\| \le \sum_{j=1}^{k} \int_{D_j} |h| \, d|\mu| = \int_A |h| \, d|\mu|.$$

Taking supremum over all finite partitions of A by decent sets, we get

$$|\nu|(A) \leq \int_A |h| \ d|\mu| \text{ for all } A \in \mathbb{D}. \qquad \#1$$

In particular, ν is of finite variation and consequently it is measure. Next, consider a decent map $g = \sum_{i=1}^m \alpha_i \rho_{B_i}$ where $\alpha_i \in F$ and $B_1, B_2, \cdots$ are *disjoint* nonempty decent sets. Let $\pi D = \int_D g d\mu$ for all $D \in \mathbb{D}$ and let $B = \bigcup_{i=1}^m B_i$. Because $g = 0$ on $D \subset A \setminus B$, we obtain

$$|\pi|(A \setminus B) = \sup_{P(A\setminus B)} \sum_{D\in P(A\setminus B)} \|\pi D\| = \sup_{P(A\setminus B)} \sum_{D\in P(A\setminus B)} \left\| \int_D g d\mu \right\| = 0.$$

Next, consider any partition $P(A \cap B_i)$ by decent sets. For any $D \in P(A \cap B_i)$, we have $\pi D = \displaystyle\int_D g d\mu = \int g \rho_D d\mu = \int \alpha_i \rho_D d\mu = \alpha_i \mu D$. Because the bilinear map $F \times E \to FE$ is scalar, we have $\|\pi D\| = \|\alpha_i\| \ \|\mu D\|$ and hence

$$|\pi|(A) = |\pi|(A \setminus B) + \sum_{i=1}^m |\pi|(A \cap B_i) = 0 + \sum_{i=1}^m \sup_{P(A\cap B_i)} \sum_{D\in P(A\cap B_i)} \|\pi D\|$$

$$= \sum_{i=1}^m \sup_{P(A\cap B_i)} \sum_{D\in P(A\cap B_i)} \|\alpha_i\| \ \|\mu D\| = \sum_{i=1}^m \|\alpha_i\| \ |\mu|(A \cap B_i) = \int_A |g| \ d|\mu|.$$

Therefore the required result holds for every decent map g. Now return to the consideration of locally μ-integrable map h. By Density Theorem, for every $\varepsilon > 0$ there is a decent map g such that $\int_A |g-h| \ d|\mu| \leq \varepsilon$. Define $\pi D = \int_D g d\mu$ for all $D \in \mathbb{D}$. Then $(\pi - \nu)(A) = \int_A (g-h) d\mu$. By the inequality #1, we have $|\pi - \nu|(A) \leq \int_A |g-h| \ d|\mu| \leq \varepsilon$. Since $|\pi| \leq |\pi - \nu| + |\nu|$, we obtain

$$\int_A |g| \ d|\mu| = |\pi|(A) \leq |\pi - \nu|(A) + |\nu|(A) \leq \varepsilon + |\nu|(A).$$

Therefore, $\displaystyle\int_A |h| \ d|\mu| \leq \int_A |h-g| \ d|\mu| + \int_A |g| \ d|\mu| \leq 2\varepsilon + |\nu|(A).$

The proof is completed by letting $\varepsilon \downarrow 0$. □

24-1.4. **Exercise** Prove $|\nu|(M) = \displaystyle\int_M |h| \ d|\mu|$ for all measurable set M. In particular if h is μ-integrable, then we have $|\nu|(X) = \|h\|_1$.

24-1.5. **Exercise** Prove that every μ-null set is a ν-null set. For $E = F = \mathbb{C}$, prove that the complex conjugate of ν is given by $\nu^-(A) = \int_A h^- d\mu^-$ for all decent sets A.

24-1.6. **Theorem** (a) If $\nu = 0$, then $h = 0$, μ-ae.
(b) If both μ, ν are real; then h is real, μ-ae.

(c) If both $\mu, \nu \geq 0$; then $h \geq 0$, μ-ae.

Proof. (a) Assume $h \neq 0$, μ-ae. Then $N = \{x \in X : \|h(x)\| > 0\}$ is not μ-null. There is a decent subset A of N such that $|\mu|(A) > 0$. Hence $|\nu|(A) = \int_A |h|\, d|\mu| > 0$. Therefore $|\nu| \neq 0$ and consequently $\nu \neq 0$.
(b) Since $\int_A (h - h^-) d\mu = \int_A h d\mu - \int_A h^- d\mu^- = (\nu - \nu^-)(A) = 0$, $\forall\, A \in \mathbb{D}$, we have $h - h^- = 0$, μ-ae. Therefore h is real μ-ae.
(c) Replace h^- by $|h|$ in (b). □

24-1.7. **Corollary** A measurable set M is ν-null iff $M \cap \{x \in X : h(x) \neq 0\}$ is μ-null.

Proof. It follows by applying the last theorem to the subspace M. □

24-1.8. To cope with the need of §24-4.8 later, assume that h is a scalar function and $g : X \to F$ a vector map so that the product gh is a map into F.

24-1.9. **Lemma** For every decent map g, we have $\int g d\nu = \int gh d\mu$.

Proof. Let $g = \sum_{i=1}^n \alpha_i \rho_{A_i}$ where $A_i \in \mathbb{D}$ and $\alpha_i \in F$. Then we have

$$\int g d\nu = \sum\nolimits_{i=1}^n \alpha_i \nu A_i = \sum\nolimits_{i=1}^n \alpha_i \int_{A_i} h d\mu = \sum\nolimits_{i=1}^n \alpha_i \int h \rho_{A_i} d\mu = \int gh d\mu. \quad \square$$

24-1.10. **Lemma** $\int |g|\, d|\nu| = \int |g|\, |h|\, d|\mu|$ for every measurable map g.

Proof. Choose simple functions $0 \leq g_n \uparrow |g|$. Then $0 \leq g_n|h| \uparrow |g|\,|h|$. By last lemma, we have $\int g_n \rho_A d|\nu| = \int g_n \rho_A |h|\, d|\mu|$ for every decent set A. It follows from §20-1.3 that

$$\int |g|\, d|\nu| = \lim_{n\to\infty} \int g_n d|\nu| = \lim_{n\to\infty} \sup_{A \in \mathbb{D}} \int g_n \rho_A d|\nu|$$

$$= \lim_{n\to\infty} \sup_{A \in \mathbb{D}} \int g_n \rho_A |h|\, d|\mu| = \lim_{n\to\infty} \int g_n |h|\, d|\mu| = \int |g|\, |h|\, d|\mu|. \quad \square$$

24-1.11. **Theorem** A measurable map g is ν-integrable iff gh is μ-integrable. In this case, we have $\int g d\nu = \int gh d\mu$.

Proof. The first statement follows immediately from last lemma because both sides are finite or infinite together. By Density Theorem, there are decent maps g_n such that $|g_n| \leq |g|$ on X and $g_n \to g$, ν-ae. Then $|g_n h| \leq |gh|$ on X and $g_n h \to gh$, μ-ae. It follows from Dominated Convergence Theorem that

$$\int g d\nu = \lim_{n\to\infty} \int g_n d\nu = \lim_{n\to\infty} \int g_n h d\mu = \int gh d\mu. \quad \square$$

24-1.12. **Exercise** For $\pi A = \int_A g d\nu$, interpret the last theorem as a chain rule: $\dfrac{d\pi}{d\mu} = \dfrac{d\pi}{d\nu}\dfrac{d\nu}{d\mu}$ almost everywhere for some measure.

24-1.13. **Exercise** Prove that a measurable map g is locally ν-integrable iff gh is locally μ-integrable.

24-1.14. **Exercise** Show that the above results also hold if g is a scalar function and $h : X \to F$ a vector map so that the product gh is a map into F.

24-1.15. **Exercise** Let $\mathbb{D}$ be the δ-ring generated by a semiring $\mathcal{S}$ over X. Prove that if a measurable map is integrable on every set in $\mathcal{S}$, then it is locally integrable.

24-1.16. **Exercise** Show that every continuous map on $\mathbb{R}^n$ is locally integrable with respect to the Lebesgue measure.

24-1.17. **Exercise** Prove that for every vector measure μ and every measurable set M, $|\mu|M = 0$ iff $\mu(M \cap A) = 0$ for every decent set A.

24-1.18. **Exercise** Let $E = F = FE = \mathbb{R}$; μ the Lebesgue measure and $f = 1$. Suppose that $(u, v) \to uv : F \times E \to FE$ is defined by $uv = 0$ for all $u \in F$ and $v \in E$. Compare the values of $|\nu|(A)$ and $\int |h| \, d|\mu|$.

24-2 Absolute Continuity

24-2.1. How do we know that one measure is an indefinite integral of another? It turns out that absolute continuity provides an elegant characterization given later in this chapter. We shall link up the classical concept of absolutely continuity of functions on the real line.

24-2.2. Let μ, ν be vector measures on a δ-space $(X, \mathbb{D})$. Then ν is said to be *μ-continuous* or *absolutely continuous* with respect to μ if every μ-null set is a ν-null set. It is denoted by $\nu \ll \mu$.

24-2.3. **Exercise** Prove that if $0 \le \nu \le \mu$ then $\nu \ll \mu$. Consider the counting measure π. Show that $2\pi \ll \pi$ is true but $2\pi \le \pi$ is false.

24-2.4. **Exercise** Prove that if $\nu \ll \mu$, then every μ-σ-finite set is also ν-σ-finite.

24-2.5. **Theorem** Let μ, ν be vector measures. If $\nu \ll \mu$, then for every ν-integrable set D and every $\varepsilon > 0$ there is $\delta > 0$ such that $|\nu|A \le \varepsilon$ whenever A is a measurable subset of D with $|\mu|(A) \le \delta$.

Proof. Without loss of generality, we may assume that $\mu, \nu \ge 0$. Let $\mathcal{K}$ be the family of all ν-integrable sets and let $D \in \mathcal{K}$ and $\varepsilon > 0$ be given. Suppose to the contrary that for every n there is a measurable set $A_n \subset D$ with $\mu(A_n) \leq 1/2^n$ and $\nu(A_n) > \varepsilon$. Then $B_k = \bigcup_{n=k}^{\infty} A_n$ and $Q = \bigcap_{k=1}^{\infty} B_k$ are measurable subsets of D. Furthermore,

$$\mu(Q) \leq \mu(B_k) \leq \sum\nolimits_{n=k}^{\infty} \mu(A_n) \leq 1/2^{k-1}.$$

Since k is arbitrary, we obtain $\mu(Q) = 0$. On the other hand, $\nu(B_k) \geq \nu(A_k) \geq \varepsilon$. Note that ν is a finite-valued measure on $\mathcal{K}$ so that §17-5.7d is applicable. Due to $B_k \downarrow Q$, we have $\nu(Q) = \lim \nu(B_k) \geq \varepsilon$ which is a contradiction to $\nu \ll \mu$. □

24-2.6. **Corollary** If f is an integrable map, then for every $\varepsilon > 0$, there is $\delta > 0$ such that $\int_M |f|\, d|\mu| \leq \varepsilon$ whenever M is a measurable set with $|\mu|M \leq \delta$.

Proof. The measure defined by $\nu M = \int_M |f|\ d|\mu|$ is finite-valued on all measurable sets. It follows from last theorem with $D = X$. □

24-2.7. **Theorem** Let μ, ν be vector measures. Suppose that the family $\mathbb{D}$ of decent sets is generated by a ring $\mathcal{R}$. Then $\nu \ll \mu$ if $\forall\ D \in \mathcal{R}$, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0, \forall\ A \in \mathcal{R}$ with $A \subset D$ and $|\mu|(A) \leq \delta$, we have $|\nu|(A) \leq \varepsilon$.

Proof. Without loss of generality, we may assume that $\mu, \nu \geq 0$. Let N be a μ-null set. Take any $D \in \mathcal{R}$ and $\varepsilon > 0$. Choose $\delta > 0$ according to given condition. Since $\mu(D \cap N) = 0$, there are disjoint sets $A_j \in \mathcal{R}$ such that $D \cap N \subset \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu(A_j) \leq \delta$. Then $B = \bigcup_{j=1}^{n}(D \cap A_j) \in \mathcal{R}$ and $\mu B \leq \delta$. Hence $\sum_{j=1}^{n} \nu(D \cap A_j) = \nu B \leq \varepsilon$. Because n is arbitrary , we have $\nu(D \cap N) \leq \sum\nolimits_{j=1}^{\infty} \nu(D \cap A_j) \leq \varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $\nu(D \cap N) = 0$. Since $D \in \mathcal{R}$ is arbitrary, N is a ν-null set. □

24-2.8. **Corollary** Let μ, ν be vector measures. Suppose that the family $\mathbb{D}$ of decent sets is generated by a semiring $\mathcal{S}$. Then $\nu \ll \mu$ if the following condition holds: for every $Q \in \mathcal{S}$ and $\varepsilon > 0$, there is $\delta > 0$ such that whenever $B_1, B_2, \cdots B_n$ are disjoint subsets of Q in $\mathcal{S}$ with $\sum_{j=1}^{n} |\mu|(B_j) \leq \delta$, we have $\sum_{j=1}^{n} \|\nu(B_j)\| \leq \varepsilon$.

Proof. Let $\mathcal{R}$ be the ring generated by $\mathcal{S}$. Then $\mathbb{D}$ is also the δ-ring generated by $\mathcal{R}$. Take any $\varepsilon > 0$ and $D = \bigcup_{i=1}^{m} Q_i \in \mathcal{R}$ where Q_i are disjoint sets in $\mathcal{S}$. Choose $\delta_i > 0$ such that for all disjoint sets $B_1, B_2, \cdots B_n \in \mathcal{S}$, if all $B_j \subset Q_i$ and if $\sum_{j=1}^{n} |\mu|(B_j) \leq \delta_i$ then $\sum_{j=1}^{n} \|\nu(B_j)\| \leq \varepsilon/m$. Let $\delta = \min \delta_i$. Suppose $A \in \mathcal{R}$ satisfies $A \subset D$ and $|\mu|(A) \leq \delta$. Write $A = \bigcup_{k=1}^{p} A_k$ as a disjoint union of sets in $\mathcal{S}$. Clearly $A = \bigcup_{i=1}^{m} \bigcup_{k=1}^{p}(A_k \cap Q_i)$ is a disjoint union of sets in $\mathcal{S}$. Let

$$A_k \cap Q_i = \cup\{B_{ijk} \in \mathcal{S} : 1 \leq j \leq n(i,k)\}$$

be any partition. Then for all j, k, we get $B_{ijk} \subset Q_i$. Observe that
$\sum_{k=1}^{p} \sum_{j=1}^{n(i,k)} |\mu|(B_{ijk}) = \sum_{k=1}^{p} |\mu|(A_k \cap Q_i) = |\mu|(A \cap Q_i) \le |\mu|(A) \le \delta \le \delta_j$.

Hence,
$$\sum_{k=1}^{p} \sum_{j=1}^{n(i,k)} \|\nu(B_{ijk})\| \le \frac{\varepsilon}{m}.$$
Since the partitions $\{B_{ijk}\}$ of $A_k \cap Q_i$ is arbitrary, we have
$$|\nu|(A \cap Q_i) = \sum_{k=1}^{p} |\nu|(A_k \cap Q_i) \le \varepsilon/m.$$
Therefore $|\nu|(A) = |\nu| \left(A \cap \bigcup_{i=1}^{m} Q_i\right) = \sum_{i=1}^{m} |\nu|(A \cap Q_i) \le \varepsilon$. It follows from last theorem that $\nu \ll \mu$. □

24-2.9. Consider the special case when $\mathcal{S}$ is the semiring of semi-intervals of the real line. A map $F : \mathbb{R} \to E$ is said to be *absolutely continuous* if for all $a < b$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all $a \le a_1 < b_1 < \cdots < a_n < b_n \le b$ we have $\sum_{j=1}^{n} \|F(b_j) - F(a_j)\| \le \varepsilon$ whenever $\sum_{j=1}^{n} (b_j - a_j) \le \delta$. Clearly, every absolutely continuous function is uniformly continuous on bounded intervals. Furthermore, a right continuous increasing function is absolutely continuous iff its induced measure is absolutely continuous with respect to the Lebesgue measure.

24-2.10. **Exercise** Prove by definition that absolutely continuous maps are of finite variation. Show that the converse is false for example by $\rho_{(-\infty,0)}$.

24-2.11. A sequence $f_n \in L_p(\mu, F)$ is *equicontinuous at* $\emptyset$ if for all measurable sets $A_j \downarrow \emptyset$, we have $\lim_{j\to\infty} \sup_n \|f_n \rho_{A_j}\|_p = 0$. This is used to characterize mean convergence in terms of convergence in measure.

24-2.12. **Lemma** If a sequence $f_n \in L_p(\mu, F)$ is equicontinuous at $\emptyset$, then $\lim_{|\mu|A\to 0} \sup_n \|f_n \rho_A\|_p = 0$ for measurable subsets A.

Proof. Suppose to the contrary that there are $\varepsilon > 0$, measurable sets C_j and a subset $\{g_j\}$ of $\{f_n\}$ such that $|\mu|C_j \le 1/2^j$ and $\|g_j \rho_{C_j}\|_p \ge \varepsilon$. All $B_k = \bigcup_{j=k}^{\infty} C_j$ and $N = \bigcap_{k=1}^{\infty} B_k$ are measurable sets. From $|\mu|N \le |\mu|B_k \le \sum_{j=k}^{\infty} |\mu|C_j \le 1/2^{k-1} \to 0$, N is a null set. Clearly $A_j = B_j \setminus N \subset C_j$ are measurable sets with $A_j \downarrow \emptyset$. There is j_0 such that for all $j \ge j_0$ and all n, we have $\|f_n \rho_{A_j}\|_p \le \frac{1}{2}\varepsilon$. Now the contradiction $0 < \varepsilon \le \|g_j \rho_{C_j}\|_p \le \|g_j \rho_{B_j}\|_p = \|g_j \rho_{A_j}\|_p \le \frac{1}{2}\varepsilon$ completes the proof. □

24-2.13. **Theorem** Let $1 \le p < \infty$ and $f_n, f_0 \in L_p(\mu, F)$. Then $f_n \to f_0$ in $L_p(\mu, F)$ iff $f_n \to f_0$ in measure and the sequence $\{f_n\}$ is equicontinuous at $\emptyset$.

Proof. Since $D = \bigcup_{n=0}^{\infty}\{x \in X : f_n(x) \neq 0\}$ is σ-finite, let $D = \bigcup_{i=0}^{\infty} D_i$ where D_0 is a null set and all $D_1 \subset D_2 \subset \cdots$ are decent sets. Assume $f_n \to f_0$ in $L_p(\mu, F)$. We have proved $f_n \to f_0$ in measure. For every $\varepsilon > 0$, there is n_0 such that $\|f_n - f_0\|_p \leq \varepsilon$ for all $n \geq n_0$. By §24-2.6, there is $\delta > 0$ such that for all n with $0 \leq n < n_0$ and for all measurable set A with $|\mu|A \leq \delta$ we have $\int_A |f_n|^p d|\mu| \leq \varepsilon^p$, i.e. $\|f_n\rho_A\|_p \leq \varepsilon$. Now let A_j be measurable sets with $A_j \downarrow \emptyset$ as $j \to \infty$. For each $i \geq 1$, $A_j \cap D_i$ are the decent sets. There is k such that for all $j \geq k$ we have $|\mu|(A_j \cap D_i) \leq \delta$. For $n \geq n_0$, $j \geq k$ and $i \geq 1$, we get $\|f_n\rho_{A_j\cap D_i}\|_p \leq \|f_n - f_0\|_p + \|f_0\rho_{A_j\cap D_i}\|_p \leq 2\varepsilon$. On the other hand for $1 \leq n < n_0$, we have $\|f_n\rho_{A_j\cap D_i}\|_p \leq \varepsilon < 2\varepsilon$ by the choice of δ. Since $|f_n\rho_{A_j\cap D_i}| \leq |f_n| \in L_p$, we obtain $\|f_n\rho_{A_j}\|_p = \lim_{i\to\infty} \|f_n\rho_{A_j\cap D_i}\|_p \leq 2\varepsilon$. Consequently, we conclude $\lim_{j\to\infty} \sup_n \|f_n\rho_{A_j}\|_p = 0$.

Conversely, by last lemma, for each i and $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set A with $|\mu|A \leq \delta$ and for every n we have $\|f_n\rho_A\|_p \leq \varepsilon$. On the other hand, since $B_i = D \setminus (D_0 \cup D_i) \downarrow \emptyset$, there is i such that for all n we have $\|f_n\rho_{B_i}\|_p \leq \varepsilon$. Observe that for all m, n we have

$$\begin{aligned}\|f_m - f_n\|_p &\leq \|(f_m - f_n)\rho_{D_0\cup D_i}\|_p + \|(f_m - f_n)\rho_{B_i}\|_p \\ &\leq \|(f_m - f_n)\rho_{D_i}\|_p + \|f_m\rho_{B_i}\|_p + \|f_n\rho_{B_i}\|_p \leq \|(f_m - f_n)\rho_{D_i}\|_p + 2\varepsilon.\end{aligned}$$

If $|\mu|D_i = 0$, then we have $\|f_m - f_n\|_p \leq 2\varepsilon$. Next assume $|\mu|D_i > 0$. Let $H_{jk} = \{x \in X : \|f_j(x) - f_k(x)\| \geq \varepsilon\,(|\mu|D_i)^{-1/p}\}$. Since $f_n \to f$ in measure, $\{f_n\}$ is Cauchy in measure. There is n_0 such that for all $j, k \geq n_0$ we have $|\mu|H_{jk} \leq \delta$. Hence $\|f_n\rho_{H_{jk}}\|_p \leq \varepsilon$ for all n. Thus for all $m, n \geq n_0$ we obtain

$$\begin{aligned}\|(f_m - f_n)\rho_{D_i}\|_p &\leq \|(f_m - f_n)\rho_{D_i\setminus H_{mn}}\|_p + \|f_m\rho_{H_{mn}}\|_p + \|f_n\rho_{H_{mn}}\|_p \\ &\leq \left[\int \left\{\varepsilon\,(|\mu|D_i)^{-1/p}\rho_{D_i\setminus H_{mn}}\right\}^p d|\mu|\right]^{1/p} + \varepsilon + \varepsilon \leq 3\varepsilon.\end{aligned}$$

Therefore for all $m, n \geq n_0$, $\|f_m - f_n\| \leq 5\varepsilon$ independent of D_i. The sequence $\{f_n\}$ is Cauchy in $L_p(\mu, F)$. Suppose $f_n \to g \in L_p(\mu, F)$. Then $\{f_n\}$ converges in measure to both f, g. Hence $f = g$, μ-ae. Consequently, we have $f_n \to f$ in $L_p(\mu, F)$ as required. □

24-3 Positive and Negative Sets

24-3.1. Let μ be a *real* measure on a δ-space $(X, \mathbb{D})$. A measurable set Q is said to be *positive* if $\mu(Q \cap A) \geq 0$ for each decent set $A \in \mathbb{D}$ and *negative* if $\mu(Q \cap A) \leq 0$. Note that null sets have been introduced. In symbols, we write $Q \geq 0$, $Q \leq 0$ and $Q = 0$ respectively.

24-3.2. **Exercise** Prove that if $Q \subset P \geq 0$ then $Q \geq 0$. Furthermore we have

$$0 \leq \mu(Q \cap A) \leq \mu(P \cap A), \qquad \forall\, A \in \mathbb{D}.$$

24-3.3. **Exercise** Prove that countable unions of positive sets are positive.

24-3.4. **Lemma** Let $\mu : \mathbb{D} \to \mathbb{R}$ be a *real* measure and let A be a decent set. For every $\varepsilon > 0$ there is a decent subset B of A such that $\mu B \geq \mu A$ and for every decent subset Q of B we have $\mu Q \geq -\varepsilon$.

Proof. Suppose to the contrary that there is $\varepsilon > 0$ such that for every decent subset B of A, if $\mu B \geq \mu A$ then $\mu Q < -\varepsilon$ for some decent subset Q of B. Firstly for $B = A$, there is a decent subset Q_1 of A satisfying $\mu Q_1 < -\varepsilon$. Hence $\mu(A \setminus Q_1) = \mu A - \mu Q_1 \geq \mu A + \varepsilon$. Assume inductively that $Q_1, Q_2, \cdots, Q_n$ are disjoint decent subsets of A such that $\mu\left(A \setminus \bigcup_{j=1}^{n} Q_j\right) \geq \mu A + n\varepsilon$. For $B = A \setminus \bigcup_{j=1}^{n} Q_j$, there is a decent subset Q_{n+1} of B satisfying $\mu Q_{n+1} < -\varepsilon$. Hence $\mu\left(A \setminus \bigcup_{j=1}^{n+1} Q_j\right) = \mu(B \setminus Q_{n+1}) = \mu B - \mu Q_{n+1} \geq \mu A + (n+1)\varepsilon$. Therefore an infinite sequence $\{Q_n\}$ has been constructed. Let $Q = \bigcup_{j=1}^{\infty} Q_j$. Since all $Q_j \subset A$, Q is a decent subset of A. Finally the countable disjoint union $A = (A \setminus Q) \cup \left(\bigcup_{j=1}^{\infty} Q_j\right)$ gives

$$-\infty < \mu A = \mu(A \setminus Q) + \sum_{j=1}^{\infty} \mu Q_j \leq \mu(A \setminus Q) + \sum_{j=1}^{\infty} (-\varepsilon)$$

which is a contradiction. This completes the proof. □

24-3.5. **Theorem** Let $\mu : \mathbb{D} \to \mathbb{R}$ be a *real* measure. Then every decent set A contains a positive set B such that $\mu B \geq \mu A$.

Proof. It follows from last lemma that there is a sequence of decent sets $B_{n+1} \subset B_n \subset A$ such that $\mu B_{n+1} \geq \mu B_n \geq \mu A$ and $\mu Q \geq -\frac{1}{n}$ for all decent subsets Q of B_n. Then $B = \bigcap_{n=1}^{\infty} B_n$ is a decent subset of A. Furthermore for each decent set Q, $Q \cap B$ is a decent subset of every B_n. Hence we have $\mu(Q \cap B) \geq -\frac{1}{n}$ for all n, that is $\mu(Q \cap B) \geq 0$. Therefore B is a positive set. Finally since $B_n \downarrow B$, $\mu B = \lim_{n \to \infty} \mu B_n \geq \mu A$. □

24-4 Existence of Derivatives

24-4.1. In this section, we prove that if $\nu \ll \mu$ are complex measures, then the μ-density of ν exists on μ-σ-finite sets.

24-4.2. **Lemma** Let μ, ν be *positive* measures on a δ-space $(X, \mathbb{D})$. If ν is absolutely continuous with respect to μ, then for each decent set M there is a finite-valued upper function f on X such that $f(X \setminus M) = 0$ and $\nu A = \int_A f d\mu$ for every decent subset A of M.

Proof. Let $\mathbb{D}(M)$ denote the family of all decent subsets of M and $\mathbb{F}$ the family of upper functions h on X such that $\int_A h d\mu \le \nu A$ for all $A \in \mathbb{D}(M)$. Note that since both h, μ are positive objects, the integral $\int_A h d\mu$ always exists. Clearly $\mathbb{F}$ is non-empty because $h = 0$ belongs to $\mathbb{F}$. Next, suppose $f, g \in \mathbb{F}$. Then $D = \{x \in X : f(x) < g(x)\}$ is a measurable set. Now for each $A \in \mathbb{D}(M)$,

$$\int_A (f \vee g) d\mu = \int_{A \cap D} (f \vee g) d\mu + \int_{A \setminus D} (f \vee g) d\mu$$
$$= \int_{A \cap D} g d\mu + \int_{A \setminus D} f d\mu \le \nu(A \cap D) + \nu(A \setminus D) = \nu A.$$

Hence the measurable function $f \vee g$ is in $\mathbb{F}$. Next, since M is a decent set, we have $\alpha = \sup\left\{\int_M h d\mu : h \in \mathbb{F}\right\} \le \nu M < \infty$. There is a sequence $h_n \in \mathbb{F}$ such that $\lim_{n\to\infty} \int_M h_n d\mu = \alpha$. It follows that $g_n = h_1 \vee h_2 \vee \cdots \vee h_n \in \mathbb{F}$. Since μ is positive, we have $\int_M h_n d\mu \le \int_M g_n d\mu \le \alpha$. Now $0 \le g_n \uparrow$ converges to an upper function, say $g = \lim g_n$. It follows from Monotone Convergence Theorem that $\int_M g d\mu = \lim_{n\to\infty} \int_M g_n d\mu = \alpha < \infty$. Hence g is μ-integrable on M. The set $N = \{x \in M : g(x) = \infty\}$ is a μ-null set. The function $f = g\rho_{M\setminus N}$ is a finite-valued upper function satisfying $f(X \setminus M) = 0$ and $f = g$, μ-ae on M. Hence $\int_M f d\mu = \int_M g d\mu = \alpha$. Take any $A \in \mathbb{D}(M)$. Since $g_n \in \mathbb{F}$ we have $\int_A g_n d\mu \le \nu A$. Since $g_n \rho_A \uparrow f\rho_A$, μ-ae, the conclusion of $f \in \mathbb{F}$ follows from the calculation: $\int_A f d\mu = \lim_{n\to\infty} \int g_n \rho_A d\mu = \lim_{n\to\infty} \int_A g_n d\mu \le \nu A$. Next, we claim $\nu A = \int_A f d\mu$ for each $A \in \mathbb{D}(M)$. Suppose to the contrary that there is $Q \in \mathbb{D}(M)$ such that $\int_Q f d\mu < \nu Q$. Define $\xi A = \nu A - \int_A f d\mu$ for every $A \in \mathbb{D}(M)$. Then ξ is a positive measure on the measurable subspace $(M, \mathbb{D}(M))$. Since $\xi Q > 0$, there is $\varepsilon > 0$ such that $\xi Q > \varepsilon\mu Q$. Now $\pi = \xi - \varepsilon\mu$ is a real measure on the measurable subspace M. There is a π-positive subset P of Q such that $\pi P \ge \pi Q$. Thus $h = f + \varepsilon\rho_P$ is an upper function on X. For any decent subset A of M, $P \cap A$ is a decent subset of P and hence $\pi(P \cap A) \ge 0$, that is $\varepsilon\mu(P \cap A) \le \xi(P \cap A)$. Observe that

$$\int_A h d\mu = \int_A f d\mu + \varepsilon \int_A \rho_P d\mu = \int_A f d\mu + \varepsilon\mu(P \cap A)$$
$$\le \int_A f d\mu + \xi(P \cap A) = \int_A f d\mu + \nu(P \cap A) - \int_{P \cap A} f d\mu$$

$$= \int_{A \setminus P} f d\mu + \nu(P \cap A) \leq \nu(A \setminus P) + \nu(P \cap A) = \nu(A).$$

Hence $h \in \mathbb{F}$. Consequently, we have

$$\alpha \geq \int_M h d\mu = \int_M (f + \varepsilon \rho_P) d\mu = \int_M f d\mu + \varepsilon \int \rho_P d\mu = \alpha + \varepsilon \mu P$$

which gives $\mu P = 0$. Since $\nu \ll \mu$, we have $\nu P = 0$, or

$$0 < \pi Q \leq \pi P = (\xi - \varepsilon\mu)(P) = \nu P - \int_P f d\mu - \varepsilon \mu P \leq 0.$$

This contradiction establishes the proof. □

24-4.3. **Lemma** Let μ be a *positive* measure on a δ-space $(X, \mathbb{D})$ and ν a complex measure. If ν is absolutely continuous with respect to μ, then for each μ-σ-finite set M there is a locally μ-integrable function f on X such that $f(X \setminus M) = 0$ and $\nu A = \int_A f d\mu$ for every decent subset A of M. Furthermore, if ν is positive, then we may choose $f \geq 0$. If ν is real, then we may choose f to be real.

Proof. Firstly assume $\nu \geq 0$. There is a sequence of disjoint measurable sets $H_0, H_1, H_2, \cdots$ such that H_0 is a μ-null set; all H_i are decent sets for $i \geq 1$ and $M = \bigcup_{n=0}^{\infty} H_n$. For each $n \geq 1$ there is a positive measurable function f_n on X such that $f_n(X \setminus H_n) = 0$ and $\nu A = \int_A f_n d\mu$ for every decent subset A of H_n. Clearly $f = \sum_{n=1}^{\infty} f_n$ is a positive measurable function on X satisfying $f(X \setminus M) = 0$. Now every decent set A can be written as a disjoint union: $A = (A \setminus M) \cup \left[\bigcup_{n=0}^{\infty}(A \cap H_n)\right]$. From $\nu \ll \mu$, $\mu(A \cap H_0) = 0$ implies $\nu(A \cap H_0) = 0$. Since

$$\int_A f d\mu = \int_{A \setminus M} f d\mu + \int_{A \cap H_0} f d\mu + \sum_{n=1}^{\infty} \int_{A \cap H_n} f d\mu$$

$$= 0 + 0 + \sum_{n=1}^{\infty} \nu(A \cap H_n) = \nu(A \cap H_0) + \sum_{n=1}^{\infty} \nu(A \cap H_n) = \nu(A \cap M) < \infty,$$

f is μ-integrable on every decent set A. In particular, f is locally μ-integrable. This proves the case when $\nu \geq 0$. Next, suppose ν is a real measure. Then $0 \leq \nu_+ \leq |\nu|$ and hence $\nu_+ \ll \mu$. Applying the above result to ν_+ and ν_- respectively, there are locally μ-integrable functions $g, h \geq 0$ on X vanishing outside M and satisfying $\nu_+(A) = \int_A g d\mu$, $\nu_-(A) = \int_A h d\mu$ for all $A \in \mathbb{D}(M)$. Then $f = g - h$ is the required function. The case for complex measure ν is left as an exercise. □

24-4.4. **Polar Form of Complex Measures** Let μ be a complex measure on a δ-space $(X, \mathbb{D})$. Then for every μ-σ-finite set M, there is a locally μ-integrable function f on X such that $f(X \setminus M) = 0$, $|f| = 1$ on M, $\mu A = \int_A f d|\mu|$ and

$|\mu|(A) = \int_A f^- d\mu$ for every decent subset A of M. Furthermore if μ is real, then we may choose f to be real. This is handy to reduce complex measures to positive measures.

Proof. Since $\mu, |\mu|$ have the same null sets, we have $\mu \ll |\mu|$. There is a locally μ-integrable function g on X such that $g(X \setminus M) = 0$; $\mu A = \int_A g d|\mu|$ for every decent subset A of M. Then we have $\int_A 1 d|\mu| = |\mu|(A) = \int_A |g| \, d|\mu|$ for all $A \in \mathbb{D}(M)$. Hence $|g| = 1$, μ-ae on M. Now $N = \{x \in M : |g|(x) \neq 1\}$ is a μ-null set. For $f = \rho_N + g\rho_{M \setminus N}$, we have $|f| = 1$ on M, $f(X \setminus M) = 0$ and for all $A \in \mathbb{D}(M)$, $\mu A = \int_A g d|\mu| = \int_A g\rho_{M\setminus N} d|\mu| = \int_A (f - \rho_N) d|\mu| = \int_A f d|\mu|$ and $\int_A f^- d\mu = \int_A f^- f d|\mu| = \int_A |f|^2 d|\mu| = \int_A 1 d|\mu| = |\mu|(A)$. Finally, suppose that μ is real. From $\mu A = \int_A f d|\mu|$ for all $A \in \mathbb{D}(M)$, f is real-valued μ-ae on M. Since $f(X \setminus M) = 0$, f is real-valued μ-ae on X. It is an exercise to construct a real function to replace f. □

24-4.5. **Radon-Nikodym Theorem** Let μ, ν complex measures. If ν is absolutely continuous with respect to μ, then for each μ-σ-finite measurable set M there is a locally μ-integrable function f on X such that $f(X \setminus M) = 0$ and $\nu A = \int_A f d\mu$ for every decent subset A of M. Furthermore, if both μ, ν are positive; then we may choose $f \geq 0$. The same is true for real case.

Proof. It is a refinement of last two lemmas. □

24-4.6. **Corollary** Let $0 \leq \nu \leq \mu$ be positive measures. Then for every μ-σ-finite set M, there is a locally μ-integrable function f on X such that $f(X \setminus M) = 0$, $0 \leq f \leq 1$ on M and $\nu A = \int_A f d\mu$ for every decent subset A of M.

Proof. Suppose that g is a locally μ-integrable positive function satisfying $g(X \setminus M) = 0$ and $\nu A = \int_A g d\mu$ for every decent subset A of M. Then

$$\int_A (1 - g) d\mu = \mu(A) - \nu(A) \geq 0, \qquad \forall \ A \in \mathbb{D}(M).$$

Hence $1 - g \geq 0$, μ-ae on M. Since $g(X \setminus M) = 0$, we have $0 \leq g \leq 1$, μ-ae on X. For $N = \{x \in X : 0 \leq g(x) \leq 1\}$, $f = g\rho_N$ is a required function. □

24-4.7. **Theorem** Let μ be a complex measure on a δ-space $(X, \mathbb{D})$ and $f : X \to E$ a measurable map. If f is integrable on a measurable set M, then we have $\int_M f d\mu \in |\mu|(M) \ \overline{\text{co}}\text{ba} \ f(M)$ where $\overline{\text{co}}\text{ba} \ f(M)$ is the closed convex balanced hull of $f(M)$.

Proof. Since $f\rho_M$ is integrable on X, the set $S = \{x \in M : f(x) \neq 0\}$ is σ-finite. There is a locally integrable function φ on X such that $\varphi(X \setminus S) = 0$,

$|\varphi| = 1$ on S and $\mu A = \int_A \varphi d|\mu|$ for every decent subset A of S. Hence we have $\int_M f d\mu = \int_S f\varphi d|\mu|$ and $|\mu|(S)\ \overline{\text{co}}\text{ba}\ f(S) \subset |\mu|(M)\ \overline{\text{co}}\text{ba}\ f(M)$. Therefore we may assume that μ is a positive measure. Suppose to the contrary that $\int_M f d\mu \notin \mu(M)\ \overline{\text{co}}\text{ba}\ f(M)$. In particular, we have $\mu M > 0$. There is a *real* linear form v on E and a real number t such that $v \int_M f d\mu < t \leq v(z)$ for all $z \in \mu(M)\ \overline{\text{co}}\text{ba}\ f(M)$. Integrating $\int_M (vf) d\mu < t \leq v[\mu(M) f(x)]$ over M, we obtain a contradiction: $(\mu M) \int_M (vf) d\mu < t\mu M \leq \mu(M) \int_M vf d\mu$. □

24-4.8. **Corollary** Let μ be a complex measure on a δ-space $(X, \mathbb{D})$; f an integrable function on X and $g : X \to E$ an essentially bounded map. Then we have $\int fg d\mu \in \|f\|_1\ \overline{\text{co}}\text{ba}\ g(M)$ where $M = \{x \in X : f(x) \neq 0\}$.

Proof. Since f is integrable, the set M is σ-finite. Now $\nu A = \int_A f d\mu$ defines a complex measure on $(X, \mathbb{D})$. Since fg is μ-integrable; by §24-1.11, the map g is ν-integrable and $\int fg d\mu = \int g d\nu \in |\nu|(M) \overline{\text{co}}\text{ba}\ g(M) = \|f\|_1\ \overline{\text{co}}\text{ba}\ g(M)$. □

24-5 Hahn and Lebesgue Decompositions

24-5.1. Let μ be a *real* measure on a δ-space $(X, \mathbb{D})$ and let M be a measurable set. A partition of M into a positive set and a negative set is called a *Hahn Decomposition* of M for μ.

24-5.2. **Hahn Decomposition Theorem** Every μ-σ-finite set M has a Hahn decomposition.

Proof. Let f be a locally μ-integrable real function on X such that $\mu A = \int_A f d|\mu|$ for every decent subset A of M. Define $P = M \cap f^{-1}[0, \infty)$ and $Q = M \setminus P$. Clearly $M = P \cup Q$ and $P \cap Q = \emptyset$. Take any decent subset A of P. Then $f \geq 0$ on A. Since $|\mu| \geq 0$, we have $\mu A = \int_A f d|\mu| \geq 0$. Therefore P is a μ-positive set. Similarly Q is a μ-negative set. □

24-5.3. **Exercise** Prove that if (P, Q) and (S, T) are Hahn decompositions of a μ-σ-finite set M then both $P \triangle S, Q \triangle T$ are μ-null sets and for every decent set A we have $\mu(A \cap P) = \mu(A \cap S)$ and $\mu(A \cap Q) = \mu(A \cap T)$.

24-5.4. Two vector measures μ, ν on a δ-space $(X, \mathbb{D})$ are said to be *singular*, in symbol $\mu \perp \nu$, if there are measurable sets S, T such that $X = S \cup T$, $S \cap T = \emptyset$, S is μ-null and T is ν-null. Note that the definition involves positive measures $|\mu|, |\nu|$ only.

24-5.5. **Theorem** Let μ, ν be vector measures on a δ-space $(X, \mathbb{D})$. Suppose X is σ-finite for both μ and ν. Then μ is singular to ν iff $|\mu| \wedge |\nu| = 0$. This justifies the notation $\mu \perp \nu$ for measures and for vector lattices.

Proof. Since everything is in terms of $|\mu|, |\nu|$; we may assume both $\mu, \nu \geq 0$. Suppose $\mu \wedge \nu = 0$. Let $\pi = \mu + \nu$. Then X is π-σ-finite. There are locally π-integrable functions $f, g \geq 0$ satisfying $\mu A = \int_A f d\pi$ and $\nu A = \int_A g d\pi$ for every decent set A. Observe that
$0 = (\mu \wedge \nu)(A) = \frac{1}{2}(\mu + \nu - |\mu - \nu|)(A) = \int_A \frac{1}{2}(f + g - |f - g|)d\pi = \int_A f \wedge g d\pi$,
or $f \wedge g = 0$, π-ae. Now define $P = f^{-1}(0, \infty)$ and $Q = X \setminus P$. Since $f, g = 0$, π-ae on Q, P respectively, we infer $\mu, \nu = 0$ on Q, P respectively. Therefore μ is singular to ν. The converse is left as an exercise. □

24-5.6. A Banach space F is said to have *Radon-Nikodym Property* if for every scalar measure μ on a δ-space $(X, \mathbb{D})$, every μ-σ-finite subset M of X, every μ-continuous vector measure $\nu : \mathbb{D} \to F$, there is a locally μ-integrable function f on X such that $f(X \setminus M) = 0$ and $\nu A = \int_A f d\mu$ for every decent subset A of M. We have proved that the scalar field $\mathbb{K}$ has Radon-Nikodym Property. In last section of this chapter, it will be shown that separable dual spaces and reflexive spaces have Radon-Nikodym Property.

24-5.7. **Lebesgue Decomposition Theorem** Let μ, ν be vector measures on a δ-space $(X, \mathbb{D})$ into Banach spaces E, F respectively. Suppose that X is σ-finite for both μ and ν. If F has Radon-Nikodym Property, then there are measures μ_a and μ_s such that $\nu = \mu_a + \mu_s$, μ_a is μ-continuous and μ_s is singular to μ. Furthermore the decomposition is unique. The pair μ_a, μ_s is called the *Lebesgue decomposition* of ν with respect to μ. Finally, if ν is real (respectively positive) then we may assume that both μ_a and μ_s are real (respectively positive).

Proof. Consider the special case when both μ, ν are positive. The positive measure $\pi = \mu + \nu$ is σ-finite on X. Clearly $\mu \ll \pi$ and $\nu \ll \pi$. There are locally π-integrable positive functions f, g on X such that $\mu A = \int_A f d\pi$ and $\nu A = \int_A g d\pi$ for all $A \in \mathbb{D}$. Define $P = f^{-1}(0, \infty)$ and $Q = f^{-1}(0)$. Then both P, Q are measurable sets satisfying $X = P \cup Q$ and $P \cap Q = \emptyset$. For every decent set B define $\mu_a(B) = \nu(B \cap P)$ and $\mu_s(B) = \nu(B \cap Q)$. Then clearly both μ_a, μ_s are positive measures on $(X, \mathbb{D})$ satisfying $\nu = \mu_a + \mu_s$. Next, take any μ-null set N. Then for every decent set B, $\int_{B \cap N} f d\pi = \mu(B \cap N) = 0$. Hence $f = 0$, π-ae on $B \cap N$, or $\pi(B \cap N \cap P) = 0$. Consequently,

$$0 \le \mu_a(B \cap N) = \nu(B \cap N \cap P) \le \pi(B \cap N \cap P) = 0,$$

that is $\mu_a(B \cap N) = 0$. Since B is any decent set, N is μ_a-null. This proves $\mu_a \ll \mu$. Finally because $f = 0$ on Q, we have $\mu(B \cap Q) = \int_{B\cap Q} f d\pi = 0$ and $\mu_s(B \cap P) = \nu(B \cap P \cap Q) = \nu(\emptyset) = 0$. Therefore $\mu \perp \mu_s$. We have proved the existence for a special case when $\mu, \nu \ge 0$. In general, suppose that μ, ν are vector measures on X into Banach spaces E, F respectively. There are positive measures ν_a, ν_s such that $|\nu| = \nu_a + \nu_s$, $\nu_a \ll |\mu|$ and $|\nu_s| \perp |\mu|$. Since F has Radon-Nikodym Property, there is a locally ν-integrable map $h : X \to F$ such that $\nu(A) = \int_A hd|\nu|$, $\forall\ A \in \mathbb{D}$. Furthermore if ν is real, we may choose h to be real. Define $\mu_a(A) = \int_A hd\nu_a$ and $\mu_s(A) = \int_A hd\nu_s$, $\forall\ A \in \mathbb{D}$. Then clearly $\nu = \mu_a + \mu_s$. Since every ν_a-null set is μ_a-null, we have $\mu_a \ll \mu$. In the same manner, we get $\mu_s \perp \mu$. For uniqueness, assume $\nu = \mu_a + \mu_s = \nu_a + \nu_s$ be two Lebesgue decompositions. Then $\xi = \mu_a - \nu_a = \nu_s - \mu_s$ is a measure on X. Because $\mu_s, \nu_s \perp \mu$, there are measurable sets P, Q, M, N such that $X = P \cup Q = M \cup N$, $P \cap Q = \emptyset$, $M \cap N = \emptyset$, P is μ_s-null, M is ν_s-null and Q, N are μ-null. Take any decent set A. Since $Q \cup N$ is μ-null we have $\mu[A \cap (Q \cup N)] = 0$. Due to $\mu_a \ll \mu$ and $\nu_a \ll \mu$, we get

$$\xi[A \cap (Q \cup N)] = \mu_a[A \cap (Q \cup N)] - \nu_a[A \cap (Q \cup N)] = 0.$$

Since P is μ_s-null and M is ν_s-null, we obtain

$$\xi[A \cap (P \cap M)] = \nu_s[(A \cap P) \cap M] - \mu_s[(A \cap M) \cap P] = 0.$$

Consequently, $\xi A = \xi[A \cap (Q \cup N)] + \xi[A \cap (P \cap M)] = 0$. Because A is arbitrary we have $\xi = 0$, that is $\mu_a = \nu_a$ and $\nu_s = \mu_s$. □

24-6 Duality of Classical Spaces

24-6.1. Let μ be a complex measure on a δ-space $(X, \mathbb{D})$; $1 \le p < \infty$; q the conjugate index defined by $\frac{1}{p} + \frac{1}{q} = 1$ and F a Banach space. In this section, we prove that under certain condition the dual of $L_p(\mu, F)$ is isomorphic to $L_q(\mu, F')$. For functions in particular, we have $L_p' = L_q$. We restrict ourselves to a framework which is natural rather than of maximum generality.

24-6.2. **Theorem** Let $1 \le p \le \infty$. For each $h \in L_q(\mu, F')$ and $f \in L_p(\mu, F)$, let $T_h(f) = \int hfd\mu$. Then T_h is a continuous linear form on $L_p(\mu, F)$ with $\|T_h\| = \|h\|_q$.

Proof. For $hf \in L_1$, $T_h(f)$ is well-defined. Clearly it is linear in f. Since $|T_h(f)| = \left|\int hfd\mu\right| \le \int |h|\ |f|\ d|\mu| \le \|h\|_q \|f\|_p$, the linear form T_h is continuous on $L_p(\mu, F)$ and $\|T_h\| \le \|h\|_q$. If $\|h\|_q = 0$ then $\|T_h\| = \|h\|_q = 0$. It remains to show $\|h\|_q \le \|T_h\|$ for $\|h\|_q > 0$.

Case 1: Assume that $1 \le q < \infty$ and $h = \sum_{j=1}^n \alpha_j \rho_{A_j}$ where $\alpha_j \ne 0 \in F'$ and A_j are disjoint nonempty decent sets. For every $\varepsilon > 0$ and each j, choose $\beta_j \in F$ satisfying $(1-\varepsilon)\|\alpha_j\| \le |\alpha_j \beta_j|$ and $\|\beta_j\| = 1$. Replacing β_j with $\beta_j\theta_j$ for some number $|\theta_j| = 1$, we may assume $\alpha_j\beta_j \ge 0$. Since $M = \bigcup_{j=1}^n A_j$ is a decent set, there is a locally integrable function φ on X such that $|\varphi| = 1$ on M, $\varphi(X \setminus M) = 0$ and $|\mu|B = \int_B \varphi^- d\mu$ for all decent subsets B of M. Then $f = \sum_{j=1}^n \beta_j \rho_{A_j} \|\alpha_j\|^{q-1}\varphi^- : X \to F$ is a measurable map. Since $|f| = \sum_{j=1}^n \|\alpha_j\|^{q-1}\rho_{A_j}$, we have $f \in L_p(\mu, F)$ and also

$$\|f\|_p^p = \int \sum_{j=1}^n \|\alpha_j\|^{p(q-1)}\rho_{A_j} d|\mu| = \int \sum_{j=1}^n \|\alpha_j\|^q \rho_{A_j} d|\mu| = \|h\|_q^q > 0.$$

From

$$\|T_h\|\ \|f\|_p \ge \left|\int hf d\mu\right| = \left|\sum_{j=1}^n \int \alpha_j\beta_j\rho_{A_j}\|\alpha_j\|^{q-1}\varphi^- d\mu\right|$$
$$= \sum_{j=1}^n \int \alpha_j\beta_j\rho_{A_j}\|\alpha_j\|^{q-1}d|\mu| \ge \sum_{j=1}^n \int (1-\varepsilon)\|\alpha_j\|\rho_{A_j}\|\alpha_j\|^{q-1}d|\mu|$$
$$= (1-\varepsilon)\|h\|_q^q = (1-\varepsilon)\|f\|_p\ \|h\|_q,$$

we have $\|T_h\| \ge (1-\varepsilon)\|h\|_q$. Letting $\varepsilon \downarrow 0$, we obtain $\|h\|_q \le \|T_h\|$.

Case 2: Assume $1 \le q < \infty$ and $h \in L_q(\mu, F')$ with $\|h\|_q > 0$. For every $\varepsilon > 0$, choose a decent map $k : X \to F'$ with $\|h - k\|_q \le \varepsilon$. It follows from last case, there is $f \in L_p(\mu, F)$ such that $\|f\|_p \le 1$ and $\|k\|_q - \varepsilon \le |\int kf d\mu|$. From

$$\|T_h\| \ge |\int hf d\mu| \ge |\int kf d\mu| - |\int (h-k)f d\mu|$$
$$\ge (\|k\|_q - \varepsilon) - \int |h-k|\ |f|\ d|\mu|$$
$$\ge (\|h\|_q - \|k-h\|_q) - \varepsilon - \|h-k\|_q\|f\|_p \ge \|h\|_q - 3\varepsilon,$$

letting $\varepsilon \downarrow 0$, we have $\|h\|_q \le \|T_h\|$.

Case 3: Assume $q = \infty$ and $h \in L_\infty(\mu, F')$ with $\|h\|_\infty > 0$. For each $0 < t < \|h\|_\infty$, the set $B = \{x \in X : \|h(x)\| > t\}$ is not null. There is a decent subset D of B with $|\mu|D > 0$. Since h is measurable, there are simple maps $h_n : X \to F'$ such that $h_n \to h$ on X. By Egorov's theorem, $h_n \to h$, μ-almost uniformly on D. There is a measurable subset N of D such that $|\mu|N < \frac{1}{2}|\mu|D$ and $h_n \to h$ uniformly on the decent set $P = D \setminus N$. For any $0 < \varepsilon < \frac{1}{2}t$, choose n such that $\|h_n(x) - h(x)\| \le \varepsilon$ for all $x \in P$. Write $h_n = \sum_{j=1}^k \alpha_j\rho_{A_j}$ where $\alpha_j \in F'$ and A_j are disjoint measurable sets. For all $x \in P \subset B$, $\|h_n(x)\| \ge \|h(x)\| - \varepsilon \ge t - \varepsilon \ge \varepsilon$ implies $x \in A_j$ for some j, that is $P \subset \bigcup_{j=1}^k A_j$, or $P = \bigcup_{j=1}^k (P \cap A_j)$. It follows that $0 < |\mu|D - |\mu|N = |\mu|P = \sum_{j=1}^k |\mu|(P \cap A_j)$, i.e. $|\mu|(P \cap A_i) > 0$ for some index i. Taking any $z \in A = P \cap A_i \subset B$, we obtain $\|\alpha_i\| = \|h_n(z)\| > t$. Also for all $x \in A$, we get $\|h(x) - \alpha_i\| = \|h(x) - h_n(x)\| \le \varepsilon$.

Pick $\beta \in F$ such that $|\alpha_i\beta| \geq (1-\varepsilon)\|\alpha_i\|$ and $\|\beta\| \leq 1$. Choose a number θ such that $|\theta| = 1$ and $\alpha_i\beta\theta = |\alpha_i\beta|$. Select an integrable function φ on X such that $|\varphi| = 1$ on A_i, $\varphi(X \setminus A_i) = 0$ and $|\mu|S = \int_S \varphi^- d\mu$ for all decent subset S of A_i. Now $f = \beta\theta\varphi^-\rho_A/|\mu|(A)$ is a decent map with $\|f\|_1 = \int |f|\, d|\mu| = \|\beta\| \leq 1$. From

$$\begin{aligned}
\|T_h\| &\geq \left|\int hf d\mu\right| = \left|\int \frac{h\beta\theta\varphi^-\rho_A}{|\mu|A}\, d\mu\right| \\
&\geq \left|\int \frac{\alpha_i\beta\theta\varphi^-\rho_A}{|\mu|A}\, d\mu\right| - \left|\int \frac{(h-\alpha_i)\beta\theta\varphi^-\rho_A}{|\mu|A}\, d\mu\right| \\
&\geq \int \frac{\alpha_i\beta\theta\rho_A}{|\mu|A}\, d|\mu| - \int \frac{|h-\alpha_i|\ \|\beta\theta\|\rho_A}{|\mu|A}\, d|\mu| \\
&\geq \int \frac{(1-\varepsilon)\|\alpha_i\|\rho_A}{|\mu|A}\, d|\mu| - \int \frac{\varepsilon\rho_A}{|\mu|A}\, d|\mu| \\
&\geq \int \frac{(1-\varepsilon)t\rho_A}{|\mu|A}\, d|\mu| - \varepsilon \geq (1-\varepsilon)t - \varepsilon
\end{aligned}$$

letting $\varepsilon \downarrow 0$, we have $\|T_h\| \geq t$. Finally letting $t \uparrow \|h\|_\infty$, we have $\|T_h\| \geq \|h\|_\infty$. □

24-6.3. **Exercise** For $F = \mathbb{K}$, $1 < q < \infty$ and $\mu \geq 0$, the above proof can be dramatically simplified. Let $f = |h|^{q-1}\text{sgn}(h^-)$. Prove that $fh = |f|^p = |h|^q$ and $\int fhd\mu = \|f\|_p^p = \|h\|_q^q = \|f\|_p\|h\|_q$.

24-6.4. **Corollary** $\|T_h\|_q = \sup\{\int |h|\ fd|\mu| : 0 \leq f \in L_p, \|f\|_p \leq 1\}$.

Proof. For every $\varepsilon > 0$, choose $g \in L_p$ such that $\|T_{|h|}\| \leq \left|\int |h|gd|\mu|\ \right| + \varepsilon$. Then we have $0 \leq f = |g| \in L_p$, $\|f\|_p = \|g\|_p \leq 1$ and

$$\|T_{|h|}\| \leq \left|\int |h|gd|\mu|\ \right| + \varepsilon \leq \int |h|\ |g|d|\mu| + \varepsilon = \int |h|fd|\mu| + \varepsilon \leq \|T_{|h|}\| + \varepsilon.$$

From $\|T_h\|_q = \|h\|_q = \|\ |h|\ \|_q = \|T_{|h|}\|_q$, the proof is completed. □

24-6.5. We have proved that the map $\pi : L_q(\mu, F') \to L_p(\mu, F)'$ given by $\pi(h) = T_h$ is an isometry. Naturally we want to know if π is an isomorphism. We have proved that the scalar field $\mathbb{K}$ has Radon-Nikodym Property. In next section of this chapter, it will be shown that separable dual spaces and reflexive spaces have Radon-Nikodym Property.

24-6.6. **Lemma** Let $1 < p < \infty$. Then for every continuous linear form T on $L_p(\mu, F)$, there is a σ-finite set M such that for each $f \in L_p(\mu, F)$, we have $T(f\rho_{X\setminus M}) = 0$. In other words, T is *concentrated* on a σ-finite set M.

Proof. By Density Theorem, we have $\|T\| = \sup\{|T(f)| : \|f\|_p = 1\}$ where f runs over the decent maps. For each n, there is a decent map f_n such that

$\|f_n\|_p = 1$ and $|T(f_n)| \geq \|T\|(1 - \frac{1}{n})$. Replacing f_n by $\theta_n f_n$ for some number $|\theta_n| = 1$, we may assume $T(f_n) \geq 0$. Now each $M_n = \{x \in X : |f_n(x)| > 0\}$ is σ-finite. Hence $M = \bigcup_{n=1}^{\infty} M_n$ is also σ-finite. Now take any decent subset A of $X \setminus M$ and any $\alpha \in F$. Choose $\theta \in \mathbb{K}$ such that $|\theta| = 1$ and $\theta T(\alpha\rho_A) \geq 0$. Then for each $r > 0$, $g = f_n + r\theta\alpha\rho_A$ is also a decent map. Observe that

$$\|g\|_p^p = \int |f_n + r\theta\alpha\rho_A|^p d|\mu| = \int |f_n|^p d|\mu| + \int r^p\|\alpha\|^p\rho_A d|\mu| = 1 + r^p\|\alpha\|^p|\mu|A$$

that is, $\|g\|_p = (1 + r^p\|\alpha\|^p|\mu|A)^{1/p}$. Hence

$$\|T\|(1 + r^p\|\alpha\|^p|\mu|A)^{1/p} = \|T\|\ \|g\|_p \geq |T(g)| = |T(f_n + r\theta\alpha\rho_A)|$$
$$= T(f_n) + r\theta T(\alpha\rho_A) \geq \|T\|(1 - \tfrac{1}{n}) + r\theta T(\alpha\rho_A),$$

or,
$$0 \leq \theta T(\alpha\rho_A) \leq \|T\| \left[(1 + r^p\|\alpha\|^p|\mu|A)^{1/p} - 1 + \frac{1}{n}\right] \frac{1}{r}.$$

Letting $n \to \infty$, we have

$$0 \leq \theta T(\alpha\rho_A) \leq \|T\| \left[(1 + r^p\|\alpha\|^p|\mu|A)^{1/p} - 1\right] \frac{1}{r}, \quad \forall\, r > 0.$$

Letting $r \downarrow 0$, it follows from $1 < p < \infty$ and the L'Hopital's rule that $\theta T(\alpha\rho_A) = 0$, that is $T(\alpha\rho_A) = 0$ for every $\alpha \in F$. Next, take any decent map $g = \sum_{j=1}^{k} \alpha_j\rho_{B_j}$ where $\alpha_j \in F$ and $B_1, B_2, \cdots$ are disjoint decent sets. Then $g\rho_{X\setminus M} = \sum_j \alpha_j\rho_{A_j}$ where $A_j = B_j \setminus M$ are decent subsets of $X \setminus M$. Hence $T(g\rho_{X\setminus M}) = \sum_j T(\alpha_j\rho_{A_j}) = 0$. Finally, take any $f \in L_p(\mu, F)$. There are decent maps g_n such that $|g_n| \leq |f|$ on X and $g_n \to f$, μ-ae. Then $|g_n\rho_{X\setminus M}| \leq |f|$ on X and $g_n\rho_{X\setminus M} \to f\rho_{X\setminus M}$, μ-ae. The L_p-Dominated Convergence Theorem ensures $g_n\rho_{X\setminus M} \to f\rho_{X\setminus M}$ in $L_p(\mu, F)$. Since T is continuous, we have $T(f\rho_{X\setminus M}) = \lim_{n\to\infty} T(g_n\rho_{X\setminus M}) = 0$. □

24-6.7. **L_p-Duality Theorem** Assume that $1 < p < \infty$ or that X is σ-finite for $p = 1$. If F' has the Radon-Nikodym Property, then the map $\pi : L_q(\mu, F') \to L_p(\mu, F)'$ is an isomorphism.

Proof. To prove that π is surjective, let $T \in L_p(\mu, F)'$. We have to find $h \in L_q(\mu, F')$ such that $Tf = \int hf d\mu$ for all $f \in L_p(\mu, F)$. Now for every $A \in \mathbb{D}$, $\beta \in F$ and $x \in X$; let $(\rho_A\beta)(x) = \rho_A(x)\beta$ and $(\nu A)\beta = T(\rho_A\beta)$. Clearly νA is a linear form on F. Since

$$\sup_{\|\beta\|\leq 1} |(\nu A)\beta| = \sup_{\|\beta\|\leq 1} |T(\rho_A\beta)| \leq \sup_{\|\beta\|\leq 1} \|T\|\ \|\rho_A\beta\|_p \leq \|T\|(|\mu|A)^{1/p},$$

we have $\nu A \in F'$ and $\|\nu A\| \leq \|T\|(|\mu|A)^{1/p}$. In particular, if $A_n \downarrow \emptyset$ in $\mathbb{D}$, then $\|\nu A_n\| \leq \|T\|(|\mu|A_n)^{1/p} \to 0$. Hence ν is countably additive. Next, to show that ν is of finite variation, suppose that $A = \bigcup_{j=1}^{n} B_j$ is a disjoint union where

$A, B_j \in \mathbb{D}$. For each j, let $\beta_j \in E$ with $\|\beta_j\| \le 1$. Choose a number θ_j such that $|\theta_j| = 1$ and $\theta_j T(\rho_{B_j}\beta_j) = |T(\rho_{B_j}\beta_j)|$. Then we have

$$\sum_{j=1}^{n} |(\nu B_j)\beta_j| = \sum_{j=1}^{n} |T(\rho_{B_j}\beta_j)| = \sum_{j=1}^{n} T(\rho_{B_j}\beta_j\theta_j)$$
$$\le \|T\| \left\|\sum_{j=1}^{n} \rho_{B_j}\beta_j\theta_j\right\|_p \le \|T\| \left\{\int \left|\sum_{j=1}^{n} \rho_{B_j}\beta_j\theta_j\right|^p d|\mu|\right\}^{1/p}$$
$$\le \|T\| \left(\int \sum_{j=1}^{n} \left|\rho_{B_j}\right|^p \|\beta_j\|^p |\theta_j|^p d|\mu|\right)^{1/p} \le \|T\|(|\mu|A)^{1/p}.$$

Hence we get $\sum_{j=1}^{n} \|\nu B_j\| \le \|T\|(|\mu|A)^{1/p}$ by taking suprema over all $\|\beta_j\| \le 1$ in E independently. Consequently, ν is of finite variation. Therefore ν is a vector measure on X. From $\|\nu A\| \le \|T\|(|\mu|A)^{1/p}$, we have $\nu \ll \mu$.

Next, for $1 < p < \infty$, there is a σ-finite set M such that $T(f\rho_{X\setminus M}) = 0$ for all $f \in L_p(\mu, F)$. For $p = 1$, let $M = X$. Since F' has Radon-Nikodym Property, there is a locally μ-integrable map $h : X \to F'$ such that $h(X \setminus M) = 0$ and $\nu A = \int_A h d\mu$ for all decent subsets A of M. Observe that if $F = \mathbb{K}$, $\beta = 1$ and $\mu, T \ge 0$; Radon-Nikodym Theorem ensures that $h \ge 0$. Now for each decent set $A \in \mathbb{D}$ and $\beta \in E$, we have

$$T(\rho_A\beta) = T(\rho_{A\cap M}\beta) + T(\rho_{A\setminus M}\beta) = \nu(A \cap M)\beta + 0$$
$$= \left(\int_{A\cap M} hd\mu + \int_{A\setminus M} hd\mu\right)\beta = \int h(\rho_A\beta)d\mu.$$

By linearity, for each decent map $f : X \to F$, we get $T(f) = \int hfd\mu$. It remains to show $h \in L_q$. In this case, the proof is completed because T and $f \to \int hfd\mu$ are continuous linear forms which agree on the dense set of decent maps must agree on $L_p(\mu, F)$.

Consider the case $1 < p < \infty$ first. Write $M = \bigcup_{k=0}^{\infty} H_k$ as a disjoint union where H_0 is a μ-null set and $H_1, H_2, \cdots$ are decent sets. Because each $M_k = \left(\bigcup_{j=1}^{k} H_j\right) \cap \{x \in X : \|h(x)\| \le k\}$ is a decent set and $|h\rho_{M_k}|^p \le k^p\rho_{M_k} \in L_1$, we have $h\rho_{M_k} \in L_q(\mu, F')$. The linear form T_k on $L_p(\mu, F)$ given by $T_k(f) = \int(h\rho_{M_k})fd\mu$ is continuous. Observe that

$$\|h\rho_{M_k}\|_q = \|T_k\| = \sup\{|\textstyle\int(h\rho_{M_k})fd\mu| : f \in L_p(\mu, F),\ \|f\|_p \le 1\}$$
$$= \sup\{|\textstyle\int h(f\rho_{M_k})d\mu| : f \in L_p(\mu, F),\ \|f\|_p \le 1\}$$
$$= \sup\{|T(f\rho_{M_k})| : f \in L_p(\mu, F),\ \|f\|_p \le 1\}$$
$$\le \sup\{\|T\|\ \|f\rho_{M_k}\|_p : f \in L_p(\mu, F),\ \|f\|_p \le 1\} \le \|T\|.$$

Since $|h\rho_{M_k}| \uparrow |h|$, μ-ae; it follows from L_q-Monotone Convergence Theorem that $h \in L_q$.

Finally for $p = 1$ and $q = \infty$, define M_k as in last case. Clearly we get

$h\rho_{M_k} \in L_\infty(\mu, F')$. Repeat the last case, we have $|h\rho_{M_k}| \leq \|h\rho_{M_k}\|_\infty \leq \|T\|$, μ-ae. Hence $|h| = \lim_{k\to\infty} |h\rho_{M_k}| \leq \|T\|$, μ-ae. Consequently $\|h\|_\infty \leq \|T\|$. This completes the proof. □

24-6.8. For the scalar case when $F = \mathbb{K}$, we can derive additional information about h from T by decomposition into positive linear forms.

24-6.9. **Lemma** For $1 \leq p \leq \infty$, the space L_p is a breakable vector lattice. Furthermore, every continuous linear form T on L_p is order bounded and can be written as a linear combination of continuous positive linear forms.

Proof. It is obvious that L_p is a vector lattice under pointwise operations. The following equalities and inequalities are considered to be almost everywhere. Let $|f| \leq g + h$ where $f, g, h \in L_p$ with $g, h \geq 0$. Write $|f| = g_1 + h_1$ where $0 \leq g_1 \leq g$, and $0 \leq h_1 \leq h$ are in L_p. Since $\operatorname{sgn}(f)$ is bounded measurable, $g_2 = g_1 \operatorname{sgn}(f)$ and $h_2 = h_1 \operatorname{sgn}(f)$ belong to L_p. Clearly $|g_2| \leq g_1 \leq g$, $|h_2| \leq h_1 \leq h$ and $f = g_2 + h_2$. Therefore L_p is breakable. Next, let T be a continuous linear form on L_p. Then for all $|g| \leq f$ in L_p, we have

$$|T(g)| \leq \|T\| \; \|g\|_p \leq \|T\| \; \|f\|_p < \infty.$$

Therefore every continuous linear form on the vector lattice L_p is order bounded. Furthermore, the variation $|T|$ exists and is also continuous. In particular, every continuous linear form T on L_p can be written in the form $T = \sum_{k=0}^{3} i^k T_k$ where each T_k is a continuous positive linear form. □

24-6.10. **Theorem** Assume that $1 < p < \infty$ or that X is σ-finite for $p = 1$. For each continuous linear form T on L_p, there is $h \in L_q$ such that $T(f) = \int hf d\mu$ for every $f \in L_p$. Furthermore if $T \geq 0$ and $\mu \geq 0$, then we may choose $h \geq 0$. If both T and μ are real, then we may choose h to be real.

Proof. The proof of the case when both T, μ are positive is built into the last theorem. Next, assume that T is positive but μ is a complex measure. Then there is $0 \leq g \in L_q$ such that $T(f) = \int gf d|\mu|$ for all $f \in L_p$. Let T be concentrated on some $|\mu|$-σ-finite set M. There is a locally μ-integrable function φ on X such that $\varphi(X \setminus M) = 0$, $|\varphi| = 1$ on M and $|\mu|(A) = \int_A \varphi^- d\mu$ for every decent subset A of M. Then $T(f) = \int gf d|\mu| = \int gf\varphi^- d\mu$, $\forall\, f \in L_p$. Let $h = g\varphi^-$. Since $|h| \leq g$ we have $h \in L_p$. Obviously, h is a required function. Finally, take any arbitrary T. Write $T = \sum_{k=0}^{3} i^k T_k$ where $T_k \geq 0$. There are $0 \leq h_k \in L_q$ such that $T_k(f) = \int h_k f d\mu, \forall\, f \in L_p$. Define $h = \sum_{k=0}^{3} i^k h_k$. Then $h \in L_q$ and $T(f) = \sum_{k=0}^{3} i^k T_k f = \sum_k i^k \int h_k f d\mu = \int hf d\mu$. We leave it as an exercise to show that if both T and μ are real, then we may choose h to be real. □

24-6.11. **Exercise** Let X be μ-σ-finite and let $1 \leq p \leq \infty$. Suppose that $h \in L_q$ is fixed and T is a continuous linear form on L_p given by $T(f) = \int hf d\mu$ for all $f \in L_p$. Prove that the valuation $|T|$ as a linear form defined in §16-5.4a is determined by $|T|(f) = \int f|h|\, d|\mu|$ for all $f \in L_p$.

24-6.12. **Theorem** Let $h : X \to F'$ be a measurable map and $1 \leq p \leq \infty$. If for every $f \in L_p(\mu, F)$ we have $hf \in L_1$, then $h \in L_q(\mu, F')$.

Proof. (a) We claim that if M is a σ-finite subset of X, then $|h|\rho_M\xi \in L_1$ for every $0 \leq \xi \in L_p$. In fact, firstly suppose $q = \infty$. Then $0 \leq \xi \in L_1$. Hence $\rho_M\xi \in L_1 = L_p$. Thus $h\rho_M\xi \in L_1$, that is $|h|\rho_M\xi = |h\rho_M\xi| \in L_1$. Next suppose $1 \leq q < \infty$. Since h is measurable, there are simple maps $h_n : X \to F'$ such that $|h_n| \leq |h|$ and $h_n \to h$. By §21-7.11, write $M = \bigcup_{j=1}^{\infty} A_j \cup N$ where N is a null set and $h_n \to h$ uniformly on each of the disjoint decent sets $A_1, A_2, \cdots$. Let $\varepsilon_j = (2^j|\mu|A_j)^{-1/q}$ if $|\mu|A_j > 0$; and $\varepsilon_j = 2^{-j/q}$ if $|\mu|A_j = 0$ so that $\varphi = \sum_{j=1}^{\infty} \varepsilon_j\rho_{A_j} \in L_q$. For each j, there is n such that for all $x \in A_j$ we obtain $\|h(x) - h_n(x)\| \leq \varepsilon_j$. Write $h_n = \sum_{i=1}^{k} \alpha_i\rho_{B_i}$ where B_i are disjoint measurable sets and $\alpha_i \in F'$. For each i, choose $\beta_i \in F$ such that $\|\beta_i\| \leq 1$ and $\|\alpha_i\| \leq |\alpha_i\beta_i| + \varepsilon_j$. Then $g_j = \sum_{i=1}^{k} \beta_i\rho_{B_i}$ is a simple map with $|g_j| \leq 1$. Observe that $h_n(x)g_j(x) = \sum_{i=1}^{k} \alpha_i\beta_i\rho_{B_i}$. For each $x \in A_j$ we have

$$\begin{aligned}\|h(x)\| &\leq \|h_n(x)\| + \varepsilon_j = \textstyle\sum_{i=1}^{k} \|\alpha_i\|\rho_{B_i} + \varepsilon_j \leq \sum_{i=1}^{k} |\alpha_i\beta_i|\rho_{B_i} + 2\varepsilon_j \\ &= |h_n(x)g_j(x)| + 2\varepsilon_j \leq |h(x)g_j(x)| + \|h_n(x) - h(x)\|\ \|g_j(x)\| + 2\varepsilon_j \\ &\leq |h(x)g_j(x)| + 3\varepsilon_j.\end{aligned}$$

Therefore we get $|h|\rho_{A_j} \leq |hg_j\rho_{A_j}| + 3\varepsilon_j\rho_{A_j}$. Now $g = \sum_{j=1}^{\infty} g_j\rho_{A_j}$ is pointwise convergent because A_j are disjoint. Clearly $|g| \leq 1$. As a limit of simple maps, g is measurable. Since $M \setminus N \subset \bigcup_{j=1}^{\infty} A_j$, we have $|h|\rho_{M\setminus N} \leq |hg| + 3\varphi$. Now take any $0 \leq \xi \in L_p$. Since $\varphi \in L_q$, we have $\varphi\xi \in L_1$. From $|g\xi| \leq \xi$, μ-ae; we get $f = g\xi \in L_p(\mu, F)$ and thus $hf = hg\xi \in L_1$ by given condition. Since $|h|\rho_M\xi \leq |hg\xi| + 3\varphi\xi$, μ-ae; the function $|h|\rho_M\xi$ is integrable as claimed.

(b) We claim that if M is a σ-finite subset of X, then

$$t = \sup\{\textstyle\int_M |h|\xi\, d|\mu| : 0 \leq \xi \in L_p,\ \|\xi\|_p \leq 1\} < \infty.$$

In fact, suppose to the contrary that for every n, there is $0 \leq \varphi_n \in L_p$ such that $\|\varphi_n\|_p \leq 1$ and $\int_M |h|\varphi_n d|\mu| \geq n^3$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}\|\varphi_n\|_p \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}\, \varphi_n$ converges to some $0 \le \xi \in L_p$. Now $|h|\rho_M\xi$ is integrable by part (a). For every n, we have $|h|\rho_M\xi \ge \frac{1}{n^2}|h|\rho_M\varphi_n$, μ-ae, that is $\infty > \int_M |h|\xi d|\mu| \ge \frac{1}{n^2}\int_M |h|\varphi_n d|\mu| \ge n$ for all n which is a contradiction. Therefore we must have $t < \infty$.

(c) We claim that if M is a σ-finite subset of X, then $|h|\rho_M \in L_q$. In fact, by §20-4.10 for the upper function $|h|$, choose decent functions $0 \le h_n \uparrow |h|\rho_M$, μ-ae. Then all $h_n \in L_q$ and $\|h_n\|_q = \sup \int h_n\xi d|\mu| \le \sup \int_M |h|\xi d|\mu| = t < \infty$ where the suprema are taken over $0 \le \xi \in L_p$ with $\|\xi\|_p \le 1$. Therefore we obtain $\sup_n \|h_n\|_q \le t < \infty$. For $1 \le q < \infty$, $|h|\rho_M \in L_q$ by L_q-Monotone Convergence Theorem. For $q = \infty$, $\|\,|h|\rho_M\|_\infty = \lim \|h_n\|_\infty \le t < \infty$ implies $|h|\rho_M \in L_q$.

(d) Assume $1 \le q < \infty$. We claim that there is σ-finite set $M \subset X$ such that $h = h\rho_M$, μ-ae. In fact, for every decent set A we have $h\rho_A \in L_q$, that is $|h|^q\rho_A \in L_1$. Let $t = \sup \int_A |h|^q d|\mu|$ for all decent sets A. There are decent sets $A_n \subset A_{n+1}$ such that $\sup_n \int_{A_n} |h|^q d|\mu| = t$. Then the set $M = \bigcup_{n=1}^{\infty} A_n$ is σ-finite. Thus $|h|^q\rho_M$ is integrable and $|h|\rho_{A_n} \le |h|\rho_M$ for every n. Hence $t = \sup_n \int_{A_n} |h|^q d|\mu| \le \int_M |h|^q d|\mu| < \infty$. Next, for every decent set B disjoint from M with $\|h(x)\| > 0$ for all $x \in B$, then

$$t + \int_B |h|^q d|\mu| = \sup \int_M |h|^q d|\mu| + \int_B |h|^q d|\mu| = \sup \int_{M\cup B} |h|^q d|\mu| \le t,$$

that is $\int_B |h|^q d|\mu| = 0$. Hence B is null. Therefore we have $h = h\rho_M$, μ-ae. Consequently, $|h| = |h|\rho_M \in L_q$, that is $h \in L_q(\mu, F')$.

(e) Finally let $q = \infty$. We claim that $h \in L_\infty(\mu, F')$. In fact, suppose to the contrary that $\|h\|_\infty = \infty$. Then for every n, the set $B_n = \{x \in X : \|h(x)\| \ge n\}$ is not null. There is a decent set $A_n \subset B_n$ with $|\mu|A_n > 0$. Then $M = \bigcup_{n=1}^{\infty} A_n$ is σ-finite. By (c), $|h|\rho_M \in L_\infty$. On the other hand, $\|h\rho_M\|_\infty \ge \|h\rho_{A_n}\|_\infty \ge n$ for every n. Thus $\|h\rho_M\|_\infty = \infty$. This contradiction completes the proof. □

24-6.13. **Corollary** Let $f : X \to F$ be a measurable map and $1 \le p \le \infty$. If for every $h \in L_q(\mu, F')$ we have $hf \in L_1$, then $f \in L_p(\mu, F)$.

Proof. Replacing F by F' and interchange h, f and p, q in last theorem; we have $f \in L_p(\mu, F'')$, that is $|f| \in L_1$ and hence $f \in L_p(\mu, F)$. □

24-6.99. **Exercise** Let F be a Banach space and let μ be a vector measure on a δ-space $(X, \mathbb{D})$ into the dual space F'. For convenience, write $\beta\alpha = \beta(\alpha) = \alpha(\beta) = \alpha\beta$ for every $\alpha \in F'$. Note that the map $(\alpha, \beta) \to \alpha\beta$:

$F' \times F \to \mathbb{K}$ is an admissible bilinear map so that the integral $\int f d\mu$ is well-defined for μ-integrable maps $f : X \to F$ in $L_1(\mu, F'; F)$. Take any $h : X \to F'$ in $L_q(\mu, F')$ and $f : X \to F$ in $L_p(\mu, F)$. Then hf is measurable because it is the limit of certain simple functions derived from h, f. From $h \in L_q(\mu, F')$, we have $|h| \in L_q(|\mu|)$ and similarly $|f| \in L_p(|\mu|)$. Since $|hf| \leq |h|\ |f|$, we obtain $hf \in L_1(|\mu|)$ and hence $S_h(f) = \int hfd|\mu|$ is well-defined. Study the map $h \to S_h$ from $L_q(\mu, F')$ into the dual space of $L_p(\mu, F)$.

24-7 Spaces with Radon-Nikodym Property

24-7.1. We assume that the dual space has Radon-Nikodym Property in order to prove L_p-Duality Theorem and Lebesgue Decomposition Theorem. Note that c_0 does not have Radon-Nikodym Property, e.g. [Diestel-77, p60]. We shall give sufficient conditions for a Banach space to have Radon-Nikodym Property by improving their work from bounded positive measures on σ-algebras into our context. Let μ be a scalar measure on a δ-space $(X, \mathbb{D})$ and ν a vector measure on X into a Banach space E.

24-7.2. **Lemma** Let $Q = \bigcup_{n=0}^{\infty} H_n$ be a disjoint union of measurable subsets H_n of X. If for each n, the map $g_n : X \to E$ is a locally μ-integrable map such that $\nu(B) = \int_B g_n d\mu$ for all decent subsets B of H_n, then the series $h = \sum_{n=1}^{\infty} g_n \rho_{H_n} : X \to E$ is a locally μ-integrable map such that $h(X \setminus Q) = 0$ and $\nu(A) = \int_A hd\mu$ for all decent subsets A of Q.

Proof. Clearly h is a measurable map such that $h(X \setminus Q) = 0$. Take any decent set A. From $\nu(B) = \int_B g_n d\mu$ for all decent subsets B of H_n, we have $|\nu|(A \cap H_n) = \int_{A \cap H_n} |g_n|\ d|\mu| = \int_A |g_n \rho_{H_n}|\ d|\mu|$. Observe that

$$\sum_{n=0}^{\infty} \int_A |g_n \rho_{H_n}|\ d|\mu| = \sum_{n=0}^{\infty} |\nu|(A \cap H_n) = |\nu| \left(A \cap \bigcup_{n=0}^{\infty} H_n\right) \leq |\nu|(A) < \infty.$$

Integration term by term gives

$$\begin{aligned} \int_A hd\mu &= \int_A \sum_{n=0}^{\infty} g_n \rho_{H_n} d\mu = \sum_{n=0}^{\infty} \int_A g_n \rho_{H_n} d\mu \\ &= \sum_{n=0}^{\infty} \nu(A \cap H_n) = \nu \left(A \cap \bigcup_{n=0}^{\infty} H_n\right) = \nu(A \cap Q). \end{aligned}$$

Since h is μ-integrable on every decent set, it is locally μ-integrable. □

24-7.3. **Theorem** If every vector measure into E has a density on every decent set with respect to its variation, then E has Radon-Nikodym property.

Proof. Let μ be a scalar measure on $(X, \mathbb{D})$ and ν a μ-continuous vector measure into E. Every μ-σ-finite set can be written as a disjoint union

$M = \bigcup_{n=0}^{\infty} H_n$ where H_0 is a μ-null set and all $H_1, H_2, \cdots$ are decent sets. By $\nu \ll \mu$, H_0 is also ν-null. Let $g_0 = 0$ so that $\int_B g_0 d\mu = 0 = \nu(B)$ for all decent subsets B of H_0. Since $|\nu| \ll \mu$ there is a locally μ-integrable function φ on X such that $|\nu|(A) = \int_A \varphi d\mu$ for every decent subset A of M. By given condition, for each $n \geq 1$ there is a locally ν-integrable map $f_n : X \to E$ such that $\nu B = \int_B f_n d|\nu|$ for all decent subsets B of H_n. Thus $g_n = f_n \varphi \rho_{H_n}$ is μ-integrable and satisfies $\nu B = \int_B g_n d\mu$ for all decent subsets B of H_n. Now the result follows from last lemma. □

24-7.4. **Theorem** Let F be a Banach space. If the dual space F' is separable, then F' has the Radon-Nikodym Property.

Proof. Let ν be a vector measure on a δ-space $(X, \mathbb{D})$ into F' and M be a decent set. We have to find a ν-integrable map $g : X \to F'$ such that $\nu B = \int_B g d|\nu|$ for every decent subset B of M. In fact, for every $\beta \in F$ and $A \in \mathbb{D}$, let $\pi_\beta A = \nu(A \cap M)\beta$. Clearly $\pi_\beta : \mathbb{D} \to \mathbb{K}$ is finitely additive. Also π_β is countably additive because if $A_n \downarrow \emptyset$ in $\mathbb{D}$, then $|\pi_\beta A_n| = |\nu(A_n \cap M)\beta| \leq \|\nu(A_n \cap M)\| \; \|\beta\| \to 0$ as $n \to \infty$. Next, π_β is of finite variation because if $A = \bigcup_{j=1}^{n} B_j$ is a disjoint union with $A, B_j \in \mathbb{D}$, then

$$\sum_{j=1}^{n} |\pi_\beta B_j| = \sum_{j=1}^{n} |\nu(B_j \cap M)\beta|$$
$$\leq \sum_{j=1}^{n} \|\nu(B_j \cap M)\| \; \|\beta\| \leq |\nu|(A \cap M)\|\beta\| < \infty$$

implies $|\pi_\beta|(A) \leq |\nu|(A \cap M)\|\beta\|$. Clearly if $|\nu|A = 0$, then $\pi_\beta A = \nu(A \cap M)\beta = 0$. Hence $\pi_\beta \ll |\nu|$. By Radon-Nikodym theorem, there is a locally integrable function f_β on X such that $f_\beta(X \setminus M) = 0$ and $\pi_\beta A = \int_A f_\beta d|\nu|$ for all $A \in \mathbb{D}$ because $\pi_\beta(A \setminus M) = 0 = \int_{A \setminus M} f_\beta d|\nu|$. Let $P = \{x \in X : \text{Re } f_\beta(x) \geq 0\}$. From

$$\int_A |\text{Re } f_\beta| d|\nu| = \int_{A \cap P} (\text{Re } f_\beta) d|\nu| + \int_{A \setminus P} -(\text{Re } f_\beta) d|\nu|$$
$$\leq |\text{Re } \nu(A \cap P \cap M)\beta| + |\text{Re } \nu[(A \setminus P) \cap M] \; \beta|$$
$$\leq \|\nu(A \cap M \cap P)\| \; \|\beta\| + \|\nu(A \cap M) \setminus P)\| \; \|\beta\|$$
$$\leq |\nu|(A \cap M) \; \|\beta\| \leq \textstyle\int_A \|\beta\| \; d|\nu|,$$

we have $|\text{Re } f_\beta| \leq \|\beta\|$, ν-ae. Similarly, we get $|\text{Im } f_\beta| \leq \|\beta\|$, ν-ae. Hence $|f_\beta| \leq 2\|\beta\|$, ν-ae. Next,

$$\int_A f_{\beta+\gamma} d|\nu| = \nu(A \cap M)(\beta + \gamma) = \nu(A \cap M)\beta + \nu(A \cap M)\gamma$$
$$= \int_A f_\beta d|\nu| + \int_A f_\gamma d|\nu| = \int_A (f_\beta + f_\gamma) d|\nu|$$

implies $f_{\beta+\gamma} = f_\beta + f_\gamma$, ν-ae. Similarly we also have $f_{t\beta} = t f_\beta$, ν-ae for all $t \in \mathbb{K}$. Since F' is separable, so is F. Let H_1 be a countable dense set in F and $\mathbb{K}_0$ a countable dense subset of the scalar field $\mathbb{K}$. Then the set H of all linear combinations $\sum_{j=1}^m t_j\beta_j$ with $\beta_j \in H_1$ and $t_j \in \mathbb{K}_0$ is countable. There is a null set N such that $|f_\beta(x)| \leq 2\|\beta\|$, $f_{\beta+\gamma}(x) = f_\beta(x) + f_\gamma(x)$ and $f_{t\beta}(x) = t f_\beta(x)$ for all $x \in M \setminus N$; $\beta, \gamma \in H$; and $t \in \mathbb{K}_0$. The same is true for $x \in X \setminus M$ because all $f_\beta(x) = 0$. For every $\beta \in F$, we have $\beta_j \to \beta$ for some sequence $\beta_j \in H$. Define $h_\beta = \lim_{j\to\infty} \rho_{X\setminus N} f_{\beta_j}$. If $\gamma_j \in H$ with $\gamma_j \to \beta$ and if $x \in X$; we get

$$|\rho_{X\setminus N}(x) f_{\beta_j}(x) - \rho_{X\setminus N}(x) f_{\gamma_j}(x)| \leq |\rho_{X\setminus N}(x) f_{\beta_j - \gamma_j}(x)| \leq 2\|\beta_j - \gamma_j\| \to 0$$

as $j \to \infty$. Thus the definition of h_β is independent of the choice of $\{\beta_j\}$ in H. Hence h_β is well-defined. Because H is dense in F, we have $|h_\beta| \leq 2\|\beta\|$ and $h_{\beta+\gamma} = h_\beta + h_\gamma$, $h_{t\beta} = t h_\beta$ for all $\beta, \gamma \in F$ and $t \in \mathbb{K}$. Thus the map $g(x) : F \to \mathbb{K}$ defined by $g(x)(\beta) = h_\beta(x)$ is a continuous linear form. As the limit of measurable functions $\rho_{X\setminus N} f_{\beta_j}$ for $\beta_j \in H$, the function h_β is measurable for each $\beta \in F$. Hence $x \to h_\beta(x) = g(x)(\beta)$ is measurable for all $\beta \in F$. Therefore $x \to g(x)$ is weak-star measurable. Because F' is separable, the map $g : X \to F'$ is strongly measurable. Next, let $\beta_j \to \beta$ in F with $\beta_j \in H$. For every $x \in X \setminus M$, $g(x)\beta = h_\beta(x) = \lim_{j\to\infty} \rho_{X\setminus N} f_{\beta_j}(x) = 0$. Hence $g(X \setminus M) = 0$. Next, since $\{\beta_j\}$ is bounded, i.e. $\|\beta_j\| \leq t$ for some $t > 0$ and for all j; we have $|\rho_{X\setminus N} f_{\beta_j}|\rho_A \leq 2t\rho_A$ and $2t\rho_A$ is integrable. It follows from Dominated Convergence Theorem that

$$\begin{aligned}\int_A h_\beta d|\nu| &= \int_A \lim_{j\to\infty} \rho_{X\setminus N} f_{\beta_j} d|\nu| = \lim_{j\to\infty} \int_A \rho_{X\setminus N} f_{\beta_j} d|\nu| \\ &= \lim_{j\to\infty} \int_A f_{\beta_j} d|\nu| = \lim_{j\to\infty} \nu(A \cap M)\beta_j = \nu(A \cap M)\beta.\end{aligned}$$

Since $\|g(x)\| \leq \sup_{\|\beta\|\leq 1} |g(x)(\beta)| \leq \sup_{\|\beta\|\leq 1} |h_\beta(x)| \leq 2\|\beta\| \leq 2$, the map g is bounded on X and hence it is integrable on every decent set A, that is locally integrable. Finally, from $\nu(A \cap M)\beta = \int_A h_\beta(x) d|\nu|(x) = \int_A g(x)(\beta) d|\nu|(x) = \left[\int_A g(x) d|\nu|(x)\right](\beta)$, we obtain $\nu(A \cap M) = \int_A g d|\nu|$. Therefore for every decent subset B of M, we have $\nu B = \int_B g d|\nu|$. This completes the proof. □

24-7.5. Let μ be a *positive* measure on a δ-space $(X, \mathbb{D})$ and π a finite partition of a decent set M by sets in $\mathbb{D}$. Let $h : X \to E$ be a measurable map which is μ-integrable on M. The π-*average* of h on M is the map $V_\pi h : X \to E$ defined by $V_\pi h = \sum_{A\in\pi} \frac{\int_A h d\mu}{\mu A} \rho_A$. We understand that $\frac{\int_A h d\mu}{\mu A}$ is replaced by zero if $\mu A = 0$. Next, let $P(M)$ denote the set of all finite partitions of M by sets in $\mathbb{D}$. For all $\sigma, \pi \in P(M)$, write $\pi \geq \sigma$ if π is finer than σ, that is, every set in σ is a union of sets in π.

24-7.6. **Lemma** (a) If h is a simple map, then $\lim_{\pi\to\infty} V_\pi h = h\rho_M$. It means that there is $\sigma \in P(M)$ such that for all $\pi \geq \sigma$ we have $V_\pi h = h\rho_M$.
(b) $V_\pi : L_\infty(\mu, E) \to L_\infty(\mu, E)$ is a continuous linear map with $\|V_\pi\| \leq 1$.
(c) If $h(X)$ is relatively compact where $h \in L_\infty(\mu, E)$, then we have $\lim_{\pi\to\infty} \|V_\pi h - h\rho_M\|_\infty = 0$, that is for every $\varepsilon > 0$, there is $\sigma \in P(M)$ such that for all $\pi \geq \sigma$ we have $\|V_\pi h - h\rho_M\|_\infty \leq \varepsilon$.

Proof. (a) Let $h = \sum_{j=1}^k \alpha_j \rho_{D_j}$ where D_j's are disjoint measurable sets and $\alpha_j \in E$. Let $\alpha_0 = 0 \in E$; $D_0 = X \setminus \bigcup_{j=1}^k D_j$ and $B_j = M \cap D_j$. Then $h = \sum_{j=0}^k \alpha_j \rho_{D_j}$ and $\sigma = \{B_j : 0 \leq j \leq k\} \in P(M)$. Take any $\pi \geq \sigma$ in $P(M)$. Write $B_j = \bigcup_i A_{ij}$ where $A_{1j}, A_{2j}, \cdots$ are disjoint sets in π. Then we have

$$V_\pi h = \sum_{A\in\pi} \frac{\int_A h d\mu}{\mu A} \rho_A = \sum_j \sum_i \frac{\int_{A_{ij}} h d\mu}{\mu A_{ij}} \rho_{A_{ij}} = \sum_j \sum_i \frac{\alpha_j \mu A_{ij}}{\mu A_{ij}} \rho_{A_{ij}}$$
$$= \textstyle\sum_j \alpha_j \sum_i \rho_{A_{ij}} = \sum_j \alpha_j \rho_{B_j} = h\rho_M.$$

(b) Observe that h is μ-integrable on every $A \in \pi$ by $|h\rho_A| \leq \|h\|_\infty \rho_M$, μ-ae. Clearly V_π is linear. Finally for each $x \in A$, we have

$$\|(V_\pi h)(x)\| = \left\| \frac{\int_A h d\mu}{\mu A} \right\| \leq \frac{\int_A |h|\, d\mu}{\mu A} \leq \frac{\int_A \|h\|_\infty d\mu}{\mu A} \leq \|h\|_\infty,$$

that is $\|V_\pi h\|_\infty \leq \|h\|_\infty$, or $\|V_\pi\| \leq 1$.
(c) Since h has relatively compact range, for every $\varepsilon > 0$ there is a simple map $g : X \to E$ such that $\|g(x) - h(x)\| \leq \varepsilon$ for all $x \in X$. There is $\sigma \in P(M)$ such that for all $\pi \geq \sigma$ we have $V_\pi g = g$. Consequently,

$$\|V_\pi h - h\|_\infty \leq \|V_\pi\|\ \|h - g\|_\infty + \|V_\pi g - g\|_\infty + \|g - h\|_\infty \leq 2\varepsilon. \qquad \square$$

24-7.7. Let μ be a scalar measure on a δ-space $(X, \mathbb{D})$ and $\overline{\mathbb{B}}$ the closed unit ball of $L_1(\mu)$. A continuous linear map $T : L_1(\mu) \to E$ is *weakly compact* if every sequence in $T\left(\overline{\mathbb{B}}\right)$ has a weakly convergent subsequence.

24-7.8. **Theorem** Let μ is a σ-finite scalar measure on X. If $T : L_1(\mu) \to E$ is a weakly compact linear map with separable range, then there is $h \in L_\infty(\mu, E)$ such that for all $f \in L_1(\mu)$ we have $T(f) = \int fh d\mu$ and that $h(X)$ is contained in the closure of $T\left(\overline{\mathbb{B}}\right)$.

Proof. By Polar Form of Complex Measures, we may assume that $\mu \geq 0$. Since X is σ-finite, write $X = \bigcup_{k=0}^\infty M_k$ as a disjoint union of a null set M_0 and decent sets $M_1, M_2, \cdots$. Let E_1 be the closure of the range of T. Then E_1 is separable and contains the weak limit of every sequence in range of T. Hence we may assume $E = E_1$. Choose a sequence $w_n \in E'$ by §7-6.14 such that $\|w_n\| = 1$ and $\|\beta\| = \sup_{n\geq 1} |w_n\beta|$ for every $\beta \in E$. Now each function $w_nT : L_1(\mu) \to \mathbb{K}$ is a continuous linear form with σ-finite measure μ. There is $\varphi_n^* \in L_\infty(\mu)$ such that $\|\varphi_n^*\|_\infty = \|w_nT\|$ and $w_nT(f) = \int f\varphi_n^* d\mu$ for all $f \in L_1(\mu)$. For each $k \geq 1$, $M = M_k$ is a decent set. Now $\varphi_n = \varphi_n^*\rho_M \in L_\infty(\mu)$, $\varphi_n(X \setminus M) = 0$ and $w_nT(f\rho_M) = \int f\varphi_n d\mu$ for all $f \in L_1(\mu)$. Without loss of generality, we may assume $|\varphi_n| \leq \|\varphi_n\|_\infty$ everywhere on X. Next define $g_\pi : X \to E$ by $g_\pi = \sum_{A\in\pi} \frac{T\rho_A}{\mu A}\rho_A$. From $w_ng_\pi = \sum_{A\in\pi} \frac{w_nT\rho_A}{\mu A}\rho_A = \sum_{A\in\pi} \frac{\int \rho_A\varphi_n d\mu}{\mu A}\rho_A = V_\pi\varphi_n$, we obtain $\lim_{\pi\to\infty} \|w_ng_\pi - \varphi_n\|_\infty = \lim_{\pi\to\infty} \|V_\pi\varphi_n - \varphi_n\|_\infty = 0$ because φ_n has relatively compact range in $\mathbb{K}$. Hence we can choose a sequence $\pi_m \in P(M)$ satisfying $\lim_{m\to\infty} \|w_ng_{\pi_m} - \varphi_n\|_\infty = 0$. There is a μ-null set N_n such that $\lim_{m\to\infty} w_ng_{\pi_m} = \varphi_n$ uniformly on $X \setminus N_n$. The set $N = \bigcup_{n=1}^\infty N_n$ is μ-null and for each n, $\lim_{m\to\infty} w_ng_{\pi_m} = \varphi_n$ uniformly on $X \setminus N$. Take any $x \in M$. Then we have $x \in A$ for some $A \in \pi$. If $\mu A > 0$, then $\|\rho_A/\mu A\|_1 = 1$ and $g_\pi(x) = T\left(\rho_A/\mu A\right) \in T\left(\mathbb{B}\right)$. If $\mu A = 0$, then $g_\pi(x) = 0 \in T\left(\mathbb{B}\right)$. Since T is weakly compact, there is a weakly convergent subsequence $g_{\pi_{m(j)}}(x) \to h_k(x)$ as $j \to \infty$. Hence $w_ng_{\pi_{m(j)}}(x) \to w_nh_k(x)$ and also $w_ng_{\pi_{m(j)}}(x) \to \varphi_n(x)$, that is $w_nh_k(x) = \varphi_n(x)$. Since the closure Q of the convex set $T\left(\mathbb{B}\right)$ contains all $g_{\pi_{m(j)}}(x) \in T\left(\mathbb{B}\right)$, Q also contains the weak limit $h_k(x)$. Now define $h_k(X \setminus M) = 0$. Then we have $h_k(X) \subset Q$ and $w_nh_k = \varphi_n$ is measurable for each n. Now h_k is strongly measurable for $h_k(X) \subset E$ is separable. Thus $h = \sum_{k=1}^\infty h_k$ is a measurable map from X into E satisfying $h(X) \subset Q$, i.e. $h \in L_\infty(\mu, E)$. Finally take any $f \in L_1(\mu)$. From $w_nT(f\rho_{M_k}) = \int f\varphi_n d\mu = \int fw_nh_k d\mu = w_n \int fh_k d\mu$, we have $T(f\rho_{M_k}) = \int fh_k d\mu$ for all $k \geq 1$. By Dominated Convergence Theorem, we have $f = \sum_{k=1}^\infty f\rho_{M_k}$ in $L_1(\mu)$. Therefore we obtain $Tf = \sum_{k=1}^\infty T(f\rho_{M_k}) = \sum_{k=1}^\infty \int fh_k d\mu = \int \sum_{k=1}^\infty (fh_k) d\mu = \int fh d\mu$. □

24-7.9. **Theorem** Let μ is a σ-finite scalar measure on X. If $T : L_1(\mu) \to E$ is a compact linear map, then there is $h \in L_\infty(\mu, E)$ such that for all $f \in L_1(\mu)$ we have $T(f) = \int fh d\mu$ and that $h(X)$ is relatively compact in E.

Proof. Since every compact linear map is weakly compact and with separable range, let $h \in L_\infty(\mu, E)$ be obtained by last theorem. Since the closure Q of $T\left(\mathbb{B}\right)$ is compact, $h(X) \subset Q$ is relatively compact. □

24-7.10. **Theorem** Let μ is a scalar measure on a δ-space X. If a measurable map $h : X \to E$ has relatively compact range, then the linear map $T : L_1(\mu) \to E$ given by $Tf = \int fhd\mu$ is compact.

Proof. Note that $Tf = \int fhd\mu$ is well-defined because $h \in L_\infty(\mu, E)$. The closed convex balanced hull Q of the relatively compact set $h(X)$ is compact. Thus $Tf = \int fhd\mu \in \|f\|_1 Q \subset Q$ for all $\|f\|_1 \leq 1$ in $L_1(\mu)$. Therefore T is a compact linear map. □

24-7.11. **Exercise** Prove the last theorem by uniform approximation of h with simple maps which provide finite dimensional approximations of T.

24-7.12. **Lemma** Let μ be a bounded positive measure on a δ-space $(X, \mathbb{D})$. For every $h \in L_\infty(\mu, E)$, the set $\{\int_A hd\mu : A \in \mathbb{D}\}$ is precompact in E.

Proof. Let $\varepsilon > 0$ be given. Since h is measurable, let $g_n \to h$ for some simple maps $g_n : X \to E$. By Egorov's Theorem, there is a measurable set P such that $\mu P \leq \varepsilon/(1 + \|h\|_\infty)$ and $g_n \to h$ uniformly on $Q = X \setminus P$. By §19-6.4, the set $(h\rho_Q)(X)$ is relatively compact in E. By §11-6.6, its closed convex balanced hull K is compact. For every $\varepsilon > 0$, write $\mu(X)K \subset \bigcup_{j=1}^k \mathbb{B}(\alpha_j, \varepsilon)$ where $\alpha_j \in E$. Now take any $A \in \mathbb{D}$. Thus we have $\int_A h\rho_Q d\mu \in \|\rho_A\|_1 K \subset \mu(X)K$, that is, $\| \int_A h\rho_Q d\mu - \alpha_j\| \leq \varepsilon$ for some j. Hence we obtain

$$\| \int_A hd\mu - \alpha_j\| \leq \| \int_A h\rho_P d\mu\| + \| \int_A h\rho_Q d\mu - \alpha_j\| \leq \|h\|_\infty \mu P + \varepsilon \leq 2\varepsilon.$$

Therefore the given set can be covered by a finite number of (2ε)-balls. Consequently it is precompact. □

24-7.13. **Theorem** Every reflexive space E has Radon-Nikodym property.

Proof. Let ν be a vector measure on a δ-space $(X, \mathbb{D})$ into E, $\mu = |\nu|$ and M a decent subset of X. It suffices to find a μ-integrable map $h : M \to E$ such that $\nu A = \int_A hd\mu$ for all decent subset A of M because h can be extended over X by defining $h(X \setminus M) = 0$. Therefore without loss of generality, we may assume that $X = M$ and that μ is a bounded positive measure. For each decent function $f = \sum_{j=1}^k \alpha_j \rho_{D_j}$ where $\alpha_j \in \mathbb{K}$ and D_j's are disjoint decent sets, let $Tf = \sum_{j=1}^k \alpha_j \nu D_j$. From $\|Tf\| \leq \sum_{j=1}^k |\alpha_j|\ |\nu| D_j \leq \sum_{j=1}^k |\alpha_j| \mu D_j = \|f\|_1$, the map T is continuous linear on the decent functions into E. By Density Theorem, it has a continuous linear extension over $L_1(\mu)$ which is also denoted by T for convenience. We claim that the set $H = \{T\rho_A : A \in \mathbb{D}\}$ is precompact in E, that is, every sequence has a Cauchy subsequence. Let $\{A_n \in \mathbb{D} : n \geq 1\}$ be given and $\mathcal{F}$ the δ-ring generated by $\{A_n : n \geq 1\}$. Clearly the restriction of

μ on $\mathcal{F} \subset \mathbb{D}$ is a measure. By §21-4.7, the set $L_1(\mathcal{F}, \mu)$ of integrable functions on $(X, \mathcal{F})$ is a separable subset of $L_1(\mathbb{D}, \mu)$ and hence $T[L_1(\mathcal{F}, \mu)]$ is separable in E. Let $\overline{\mathbb{B}}$ be the closed unit ball of $L_1(\mathcal{F}, \mu)$. Since $T\left(\overline{\mathbb{B}}\right)$ is bounded in the reflexive space E, the map T is weakly compact linear. There is $g \in L_\infty(\mathcal{F}, \mu)$ such that $T(f) = \int fgd\mu$ for all $f \in L_1(\mathcal{F}, \mu)$. By last lemma, the image $\{T\rho_{A_n} : n \geq 1\}$ is precompact in E. Thus it has a Cauchy subsequence. Therefore H is precompact and hence also separable in E. Let $\mathbb{K}_0$ be a countable dense subset of the scalar field $\mathbb{K}$. Then the set of all finite sums of $\{k\rho_A : k \in \mathbb{K}_0, A \in \mathbb{D}\}$ is dense in $L_1(\mu, E)$. Thus the range of T is separable because the set of all finite sums of $\{kT\rho_A : k \in \mathbb{K}_0, A \in \mathbb{D}\}$ is countable dense. There is $h \in L_\infty(\mu, E)$ such that $T(f) = \int fhd\mu$ for all $f \in L_1(\mu, E)$. In particular for $f = \rho_A$ with $A \in \mathbb{D}$, we have $\nu A = \int_A hd\mu$. This completes the proof. □

24-7.14. **Corollary** Let μ be a scalar measure on a δ-space $(X, \mathbb{D})$ and let $1 < p < \infty$. If E is a reflexive Banach space, then $L_p(\mu, E)$ is also reflexive.

24-99. **References and Further Readings** : Ricker, Ballve, Huff, Gilliam, Croitoru, Zhao, Fernandez, Sambucini and Doss.

Chapter 25

Differentiation of Measures

25-1 Geometrical Expression of Radon-Nikodym Derivatives

25-1.1. In this section, we shall prove that except a null set, every complex measure on $\mathbb{R}^n$ has a derivative in terms of the limit of difference quotient. Furthermore, it coincides with its density. We shall adopt appropriate rather than maximum generality.

25-1.2. Let λ denote the Lebesgue measure and μ a *positive* measure on $\mathbb{R}^n$. We shall use the max-norm on $\mathbb{R}^n_\infty$ and the notation of cubes defined in chapter 23. The radius of a cube A will be denoted by $\triangle A$. Take any $x \in \mathbb{R}^n$. The families of all semi-cubes, open and closed semi-cubes containing x in their interiors and with radii $\leq r$ are denoted by $S(x,r), S^o(x,r), S^-(x,r)$ respectively. The *upper* and *lower derivatives* of μ at x are defined by

$$D^*\mu(x) = \lim_{r\to 0} \sup_{A\in S(x,r)} \frac{\mu A}{\lambda A} \qquad \text{and} \qquad D_*\mu(x) = \lim_{r\to 0} \inf_{A\in S(x,r)} \frac{\mu A}{\lambda A}$$

respectively. Clearly, $0 \leq D_*\mu(x) \leq D^*\mu(x) \leq \infty$. A complex measure ν on $\mathbb{R}^n$ is said to be *differentiable* at x if there is a number $D\nu(x)$ such that

$$\lim_{r\to 0} \sup_{A\in S(x,r)} \left| \frac{\nu A}{\lambda A} - D\nu(x) \right| = 0.$$

In this case, $D\nu(x)$ is called the *derivative* of μ at x.

25-1.3. **Exercise** Prove that a positive measure μ on $\mathbb{R}^n$ is differentiable at x iff $D^*\mu(x) = D_*\mu(x) < \infty$. In this case, we have $D^*\mu(x) = D_*\mu(x) = D\mu(x)$.

25-1.4. **Example** Let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by $\varphi(x) = x$ if $x > 0$ and $\varphi(x) = 0$ otherwise. Let $\mu(a,b] = \varphi(b) - \varphi(a)$ for all $a < b$. Then μ is a positive measure on $\mathbb{R}$. Considering the intervals $(-t, t^2]$ and $(-t^2, t]$ for $t \downarrow 0$, it is easy to show that $D^*\mu(x) \neq D_*\mu(x)$.

25-1.5. **Exercise** Let μ be the measure on $\mathbb{R}$ induced by $\varphi(x) = x^2$. Find both $D^*\mu(1)$ and $D_*\mu(1)$.

25-1.6. **Lemma** The following formulas allow us to work with open or closed semi-cubes: $D^*\mu(x) = \lim_{r\to 0} \sup_{C\in S^-(x,r)} \frac{\mu C}{\lambda C} = \lim_{r\to 0} \sup_{B\in S^o(x,r)} \frac{\mu B}{\lambda B}$

and $D_*\mu(x) = \lim_{r\to 0} \inf_{C\in S^-(x,r)} \frac{\mu C}{\lambda C} = \lim_{r\to 0} \inf_{B\in S^o(x,r)} \frac{\mu B}{\lambda B}$.

Proof. Let A, B, C be semi-cube, open and closed cubes of the same center and radius respectively. Note that $\lambda A = \lambda B = \lambda C$. Since μ is positive, we have $\frac{\mu B}{\lambda B} \le \frac{\mu A}{\lambda A} \le \frac{\mu C}{\lambda C}$ from which we obtain

$$\lim_{r\to 0} \sup_{B\in S^o(x,r)} \frac{\mu B}{\lambda B} \le D^*\mu(x) \le \lim_{r\to 0} \sup_{C\in S^-(x,r)} \frac{\mu C}{\lambda C}.$$

On the other hand, for any $\delta > 0$ take any closed cube C containing x in its interior C^o and with radius $\triangle C \le \delta$. For each $m \ge 1$, let B_m be an open cube of the same center of C but with radius $\triangle B_m = \triangle C + \delta/m$. Then $B_m \downarrow C$. Hence $\frac{\mu B_m}{\lambda B_m} \to \frac{\mu C}{\lambda C}$ as $m \to \infty$. For every $\varepsilon > 0$, there is m such that

$$\frac{\mu C}{\lambda C} \le \frac{\mu B_m}{\lambda B_m} + \varepsilon \le \sup_{B\in S^o(x,2\delta)} \frac{\mu B}{\lambda B} + \varepsilon,$$

hence, $$\sup_{C\in S^-(x,r)} \frac{\mu C}{\lambda C} \le \sup_{B\in S^o(x,2\delta)} \frac{\mu B}{\lambda B} + \varepsilon.$$

Letting $\delta \to 0$ first and then $\varepsilon \to 0$, we have

$$\lim_{r\to 0} \sup_{C\in S^-(x,r)} \frac{\mu C}{\lambda C} \le \lim_{r\to 0} \sup_{B\in S^o(x,r)} \frac{\mu B}{\lambda B}.$$

This proves the first formula. The second is left as an exercise. □

25-1.7. **Lemma** Both $D^*\mu$ and $D_*\mu$ are measurable.

Proof. We want to show that $M = \{x \in X : D^*\mu(x) \ge t\}$ is measurable for every $t \in \mathbb{R}$. Now let Q_{mk} be the family of open cubes B satisfying $\frac{\mu B}{\lambda B} > t - \frac{1}{k}$ and with radius $\triangle B \le 1/m$. Obviously $\lim_{r\to 0} \sup_{B\in S^o(x,r)} \frac{\mu B}{\lambda B} \ge t$ iff $x \in \bigcap_{m,k=1}^{\infty} \cup Q_{mk}$. As a countable intersection of open sets, M is measurable. Therefore $D^*\mu$ is measurable. Likewise, $D_*\mu$ is also measurable. □

25-1.8. In order to show that $D\mu(x)$ exists almost everywhere, we need the following tool. A family $\mathcal{F}$ of cubes is called a *Vitali's cover* of a set M if for every $x \in M$ and any $\delta > 0$ there is $A \in \mathcal{F}$ such that $x \in A$ and $\triangle A \le \delta$.

25-1.9. **Vitali's Covering Theorem** For every Vitali's cover $\mathcal{F}$ of a bounded set $M \subset \mathbb{R}^n$, there is a sequence of disjoint cubes $C_j \in \mathcal{F}$ such that

(a) $\lambda(M \setminus \cup_j C_j) = 0$ and
(b) $M \subset \bigcup_{j=1}^k A_j \cup \bigcup_{j>k} B_j$ where A_j is the closure of C_j and B_j is the closed cube of the same center of A_j with radius $\triangle B_j = 5 \triangle A_j$.
Note that the sequence may be finite or infinite.

Proof. Let V be a bounded open set containing M. Let $\mathcal{G}$ be the family of closures of cubes $C \in \mathcal{F}$ contained in V. Suppose that there is a sequence $A_1, A_2, \cdots$ of disjoint sets in $\mathcal{G}$ such that $M \subset \bigcup_{j=1}^k A_j$. Then (b) is satisfied and (a) follows from $\lambda(M \setminus \cup_j C_j) \leq \lambda(M \setminus \cup_j A_j) + \sum_j(\lambda A_j - \lambda C_j) = 0$. Hence assume that M cannot be covered by any finite sequence of disjoint sets in $\mathcal{G}$. Clearly $\mathcal{G}$ is a Vitali's cover of M and also $0 < r_1 = \sup\{\triangle D : D \in \mathbb{G}\} < \infty$. There is $A_1 \in \mathcal{G}$ such that $\triangle A_1 \geq r_1/2$. Suppose that r_j, A_j are constructed inductively for all $j \leq k$. Since $M \not\subset \bigcup_{j=1}^k A_j$, there is some $x \in M \setminus \bigcup_{j=1}^k A_j$. Since $\bigcup_{j=1}^k A_j$ is closed and $\mathcal{G}$ is a Vitali's cover, there is $C \in \mathcal{G}$ with $x \in C$ and $C \cap \bigcup_{j=1}^k A_j = \emptyset$. Hence we get

$$0 < r_{k+1} = \sup \{ \triangle D : D \in \mathcal{G} \text{ and } D \cap \textstyle\bigcup_{j=1}^k A_j = \emptyset \} < \infty.$$

There is $A_{k+1} \in \mathcal{G}$ such that $\triangle A_{k+1} \geq r_{k+1}/2$ and $A_{k+1} \cap \bigcup_{j=1}^k A_j = \emptyset$. Now two infinite sequences r_k, A_k have been constructed. For each j, let B_j be the closed cube with the same center as A_j and with radius $\triangle B_j = 5 \triangle A_j$. Since $\{A_j\}$ are disjoint subsets of the bounded open set V, we have $\sum_{j=1}^\infty \lambda A_j \leq \lambda V < \infty$, or $0 < r_j \leq 2 \triangle A_j = (\lambda A_j)^{1/n} \to 0$ as $j \to \infty$. We claim $M \setminus \bigcup_{j=1}^k A_j \subset \bigcup_{j=k+1}^\infty B_j$ for all $k > 1$. In fact, for each $x \in M \not\subset \bigcup_{j=1}^k A_j$, choose C as above. Since $r_j \to 0$, there is p such that $r_{p+1} < \triangle C$. Then by the choice of r_{p+1}, we have $C \cap \bigcup_{j=1}^p A_j \neq \emptyset$. Let p be the minimum one. Then we get $C \cap \bigcup_{j=1}^{p-1} A_j = \emptyset$. Hence there is some $b \in C \cap A_p$. Clearly, $\triangle C \leq r_p \leq 2 \triangle A_p$. Let a, c be the centers of A_p, C respectively. Then, $\|x-a\| \leq \|x-c\| + \|c-b\| + \|b-a\| \leq \triangle C + \triangle C + \triangle A_p \leq 5 \triangle A_p = \triangle B_p$, i.e. $x \in B_p$. Because $C \cap \bigcup_{j=1}^k A_j = \emptyset$, we have $p > k$. Therefore $x \in \bigcup_{j=k+1}^\infty B_j$. Consequently (b) follows. The proof is completed by the observation

$$\begin{aligned}\lambda\left(M \setminus \bigcup_{j=1}^\infty C_j\right) &\leq \lambda\left(M \setminus \bigcup_{j=1}^k A_j\right) + \sum_{j=1}^k (\lambda A_j - \lambda C_j) \\ &\leq \lambda\left(\bigcup_{j=k+1}^\infty B_j\right) + 0 \leq \sum_{j=k+1}^\infty \lambda B_j \leq 5^n \sum_{j=k+1}^\infty \lambda A_j \to 0. \qquad \square\end{aligned}$$

25-1.10. **Lemma** Let M be a decent set in $\mathbb{R}^n$ and let $t > 0$.
(a) If $D^*\mu(x) \geq t$ for all $x \in M$, then $t\lambda M \leq \mu M$.
(b) Suppose $\mu \ll \lambda$. If $D_*\mu(x) \leq t$ for all $x \in M$, then $t\lambda M \geq \mu M$.

Proof. Without loss of generality, we may assume $M \neq \emptyset$. Take any bounded open set V containing M.

(a) For any $0 < \varepsilon < t$, let $\mathcal{F}$ be the family of those cubes $C \subset V$ satisfying $(t-\varepsilon)\lambda C \leq \mu C$. Since $D^*\mu(x) \geq t$ for all $x \in M$, $\mathcal{F}$ is a Vitali's cover of M. There is a sequence $\{C_j\}$ of disjoint cubes in $\mathcal{F}$ with $\lambda(M \setminus \cup_j C_j) = 0$. Observe

$$(t-\varepsilon)\lambda M \leq (t-\varepsilon)\lambda(M \cap \cup_j C_j) + (t-\varepsilon)\lambda(M \setminus \cup_j C_j)$$
$$= \sum_j (t-\varepsilon)\lambda C_j + 0 \leq \sum_j \mu C_j \leq \mu(\cup_j C_j) \leq \mu V.$$

Since every positive measure is regular, taking infimum over all open covers V of M we have $(t-\varepsilon)\lambda M \leq \mu M$. Since $\varepsilon > 0$ is arbitrary, we prove (a).

(b) As in (a), for any $\varepsilon > 0$, let $\mathcal{F}$ be the family of those cubes $C \subset V$ satisfying $(t+\varepsilon)\lambda C \geq \mu C$. Since $D_*\mu(x) \leq t$ for all $x \in M$, $\mathcal{F}$ is a Vitali's cover of M. There is a sequence $\{C_j\}$ of disjoint cubes in $\mathcal{F}$ with $\lambda(M \setminus \cup_j C_j) = 0$. Since $\mu \ll \lambda$, we obtain $\mu(M \setminus \cup_j C_j) = 0$. Therefore

$$(t+\varepsilon)\lambda M = (t+\varepsilon)\lambda(M \cap \cup_j C_j) + (t+\varepsilon)\lambda(M \setminus \cup_j C_j)$$
$$\geq \sum_j (t+\varepsilon)\lambda C_j + 0 \geq \mu(M \cap \cup_j C_j) + \mu(M \setminus \cup_j C_j) = \mu M.$$

Letting $\varepsilon \downarrow 0$, the proof is completed. □

25-1.11. **Exercise** Prove that if two measures ν_1, ν_2 are differentiable at x, then for all $t_1, t_2 \in \mathbb{K}$, the measure $t_1\nu_1 + t_2\nu_2$ is differentiable at x and its derivative is given by $D(t_1\nu_1 + t_2\nu_2)(x) = t_1 D\nu_1(x) + t_2 D\nu_2(x)$.

25-1.12. **Theorem** (a) Every complex measure ν on $\mathbb{R}^n$ is differentiable λ-ae.
(b) If $\nu \perp \lambda$ then $D\nu = 0, \lambda$-ae.
(c) If $\nu \ll \lambda$ then $D\nu = d\nu/d\lambda$, λ-ae, the μ-density of ν.

Proof. As a result of last exercise, it suffice to prove the theorem for a positive measure μ which is one of $(\text{Re }\nu)_\pm, (\text{Im }\nu)_\pm$. Choose any semi-interval A. Consider the case $\mu \perp \lambda$. Then μ is null on some measurable set M while λ is null on its complement. Observe that

$$\lambda\{x \in A \cap M : D^*\mu(x) \geq 1/j\}$$
$$\leq j\mu\{x \in A \cap M : D^*\mu(x) \geq 1/j\} \leq j\mu(A \cap M) = 0$$

and $\lambda(A \setminus M) = 0$; hence $\lambda\{x \in A : D^*\mu(x) \geq 1/j\} = 0$. Since j is arbitrary, $\lambda\{x \in A : D^*\mu(x) > 0\} = 0$. Therefore we have $0 \leq D_*\mu \leq D^*\mu = 0, \lambda$-ae. Consequently, μ is differentiable λ-ae and $D\mu = 0, \lambda$-ae. Next, consider the case $\mu \ll \lambda$. Let $f = d\mu/d\lambda : \mathbb{R} \to [0, \infty)$ be the λ-density of μ. For all rationals $t < s$, let $M_{st} = \{x \in \mathbb{R}^n : f(x) < t < s < D^*\mu(x)\}$. Then

$$s\lambda(A \cap M_{st}) \leq \mu(A \cap M_{st}) = \int_{A \cap M_{st}} f d\lambda \leq \int_{A \cap M_{st}} t d\lambda = t\lambda(A \cap M_{st})$$

gives $\lambda(A \cap M_{st}) = 0$. Since A is arbitrary, M_{st} is λ-null. Hence the countable union $M = \cup_{st} M_{st} = \{x \in \mathbb{R}^n : f(x) < D^*\mu(x)\}$ is also λ-null, that is $D^*\mu \le f$, λ-ae. Similarly, $f \le D_*\mu$, λ-ae. Consequently, μ is differentiable λ-ae and moreover, $D\mu = f$, λ-ae. Finally, since every measure is a sum of its absolutely continuous and singular parts, it is differentiable. □

25-1.13. **Corollary** For every locally integrable function f on $\mathbb{R}^n$, we have $f(x) = \lim_{r \to 0} \frac{1}{\lambda A} \int_A f(t)dt$ and $\lim_{r \to 0} \sup_{A \in S(x,r)} \frac{1}{\lambda A} \int_A |f(t) - f(x)|dt = 0$ for almost all x where A are cubes with center x and radius r.

Proof. For the measure $\nu A = \int_A f(t)dt$ for all decent sets A, we obtain $\lim_{r \to 0} \frac{1}{\lambda A} \int_A f(t)dt = D\nu = d\nu/d\lambda = f$, λ-ae. Next, let K be a dense subset of $\mathbb{C}$. For each $j \in K$, let $\nu_j A = \int_A |f(t) - j|dt$ for every decent set A. There is a λ-null set N_j such that $|f(x) - j| = D\nu_j(x)$ for all $x \notin N_j$. The set $N = \cup_{j \in K} N_j$ is λ-null. Let $\varepsilon > 0$ be given. For every $x \in \mathbb{C} \setminus N$, choose $j \in K$ with $|f(x) - j| \le \varepsilon$. There is $\delta > 0$ such that $\left| \frac{1}{\lambda A} \int_A |f(t) - j|dt - D\nu_j(x) \right| \le \varepsilon$ for all $\triangle A \le \delta$. Therefore $\frac{1}{\lambda A} \int_A |f(t) - j|dt \le |D\nu_j(x)| + \varepsilon = |f(x) - j| + \varepsilon \le 2\varepsilon$. This completes the proof. □

25-1.14. **Exercise** Investigate the possible extension to vector measures and maps from $\mathbb{R}^n$ into Banach spaces with suitable assumptions.

25-2 Jumps of Increasing Functions

25-2.1. In this section, we shall prove that every function of finite variation can be expressed as a sum of its continuous and discrete parts. Because every function G of finite variation can be decomposed into increasing functions: $G = V_+(\text{Re } G) - V_-(\text{Re } G) + iV_+(\text{Im } G) - iV_-(\text{Im } G)$, it suffices to work with increasing functions only.

25-2.2. Throughout this and next sections, let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function. Let $U(a) = \inf\{F(x) : x > a\}$ be the upper bound at $a \in \mathbb{R}$ and $L(a) = \sup\{F(x) : x < a\}$ the lower bound at $a \in \mathbb{R}$.

25-2.3. **Exercise** Prove that if $a < b$, then $L(a) \le F(a) \le U(a) \le L(b)$.

25-2.4. **Lemma** (a) L is left continuous and U is right continuous.

(b) $F(a) = U(a)$ iff F is right continuous at a.
(c) $F(a) = L(a)$ iff F is left continuous at a.
(d) $L(a) < U(a)$ iff F is discontinuous at a.

Proof. (a) Let $\varepsilon > 0$ be given. There is $b > a$ such that $F(b) \leq U(a) + \varepsilon$. For every x with $a < x < b$, we have $U(x) \leq F(b) \leq U(a) + \varepsilon$. Hence U is right continuous at a.

(b) For every $\varepsilon > 0$, there is $\delta > 0$ by (a) such that for all $a < x < a + \delta$, we have $U(x) \leq U(a) + \varepsilon$. Hence if $F(a) = U(a)$, then $F(a) = U(a) \leq F(x) \leq U(x) \leq U(a) + \varepsilon = F(a) + \varepsilon$ implies the right continuity of F at a. Conversely if F is right continuous at a, for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $a < x < a+\delta$, we have $F(x) \leq F(a)+\varepsilon$, that is $F(a) \leq U(a) \leq F(x) \leq F(a)+\varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $F(a) = U(a)$. It is an exercise to complete the proof.□

25-2.5. **Theorem** Functions of finite variation are measurable.

Proof. It suffice to work with an increasing function F on $\mathbb{R}$. If $a \in F^{-1}(t,\infty)$, then for every $x > a$ we have $t < F(a) \leq F(x)$, i.e. $x \in F^{-1}(t,\infty)$. Since $F^{-1}(t,\infty)$ is an open interval for every $t \in \mathbb{R}$, F is measurable. □

25-2.6. **Theorem** Let F be a function of finite variation. Then the set $\mathcal{E}$ of points where F is right discontinuous, is countable. Similar result holds for left and two-sided continuity. Therefore F is continuous almost everywhere.

Proof. It suffice to work with an increasing function F on $\mathbb{R}$. For each $a \in \mathcal{E}$, let t_a be a rational number in the open interval $(F(a), U(a))$. Then the map $\xi(a) = t_a$ for every $a \in \mathcal{E}$ is an injection from $\mathcal{E}$ into the set of all rationals. Consequently $\mathcal{E}$ is countable. □

25-2.7. A function $f : \mathbb{R} \to \mathbb{K}$ is called a *pulse function* if for every semi-interval $(a,b]$, the set $\{x \in (a,b] : f(x) \neq 0\}$ is countable and $\nu_{|f|}(a,b] = \sum\{|f(x)| : x \in (a,b]\} < \infty$. Let f be a pulse function and for all $a < b$, let $\nu_f(a,b] = \sum\{f(x) : x \in (a,b]\}$.

25-2.8. **Exercise** Prove that ν_f is a measure with $|\nu_f| = \nu_{|f|}$. The set of all pulse functions forms a conjugated vector space. The map $f \to \nu_f$ preserves linear combinations and complex conjugates.

25-2.9. **Exercise** Let $\{t_n\}$ be any enumeration of rational numbers on $\mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t_n) = (i/2)^n$ and $f(x) = 0$ for irrational x, is a pulse function where $i^2 = -1$.

25-2.10. **Lemma** The jump function $J = U - L$ of an increasing function $F : \mathbb{R} \to \mathbb{R}$ is a pulse function. Furthermore for every $a < b$, we have $\sum\{J(x) : x \in (a, b]\} \leq U(b) - F(a)$.

Proof. The set $\mathcal{E} = \{x \in (a, b] : J(x) \neq 0\}$ is countable. Take any finite subset $a < x_1 < x_2 < \cdots < x_n \leq b$ of $\mathcal{E}$. Then we have

$$\sum_{j=1}^{n} J(x_j) = \sum_{j=1}^{n} [U(x_j) - L(x_j)]$$
$$= U(x_n) - \sum_{j=2}^{n-1} [L(x_j) - U(x_{j-1})] - L(x_1) \leq U(b) - F(a). \qquad \square$$

25-2.11. A function $g : \mathbb{R} \to \mathbb{K}$ is *discrete* if there is a pulse function f such that for every $a \in \mathbb{R}$ we have

$$g(a) = \begin{cases} \sum\{f(x) : 0 < x \leq a\}, & \text{if } a > 0; \\ 0, & \text{if } a = 0; \\ -\sum\{f(x) : a < x \leq 0\}, & \text{if } a < 0. \end{cases}$$

Since g is the function induced by the measure ν_f, it is right continuous.

25-2.12. **Theorem** Every increasing function F is the difference of a right continuous increasing function and a positive pulse function. Hence every function of finite variation is the sum of a right continuous function of finite variation and a pulse function. Furthermore, this decomposition is unique.

Proof. Since $0 \leq U - F \leq U - L = J$, $F = U - (U - F)$ can be decomposed into a right continuous increasing function U and a positive pulse function $U - F$. The second statement follows from decomposition of function of finite variation into increasing functions. Finally, suppose $F = R - P$ where R is right continuous and P is a pulse function. Then $G = R - U = P - (U - F)$ is a right continuous pulse function. For all $a < b$, the set $\{x \in (a, b\ ] : G(x) \neq 0\}$ is countable. For every $c \in (a, b)$, there is a sequence $\{x_j\}$ in (a, b) such that $x_j \downarrow c$ and $G(x_j) = 0$ for every j. Since G is right continuous, $G(c) = \lim G(x_j) = 0$. Since $c \in (a, b)$ is arbitrary, $G = 0$ on $\mathbb{R}$, i.e. $R = U$ and $P = U - F$. Consequently, the decomposition is unique. $\square$

25-2.13. **Theorem** Every right continuous increasing function F is the sum of a continuous increasing function and an increasing discrete function. Hence every right continuous function of finite variation is the sum of a continuous function and a discrete function. Furthermore, this decomposition is unique.

Proof. Let $\mu(a, b] = F(b) - F(a)$ and $\nu(a, b] = \sum\{J(x) : x \in (a, b]\}$ for all $a < b$. From $\nu(a, b] \leq U(b) - F(a) = \mu(a, b]$ by right continuity, $\xi = \mu - \nu$ is a positive measure. The functions C, D induced by the positive measures ξ, ν respectively are increasing and right continuous. Clearly, $F(x) = F(0) + C(x) + D(x)$ for all $x \in \mathbb{R}$. For all $a < b$, we have $C(b) - C(a) = F(b) - F(a) - \nu(a, b]$. Since $L(a) \leq F(a) \leq L(b)$ and L is left continuous, letting $a \uparrow b$ we obtain

$$C(b) - C(a) = F(b) - F(a) - \sum\{J(x) : x \in (a, b]\} \to U(b) - L(b) - J(b) = 0.$$

Thus C is also left continuous. Therefore $F(0) + C$ is a continuous function. This proves the existence. To prove the uniqueness, it suffices to show that every continuous discrete function G is zero. Let g be a pulse function on $\mathbb{R}$ such that $G(b) - G(a) = \sum\{g(x) : x \in (a, b]\}$ for all $a < b$. Letting $a \uparrow b$, we have $0 = g(b)$ for all b. Therefore $G = 0$ by definition §25-2.11. □

25-2.14. **Exercise** Let $\{r_n\}$ be an enumeration of rationals on $\mathbb{R}$. Let $g_0(x) = h_0(x) = 0$ for all $x \in \mathbb{R}$. Inductively, define for each $x \in \mathbb{R}$,

$$g_n(x) = \begin{cases} g_{n-1}(x), & \text{if } x < r_n \text{ ;} \\ \frac{1}{n(n+1)} + g_{n-1}(x), & \text{if } x \geq r_n; \end{cases}$$

and

$$h_n(x) = \begin{cases} h_{n-1}(x), & \text{if } x \leq r_n \text{ ;} \\ \frac{1}{n(n+1)} + h_{n-1}(x), & \text{if } x > r_n. \end{cases}$$

Prove that g_n, h_n converge pointwise to strictly increasing bounded discontinuous functions g, h respectively. Investigate their left, right continuity. Can you write explicit formulas for the decomposition of last theorem?

25-3 Fundamental Theorems of Real Analysis

25-3.1. Application of general measure theory to classical analysis provides simplification to old material, justification of our approach, and possible stimulation to further research and development especially in vector measures and integration on infinite dimensional vector spaces. Topics such as absolute continuity, density, singular measures, Lebesgue decompositions will be covered in this section.

25-3.2. Every measure can be decomposed into absolutely continuous and singular parts with respect to the Lebesgue measure λ. We have proved that absolutely continuous measures induce absolutely continuous functions. A function g is said to be ***singular*** if its derivative exists and vanishes λ-ae. The following lemma shows that singular measures induce singular functions.

25-3.3. **Lemma** Let $F : \mathbb{R} \to \mathbb{R}$ be a right continuous increasing function and ν its induced measure on semi-intervals. Then F is differentiable at $c \in \mathbb{R}$ iff ν is differentiable at c. In this case, we have $F'(c) = D\nu(c)$.

Proof. Let F be differentiable at c. For every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{F(x)-F(c)}{x-c} - F'(c)\right| \le \varepsilon$$

for all $0 < |x-c| \le \delta$. Now for all $a < c < b$ with $b-a \le \delta$, we have

$$\left|\frac{\nu(a,b]}{\lambda(a,b]} - F'(c)\right| = \left|\frac{F(b)-F(a)}{b-a} - F'(c)\right|$$
$$\le \frac{1}{b-a}\left(\left|\frac{F(b)-F(c)}{b-c} - F'(c)\right|(b-c) + \left|\frac{F(c)-F(a)}{c-a} - F'(c)\right|(c-a)\right) \le \varepsilon.$$

Therefore the measure ν is differentiable at c and $D\nu(c) = F'(c)$. Conversely, suppose that ν is differentiable at c. For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{F(b)-F(a)}{b-a} - D\nu(c)\right| = \left|\frac{\nu(a,b]}{\lambda(a,b\,]} - D\nu(c)\right| \le \varepsilon$$

for all $a < c < b$ with $b - a \le \delta$. By right continuity of F, let $b \downarrow c$ to get

$$\left|\frac{F(c)-F(a)}{c-a} - D\nu(c)\right| \le \varepsilon.$$

Hence F is left differentiable and consequently *left* continuous at c. By symmetry of left and right sides, F is differentiable and $F'(c) = D\nu(c)$. □

25-3.4. **Theorem** Every function F of finite variation is differentiable almost everywhere and its derivative is locally integrable. Furthermore, if F is increasing, then for all $a < b$ we have $\int_{(a,b]} F'(x)dx \le L(b) - U(a)$.

Proof. Without loss of generality, we may assume that F is increasing. Since the measure induced by the right continuous function U is differentiable λ-ae, U is differentiable λ-ae. Next, suppose that U is differentiable at c. Then U is continuous at c and hence $U(c) = F(c)$. For every $\varepsilon > 0$, there is $\delta > 0$ such that $\left|\frac{U(x)-U(c)}{x-c} - U'(c)\right| \le \varepsilon$ for all $0 < |x-c| \le \delta$. Fix any such x. Since F is continuous λ-ae, there is a sequence $x_j \uparrow x$ such that $0 < |x_j - c| \le \delta$ and F is continuous at each x_j. From $U(x_j) = L(x_j)$, we get

$$U'(c) - \varepsilon \le \frac{U(x_j)-U(c)}{x_j-c} = \frac{L(x_j)-F(c)}{x_j-c}.$$

Since L is left continuous, we have as $j \to \infty$

$$U'(c) - \varepsilon \le \frac{L(x)-F(c)}{x-c} \le \frac{F(x)-F(c)}{x-c} \le \frac{U(x)-U(c)}{x-c} \le U'(c) + \varepsilon$$

by considering $x > c$ and $x < c$ separately. Therefore F is differentiable at c and $F'(c) = U'(c)$. It follows that F is differentiable λ-ae and $F' = U'$, λ-ae. Define $g_n(x) = n[F(x + 1/n) - F(x)]$ for all $x \in \mathbb{R}$ and all integers $n > 0$.

Since F is increasing, g_n is an upper function and so is $\liminf g_n = F', \lambda$-ae. By Fatou's lemma, we have $\int_{(a,b]} F'(x)dx \leq \liminf_{n\to\infty} \int_{(a,b]} g_n dx$ for every $a < b$. After simple change of variables, we obtain

$$\begin{aligned}\int_{(a,b]} g_n dx =& n\int_{(b,b+\frac{1}{n}]} F(x)dx - n\int_{(a,a+\frac{1}{n}]} F(x)dx \\ \leq& n\int_{(b,b+\frac{1}{n}]} U\left(b+\frac{1}{n}\right)dx - n\int_{(a,a+\frac{1}{n}]} U(a)dx \\ =& U\left(b+\frac{1}{n}\right) - U(a) \to U(b) - U(a)\end{aligned}$$

as $n \to \infty$. Thus we have

$$\int_{(a,b]} F'(x)dx \leq U(b) - U(a) < \infty.$$

Therefore F' is integrable on $(a, b]$. Since $a < b$ is arbitrary, it is locally integrable on $\mathbb{R}$. Finally take any $\beta \in (a, b)$. Note that the integral over the null set $\{b\}$ is zero. Letting $\beta \uparrow b$, we have

$$\begin{aligned}\int_{(a,b]} F'(x)dx = \int_{(a,b)} F'(x)dx = \sup_\beta \int_{(a,\beta]} F'(x)dx \\ \leq \sup_\beta[U(\beta) - U(a)] \leq L(b) - U(a).\end{aligned}$$ □

25-3.5. **Theorem** If $F : \mathbb{R} \to \mathbb{K}$ is an absolutely continuous function, then for all $a < b$ we have $F(b) - F(a) = \int_{(a,b]} F'(x)dx$.

Proof. Since the measure μ induced by F is λ-continuous, for all $a < b$ we have $F(b) - F(a) = \mu(a, b] = \int_{(a,b]} \frac{d\mu}{d\lambda} d\lambda = \int_{(a,b]} F' d\lambda$. □

25-3.6. **Theorem** If an absolutely continuous function $F : \mathbb{R} \to \mathbb{K}$ is also singular, then it is a constant function.

Proof. Since F is singular, $F'(x) = 0, \lambda$-ae. Last theorem ensures that for all $a < b$, $F(b) - F(a) = 0$. Therefore F is a constant function. □

25-3.7. **Theorem** Every continuous increasing function F is the sum of an absolutely continuous increasing function and a singular increasing continuous function. Furthermore, this decomposition is unique up to a constant.

Proof. Let μ be the positive measure induced by F. Lebesgue Decomposition Theorem ensures $\mu = \mu_a + \mu_s$ where μ_a, μ_s are positive measures which are absolutely continuous and singular to λ respectively. Let A, S be functions induced by μ_a, μ_s respectively. Then $F = A + S + F(0)$. Since both F, A are continuous, so is S. Finally, A, S are increasing because μ_a, μ_s are positive. The uniqueness follows immediately from last theorem. □

25-3.8. **Theorem** Every function G of finite variation can be decomposed uniquely into the form: $G = G(0) + P + D + A + S$ where P is a pulse function, D a discrete function, A an absolutely continuous function and S a singular continuous function satisfying $P(0) = D(0) = A(0) = S(0) = 0$. Furthermore, if G is real, then so are all P, D, A, S; and if G is an increasing real function, then so are all D, A, S.

25-3.9. **Theorem** Let f be a locally integrable function on $\mathbb{R}$ and for all $x \in \mathbb{R}$ let $F(x) = \displaystyle\int_\alpha^x f(t)dt$. Then F is absolutely continuous and $F' = f$, λ-ae.

Proof. Without loss of generality, we may assume that $f \geq 0$. Now $\mu(A) = \int_A f d\lambda$, for every decent set A, is an absolutely continuous positive measure on $\mathbb{R}$. It induces an absolutely continuous function G on $\mathbb{R}$. Then for all $a < x$, we have

$$F(x) - F(a) = \int_a^x f d\lambda = \mu(a, x] = G(x) - G(a) = \int_a^x G' d\lambda$$

that is $\displaystyle\int_{(a,x]} (f - G')d\lambda = 0$. Since $a < x$ are arbitrary, we have $f = G'$, λ-ae. It follows from $F(x) - F(a) = G(x) - G(a)$, F is absolutely continuous and $F' = G' = f$, λ-ae. □

25-3.10. **Exercise** Last theorem is a global version. The following is a locally version of which the proof is practically identical with those in elementary calculus. Let f be a locally integrable function on $\mathbb{R}$ and let $F(x) = \int_\alpha^x f(t)dt$ for all $x \in \mathbb{R}$. Prove that if f is continuous at $b \in \mathbb{R}$, then F is differentiable at b and $F'(b) = f(b)$. This provides an important method to evaluate integrals in terms of anti-derivatives.

25-3.11. **Exercise** Prove the following integration by parts: If f, g are absolutely continuous, then for all $a < b$ we have

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a).$$

25-3.12. **Theorem** Let $F : \mathbb{R} \to \mathbb{R}$ be a right continuous increasing function and μ its induced measure on the semi-intervals. Then F is a singular function iff μ is a singular measure.

Proof. Let F be a singular function. Write $\mu = \mu_a + \mu_s$ where μ_a, μ_s are positive measures which are absolutely continuous and singular to λ respectively. Let F_a, F_s be functions induced by μ_a, μ_s respectively. Then

$$F(x) = F(0) + F_a(x) + F_s(x)$$

for all $x \in \mathbb{R}$. Hence $F_a = F - F_s - F(0)$ is absolutely continuous and singular. Consequently it is a constant, say α. Then $F = F_s + \alpha + F(0)$ is singular. The converse is left as an exercise. □

25-3.13. All our functions were defined on the whole line. Suppose that G is a real function defined on $[a, b]$ only. Let $H : \mathbb{R} \to \mathbb{K}$ be given by

$$H(x) = \begin{cases} G(x), & \text{if } x \in [a, b], \\ G(a), & \text{if } x < a, \\ G(b), & \text{if } x > b. \end{cases}$$

Applying results to H, we get information about G.

25-3.14. **Exercise** Show that linear combinations and products of absolutely continuous functions are absolutely continuous. Show that every continuously differentiable function is absolutely continuous.

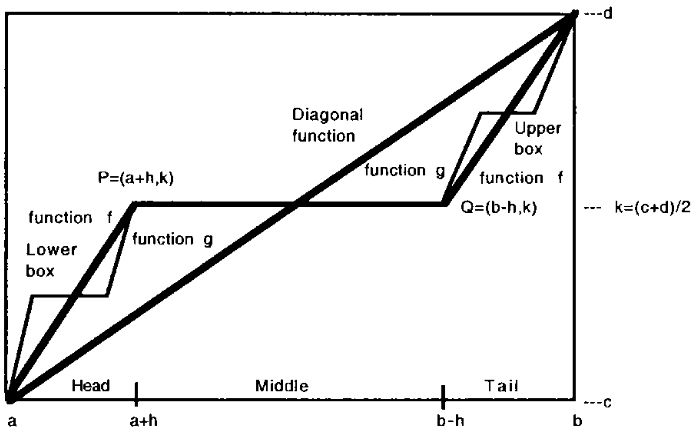
25-3.15. **Exercise** Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(0) = 0$ and $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ is differentiable but the derivative is not integrable. Show that it is continuous but not absolutely continuous.

25-4 Cantor Set and Function

25-4.1. Cantor set is an uncountable compact null set without any interior point. Cantor function is a continuous increasing function with zero-derivative almost everywhere. Explicit construction is given in this section.

25-4.2. Recall that every real number $x \in [0, 1]$ as a unique *infinite decimal* representation $0.x_1x_2x_3\cdots$ where the j-digit x_j is an integer between 0 and 9 inclusively. For example, 0.3 is rejected but $0.299\cdots$ is accepted. We prove in this section that the Cantor set in decimal form contains all numbers $x \in [0, 1]$ with every $x_j = 0$ or 9 only.

25-4.3. Consider the diagram in $\mathbb{R}^2$ below. Let $B = [a, b] \times [c, d]$ be a box. Define $h = 10\%(b - a)$, $k = (c + d)/2$, the head $H_B = [a, a + h]$, the tail $T_B = [b - h, b]$ and the middle $M_B = [a, b] \setminus (H_B \cup T_B)$. The diagonal of the box B defines a function $\triangle : [a, b] \to [c, d]$. Also the diagonal of the lower box $L_B = H_B \times [c, k]$; the upper box $U_B = T_B \times [k, d]$ together with the horizontal line from $P = (a + h, k)$ to $Q = (b - h, k)$ defines a function $f_B : [a, b] \to [c, d]$. Clearly, f_B is continuous increasing. Furthermore we have $|f_B(x) - \Delta(x)| \leq \frac{1}{2}(d - c)$ for all $x \in [a, b]$ and $f'_B(x) = 0$ for all $x \in M_B$. If we apply the same procedure to the lower and upper boxes, we get a new continuous increasing function g which is identical to f_B on the middle M_B by sharing PQ. Obviously, $|f_B(x) - g(x)| \leq \frac{1}{4}(d - c)$ for all $x \in [a, b]$. The family of lower and upper boxes is denoted by $\mathcal{F}_B = \{L_B, U_B\}$ and the family of head and tail by $\mathcal{K} = \{H_B, T_B\}$.



25-4.4. Apply the above procedure to the box $A = [0, 1] \times [0, 1]$. Then the head $K_0 = H_A = [0.00, 0.099\cdots]$ contains all numbers x with $x_1 = 0$ and $K_9 = T_A = [0.9, 0.99\cdots]$ all numbers x with $x_1 = 9$. Hence $F_1 = K_0 \cup K_9$ is a compact set containing all numbers x with $x_1 = 0$ or 9 only and has measure $\lambda F_1 = 2 * 10\% = 1/5$. The excluded middle $G_1 = M_A = [0, 1] \setminus F_1$ has measure $1 - 1/5$. The derivative of the continuous increasing function $f_1 = f_A$ vanishes on G_1. The family $\mathcal{F}_1 = \mathcal{F}_A$ contains 2 boxes $B_0 = L_A$ and $B_9 = U_A$ for next inductive step.

25-4.5. Applying the above procedure again to the box B_0, then the head $K_{00} = H_{B_0} = [0.000, 0.0099\cdots]$ contains all numbers x with $x_1x_2 = 00$ and $K_{09} = T_{B_0} = [0.09, 0.099\cdots]$ all numbers x with $x_1x_2 = 09$. To B_9, we have the head $K_{90} = H_{B_9} = [0.900, 0.9099\cdots]$ contains all numbers x with $x_1x_2 = 90$ and $K_{99} = T_{B_9} = [0.99, 0.999\cdots]$ all numbers x with $x_1x_2 = 99$. Hence $F_2 = K_{00} \cup K_{09} \cup K_{90} \cup K_{99}$ is a compact set containing all numbers x with

$x_1, x_2 = 0$ or 9 only and has measure $\lambda F_1 = 2^2 * (10\%)^2 = 1/5^2$. The excluded middle $G_2 = G_1 \cup M_{B_0} \cup M_{B_9} = [0,1] \setminus F_1$ has measure $1 - 1/5^2$. The derivative of the continuous increasing function f_2 obtained this way vanishes on G_2. Also $|f_2(x) - f_1(x)| \leq 1/2^2$ for all $x \in [0,1]$. The family $\mathcal{F}_2 = \cup\{\mathcal{F}_B : B \in \mathcal{F}_1\}$ contains 2^2 boxes for next inductive step.

25-4.6. In general, we construct a compact set F_n containing all numbers x with $x_1, x_2, \cdots, x_n = 0$ or 9 only and has measure $\lambda F_n = 1/5^n$. The excluded middle $G_n = [0,1] \setminus F_1$ has measure $1 - 1/5^n$. The derivative of the continuous increasing function f_n obtained this way vanishes on G_n. Also $|f_n(x) - f_{n-1}(x)| \leq 1/2^n$ for all $x \in [0,1]$. The *decimal Cantor set* $F = \bigcap_{n=1}^{\infty} F_n$ is *compact* and *contains exactly* all numbers x with every $x_j = 0$ or 9 only. From $\lambda F \leq \lambda F_n \leq 1/5^n$, F is a *null set* and hence cannot contain any *interior point*. Since $\{f_n\}$ is uniformly Cauchy on $[0,1]$, it converges uniformly to a *continuous* function f on $[0,1]$. It is called the *decimal Cantor function*. Since $f = f_n$ on G_n for all large n, $f'(x) = 0$ on $[0,1] \setminus F$, that is $f' = 0$ almost everywhere on $[0,1]$. Therefore f is a *singular* function.

25-4.7. The decimal Cantor set is uncountable. In fact, suppose to the contrary that $x^n = 0.x_1^n x_2^n x_3^n \cdots$ is an enumeration of all points in the decimal Cantor set where $x_j^n = 0$ or 9 only. Define $y = 0.y_1 y_2 y_3 \cdots$ where $y_n = 0$ if $x_n^n = 9$ and $y_n = 9$ if $x_n^n = 0$. Since every digit of y is either 0 or 9, y is a member of the decimal Cantor set. On the other hand, $y_n \neq x_n^n$. Hence $y \neq x^n$ of all n. This contradiction completes the proof.

25-4.8. The decimal Cantor function is *increasing*. In fact, from the diagram, all f_n are increasing. It is easy to show that $|f_{n+k}(x) - f_n(x)| \leq 1/2^n$ for all $k > 0$. Hence $|f(x) - f_n(x)| \leq 1/2^n$. Now suppose $0 \leq a < b \leq 1$. Then $f(a) \leq f_n(a) + 1/2^n \leq f_n(b) + 1/2^n \leq f(b) + 2/2^n$ for all n, that is $f(a) \leq f(b)$ as required. Finally, from $f(0) = 0$ and $f(1) = 1$, the *range* of f is $[0,1]$.

25-99. **References and Further Readings** : Rudin-74, Swartz-94, Brown-69, Freilich, Takacs, Zamfirescu, Cater and Varberg.

Chapter 26

Spectral Measures

26-1 Construction from Self-Adjoint Operators

26-1.1. Let A be a self-adjoint operator on a (complex) Hilbert space H. In §14-6, we have constructed a functional calculus for continuous functions on the spectrum $\sigma(A)$ of A. We want to extend it to wider classes of functions starting with characteristic functions of semi-intervals from which we construct a special spectral measures as an example for the general study in coming sections. Our approach is more intuitive and geometrical but longer because identification of the dual space of continuous functions with compact support as regular measures is not available until later chapters.

26-1.2. In this chapter, the universal set is denoted by X which is normally $\mathbb{R}^p$ essentially for $p = 1, 2$ only; members of X by s, t; $\mathcal{S}$ a semiring over X; $\mathbb{D}$ the δ-ring of decent sets J, K generated by $\mathcal{S}$; $\mathbb{M}$ the family of measurable sets M, N localized by $\mathbb{D}$; A, B operators on H; P, Q projectors on H; $x, y \in H$ and u, v, θ scalar coefficients. Hopefully this would resolve the notational conflicts from different parts of this book. Compare the following lemma with §20-1.4.

26-1.3. **Lemma** Let f_n, f, g be continuous *real* functions on $\mathbb{R}$.
(a) $f_n(A), f(A), g(A)$ are self-adjoint operators.
(b) If $f \leq g$, then $f(A) \leq g(A)$.
(c) If $f_n \downarrow f$ everywhere, then $f_n(A) \downarrow f(A)$ strongly as $n \to \infty$.

Proof. (a) follows from §14-6.12a and (b) from §14-6.12d. To prove (c), by Dini's theorem $f_n \downarrow f$ uniformly on the compact set $\sigma(A)$. It follows from §14-6.6 that $f_n(A) \to f(A)$ uniformly and hence strongly by §13-10.11. □

26-1.4. **Lemma** Let $f_n, g_n \geq 0$ be continuous functions on $\mathbb{R}$. Suppose that $f_n \downarrow f$ and $g_n \downarrow g$ on $\mathbb{R}$ but f, g need not be continuous.
(a) Both s-lim $f_n(A)$ and s-lim $g_n(A)$ exist.
(b) If $f \leq g$, then s-lim $f_n(A) \leq$ s-lim $g_n(A)$. See §20-1.5.
(c) If $f = g$, then s-lim $f_n(A) =$ s-lim $g_n(A)$.

Proof. Part (a) follows because bounded monotone sequence of self-adjoint operators is *strongly* convergent. To prove (b), observe that $f_m \vee g_n$ is a continuous function on $\mathbb{R}$. Letting $m \to \infty$ in $f_m \le f_m \vee g_n$, we obtain $f_m \vee g_n \downarrow f \vee g_n = g_n$. Hence $\text{s-}\lim_{m\to\infty} f_m(A) \le \text{s-}\lim_{m\to\infty} (f_m \vee g_n)(A) = g_n(A)$. Letting $n \to \infty$, we get (b). Part (c) follows by symmetry. □

26-1.5. Limits of decreasing sequences of continuous functions are called *upper semi-continuous* functions. Let f be an upper semi-continuous function and f_n be continuous functions with $f_n \downarrow f$. Then $f(A) = \text{s-}\lim_{m\to\infty} f_n(A)$ is well-defined independent of the choice of $\{f_n\}$. In particular, if f is continuous, the new definition agrees with the old one. For analogy to upper functions, see §20-1.6.

26-1.6. **Lemma** Let $f, g \ge 0$ be upper semi-continuous functions on $\mathbb{R}$.
(a) Monotonicity: If $f \le g$, then $f(A) \le g(A)$.
(b) Linearity: $(f+g)(A) = f(A) + g(A)$ and $(rf)(A) = r[f(A)]$ for every $r \ge 0$.
(c) Product: $(fg)(A) = f(A)g(A)$.
(d) Commutativity: If $AB = BA$, then $f(A)B = Bf(A)$ for every operator B.

Proof. Let f_n, g_n be continuous functions on $\mathbb{R}$ with $f_n \downarrow f$ and $g_n \downarrow g$.
(c) Since $f_n \ge f \ge 0$ and $g_n \ge g \ge 0$, we have $f_n g_n \downarrow fg$. Hence we have $(f_n g_n)(A) \downarrow (fg)(A)$. Also $(f_n g_n)(A) = f_n(A) g_n(A) \to f(A)g(A)$ strongly by §14-6.5b. Therefore $(fg)(A) = f(A)g(A)$.
(d) Since $f_n(A)B \to f(A)B$ and $Bf_n(A) \to Bf(A)$, the result follows from §14-6.7a immediately. □

26-1.7. **Theorem** Let f_n be upper semi-continuous functions. If $f_n \downarrow f$, then f is also an upper semi-continuous function with $f_n(A) \downarrow f(A)$ strongly. See §20-1.8.

Proof. For each n, there are continuous functions $g_{mn} \downarrow f_n$ as $m \to \infty$. Define $h_m = g_{m1} \wedge g_{m2} \wedge \cdots \wedge g_{mm}$. Then $h_m \ge h_{m+1}$ and $g_{mn} \ge h_m \ge f_m$ for all $n \le m$. For $m \to \infty$, we have $f_n \ge \lim h_m \ge f$. Next letting $n \to \infty$, we get $h_m \downarrow f$. Hence f is an upper semi-continuous function approximated by the continuous functions h_m. Since $\{f_m(A)\}$ is a decreasing sequence of self-adjoint operator bounded below by $f(A)$, the strong limit of $\{f_m(A)\}$ exists. For all $n \le m$, we get $g_{mn}(A) \ge h_m(A) \ge f_m(A)$. As $m \to \infty$, $f_n(A) \ge f(A) \ge \text{s-lim}\, f_m(A)$. Finally letting $n \to \infty$ we obtain $f(A) = \text{s-lim}\, f_n(A)$. □

26-1.8. A function f is *semi-continuous* if f is of the form $f = f_1 - f_2$ where f_1, f_2 are upper semi-continuous. Clearly the set of all semi-continuous functions forms an algebra over the reals. Define $f(A) = f_1(A) - f_2(A)$.

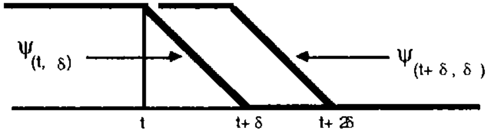
26-1.9. **Theorem** Let f, g be semi-continuous functions on $\mathbb{R}$.
(a) $f(A)$ is independent of the choice of the representation.
(b) Monotonicity: If $f \leq g$, then $f(A) \leq g(A)$.
(c) Linearity: $(f+g)(A) = f(A) + g(A)$ and $(rf)(A) = r[f(A)]$ for every $r \in \mathbb{R}$.
(d) Product: $(fg)(A) = f(A)g(A)$.
(e) Commutativity: If $AB = BA$, then $f(A)B = Bf(A)$ for every operator B.

Proof. Let $f = f_1 - f_2 \leq g_1 - g_2 = g$ where f_1, f_2, g_1, g_2 are upper semi-continuous functions. Then $f_1 + g_2 \leq g_1 + f_2$ gives $f_1(A) + g_2(A) \leq g_1(A) + f_2(A)$, that is $f_1(A) - f_2(A) \leq g_1(A) - g_2(A)$. For (a), letting $g = f$ we have $f_1(A) - f_2(A) = g_1(A) - g_2(A)$ and hence $f(A)$ is well-defined. We leave it as an exercise to complete the proof. □

26-1.10. For every $t \in \mathbb{R}$ and $\delta > 0$, the *spectral function* $\psi_{t,\delta}$ given by the following picture is continuous. More precisely,

$$\psi_{t,\delta}(s) = \begin{cases} 1 & \text{if } \ s \leq t \ ; \\ 1 + (t-s)/\delta & \text{if } \ t < s \leq t + \delta \ ; \\ 0 & \text{if } \ s > t + \delta \ . \end{cases}$$

The function $\rho_{(-\infty,t]}$ is upper semi-continuous because $\psi_{t,1/n} \downarrow \rho_{(-\infty,t]}$. Hence $P(t) = \rho_{(-\infty,t]}(A)$ is a self-adjoint operator on H.



26-1.11. **Lemma** (a) $P(t)$ is a projector for each $t \in \mathbb{R}$.
(b) $P(t)$ is increasing in t.
(c) If $t < \inf \sigma(A)$, then $P(t) = 0$.
(d) If $t > \sup \sigma(A)$, then $P(t) = I$.
(e) $P(t)$ is *strongly* right continuous in t.
The map $P : \mathbb{R} \to \mathbb{P}(H)$ is called the *spectral resolution of* A.

Proof. (a) Because $\rho^2_{(-\infty,t]} = \rho_{(-\infty,t]}$, we have $P(t)^2 = P(t)$. Since $P(t)$ is self-adjoint, it is a projector.
(b) If $s \leq t$, then $\rho_{(-\infty,s]} \leq \rho_{(-\infty,t]}$, or $P(s) = \rho_{(-\infty,s]}(A) \leq \rho_{(-\infty,t]}(A) = P(t)$.
(c) If $t < \inf \sigma(A)$, then $\rho_{(-\infty,t]}(\sigma A) = 0$ and hence $P(t) = 0$ by §14-6.5c.

(d) If $t > \sup \sigma(A)$, then $\rho_{(-\infty,t]}(\sigma A) = 1$ and hence $P(t) = I$.
(e) For every $\delta_n \downarrow 0$, the function $\psi_{t+\delta_n,\delta_n}$ is continuous. By sketching a picture, we have $\psi_{t+\delta_n,\delta_n} \downarrow \rho_{(-\infty,t]}$. From $\rho_{(-\infty,t]} \leq \rho_{(-\infty,t+\delta_n]} \leq \psi_{t+\delta_n,\delta_n}$, we have $P(t) \leq P(t+\delta_n) \leq \psi_{t+\delta_n,\delta_n}(A)$. Letting $n \to \infty$, we get

$$P(t) \leq \text{s-}\lim_{n\to\infty} P(t+\delta_n) \leq \text{s-}\lim_{n\to\infty} \psi_{t+\delta_n,\delta_n}(A) = \rho_{(-\infty,t]}(A) = P(t),$$

that is $P(t+\delta_n) \to P(t)$ strongly as $n \to \infty$. Since $\delta_n \downarrow 0$ is arbitrary, $P(t)$ is right continuous in t. □

26-1.12. **Theorem** (a) For every $a < b$ in $\mathbb{R}$, let $\mu(a,b] = P(b) - P(a)$. Then $\mu(a,b] = \rho_{(a,b]}(A)$ is a projector.
(b) μ is additive on the semiring of semi-intervals.
(c) For all $x \in H$, $\mu_x(a,b] =< \mu(a,b]x, x >$ defines a positive measure on semi-intervals. Note that we understand $\mu(a,b]x = \{\mu(a,b]\}x$.
(d) If $a < \inf \sigma(A)$ and $b > \sup \sigma(A)$; then $\mu(a,b] = I$.
(e) If B is an operator commuting with A, then B commutes with $\mu(a,b]$.

Proof. (a) Since $P(t)$ is increasing, $P(b) - P(a)$ is a projector by §13-10.7. Next, $P(b) - P(a) = \rho_{(-\infty,b]}(A) - \rho_{(-\infty,a]}(A) = \{\rho_{(-\infty,b]} - \rho_{(-\infty,a]}\}(A) = \rho_{(a,b]}(A)$.
(b) It is the charge induced by P. It also follows from §26-1.9c.
(c) Since the function $< P(t)x, x >$ is increasing and right continuous in t, its induced charge μ_x is a positive measure by §17-7.8.
(d) follows from (c) and (d) of last lemma.
(e) follows from §26-1.9e. □

26-2 Extension of Spectral Measures

26-2.1. Let H be a Hilbert space and $\mathbb{P}(H)$ all projectors. Let $\mathcal{K}$ be an *upward directed* semiring over X, that is for all $J, K \in \mathcal{K}$ there is $M \in \mathcal{K}$ such that $J \cup K \subset M$. Suppose that $\mu : \mathcal{K} \to \mathbb{P}(H)$ is a projector-valued charge. As a result of §17-2.10, the integral $\int f d\mu$ is defined for all $\mathcal{K}$-step functions f with $\int \rho_J d\mu = \mu J$.

26-2.2. **Product Formula** $\mu J \mu K = \mu(J \cap K)$ for all $J, K \in \mathcal{K}$.

Proof. Case 1: Suppose that $J \cap K = \emptyset$. We assume that $\mathcal{K}$ is directed upward. There is $M \in \mathcal{K}$ containing both J, K. Write $M \setminus J \setminus K = \bigcup_{j=1}^n D_j$ as a disjoint union of sets $D_i \in \mathcal{K}$. Then $M = J \cup K \cup \bigcup_{j=1}^n D_j$ is a disjoint union. Since μ is additive, we obtain $\mu M = \mu J + \mu K + \sum_{j=1}^n \mu D_j$. Because every term is a projector, we have $(\mu J)(\mu K) = 0$ by §13-9.8.

Case 2: In general by Semiring Formula, there are disjoint $D_i \in \mathcal{K}$ such that $\rho_J = \sum_{i=1}^m s_i \rho_{D_i}$ and $\rho_K = \sum_{i=1}^m t_i \rho_{D_i}$ where s_i, t_i are either 0 or 1. Then,

$$\rho_{J\cap K} = \rho_J \rho_K = (\sum_{j=1}^m s_j \rho_{D_j})(\sum_{k=1}^m t_k \rho_{D_k}) = \sum_{j,k=1}^m s_j t_k \rho_{D_j} \rho_{D_k}$$
$$= \sum_{j=1}^m s_j k_j \rho_{D_j} \qquad \text{because } D_j\text{'s are disjoint.}$$

By case 1, we have $\mu D_j \mu D_k = \delta_{jk} \mu D_j$. Hence

$$(\mu J)(\mu K) = (\int \rho_J d\mu)(\int \rho_K d\mu) = (\sum_{j=1}^m s_j \mu D_j)(\sum_{k=1}^m t_k \mu D_k)$$
$$= \sum_{j,k=1}^m s_j t_k \mu D_j \mu D_k = \sum_{j=1}^m s_j t_j \mu D_j = \int \sum_{j=1}^m s_j t_j \rho_{D_j} d\mu$$
$$= \int \rho_{J\cap K} d\mu = \mu(J \cap K). \qquad \square$$

26-2.3. **Corollary** If $J \subset K$, then $\mu J \leq \mu K$.

Proof. From $\mu J \mu K = \mu(J \cap K) = \mu J$, the result follows by §13-10.7. □

26-2.4. **Example** In ℓ_2, for each integer $n \geq 1$ let $e_n = (\delta_{1n}, \delta_{2n}, \delta_{3n}, \cdots)$ and $P_n x =< x, e_n > e_n$. Consider the semiring $\mathcal{G}$ consisting of all subsets of $\mathbb{R}$. For each $J \in \mathcal{G}$, let $\mu J x = \sum\{P_n x : n \in J\}$. By §14-5.7, $\mu : \mathcal{G} \to \mathbb{P}(\ell_2)$ is a charge. For $J_k = (k, \infty)$, we have $J_k \downarrow \emptyset$. Since $\|\mu J_k\| = 1$, it is impossible to have $\mu J_k \to 0$ in norm.

26-2.5. **Exercise** Prove that the map $\mu J : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ given by $(\mu J)x = x\rho_J$ for every semi-interval J defines a projector-valued charge μ on the semiring of semi-intervals. Show that μ is *not* of finite variation.

26-2.6. As a result, it is impossible to expect any theory of projector-valued measure with convergence in norm. This is why vectors have to play a crucial role. Write $\mu_{xy} J =< \mu J x, y >$ and $\mu_x = \mu_{xx}$ for all $x, y \in H$ and $J \in \mathcal{K}$. We understand $\mu J x = [\mu(J)]x$ because Jx makes no sense at all. The following formula is a simple result of §13-1.4c.

26-2.7. **Polarization Formula** $4\mu_{xy} = \sum_{n=0}^{3} i^n \mu_{x+i^n y}$ where $i^2 = -1$.

26-2.8. **Lemma** (a) μ_{xy} is sesquilinear in (x, y). We also have $\mu_{yx} = \mu_{xy}^{-}$.
(b) μ_x is a positive charge on $\mathcal{K}$ and $\mu_x X \leq \|x\|^2$.
(c) μ_{xy} is a charge of finite variation.
(d) $|\mu_{xy}|(J) \leq \|x\| \, \|y\|$ for all $J \in \mathcal{S}$. Hence μ_{xy} is a bounded charge.
(e) $|\mu_{xy}| \leq \frac{1}{2}(\mu_x + \mu_y)$.

Proof. (a) To prove $\mu_{yx} = \mu_{xy}^{-}$, for every $J \in \mathcal{K}$ we have

$$\mu_{yx} J =< \mu J y, x >=< y, \mu J x >=< \mu J x, y >^{-} = \mu_{xy}^{-} J.$$

(d) Let $J = \sum_{i=1}^m K_i$ be a disjoint union where $J, K_i \in \mathcal{K}$. Because μK_i are projectors, we have

$$\begin{aligned}
&\textstyle\sum_{i=1}^m |\mu_{xy} K_i| = \sum_{i=1}^m | < \mu K_i x, y > | = \sum_{i=1}^m | < \mu K_i x, \mu K_i y > | \\
&\le \textstyle\sum_{i=1}^m \|\mu K_i x\| \; \|\mu K_i y\| \le \sqrt{\sum_{i=1}^m \|\mu K_i x\|^2} \; \sqrt{\sum_{i=1}^m \|\mu K_i y\|^2} \\
&= \sqrt{\textstyle\sum_{i=1}^m < \mu K_i x, x >} \; \sqrt{\textstyle\sum_{i=1}^m < \mu K_i y, y >} \\
&= \sqrt{< \mu J x, x >} \; \sqrt{< \mu J y, y >} = \|\mu J x\| \; \|\mu J y\|.
\end{aligned}$$

Hence μ_{xy} is of finite variation with $|\mu_{xy}| J \le \|\mu J x\| \; \|\mu J y\| \le \|x\| \; \|y\|$. In particular for $J = X$ and $y = x$, we have (b).

(e) $$\begin{aligned} |\mu_{xy}| J &\le \|\mu J x\| \; \|\mu J y\| \le \tfrac{1}{2}(\|\mu J x\|^2 + \|\mu J x\|^2) \\ &= \tfrac{1}{2}(< \mu J x, x > + < \mu J y, y >) = \tfrac{1}{2}(\mu_x + \mu_y) J. \end{aligned}$$ □

26-2.9. A charge $\mu : \mathcal{K} \to \mathbb{P}(H)$ is a *spectral* measure if μ_{xy} are complex measures on $\mathcal{K}$ for all $x, y \in H$. By polarization formula, if μ_x is a positive measure for each $x \in H$, then μ is a spectral measure. In last section, we constructed a spectral measure on the semiring of semi-intervals associated with each self-adjoint operator. Now we continue in general setting. Let μ be a spectral measure on a semiring $\mathcal{S}$ over a set X.

26-2.10. **Theorem** Every spectral measure μ on $\mathcal{S}$ has a unique extension to a spectral measure ν on the δ-ring $\mathbb{D}$ generated by $\mathcal{S}$.

Proof. Every μ_{xy} has a unique extension ν_{xy} over $\mathbb{D}$. For every $D \in D$, define $\varphi_D : H \times H \to \mathbb{C}$ by $\varphi_D(x, y) = \nu_{xy} D$. Then for every $J \in \mathcal{S}$ we obtain $\nu_{x+y,z} J = \mu_{x+y,z} J = \mu_{xz} J + \mu_{yz} J = \nu_{xz} J + \nu_{yz} J$. By uniqueness of extension, we get $\nu_{x+y,z} D = \nu_{xz} D + \nu_{yz} D$ for all $D \in \mathbb{D}$, that is $\varphi_D(x + y, z) = \varphi_D(x, z) + \varphi_D(y, z)$. Similarly we can verify all other conditions to show that φ_D is a sesquilinear form on H. Next, because $\mathbb{D}$ is the δ-ring generated by $\mathcal{S}$, D is contained in a finite union of sets in $\mathcal{S}$. Since $\mathcal{S}$ is directed upward, there is $J \in \mathcal{S}$ with $D \subset J$. For the *positive* measure ν_x, we have

$$\varphi_D(x, x) = \nu_x D = \textstyle\int \rho_D d\nu_x \le \int \rho_J d\nu_x = \int \rho_J d\mu_x =< \mu J x, x >\le \|x\|^2.$$

Hence the sesquilinear form φ_D is continuous. There is an operator νD such that $< \nu D x, y >= \varphi_D(x, y) = \nu_{xy} D$. It is an exercise to show that ν is additive on $\mathbb{D}$. Next, for every $J \in \mathcal{S}$, $\mu_x J \ge 0$ because μJ is a projector. The unique extension ν_x of $\mu_x \ge 0$ is a positive measure, i.e. $< \nu D x, x >= \mu_x D \ge 0$. Hence $\nu D \ge 0$. Finally, we have to prove that νD is an idempotent for every $D \in \mathbb{D}$. Let $K \in \mathcal{S}$ and $z = \mu K x$. Clearly $\xi : \mathbb{D} \to \mathbb{C}$ given by $\xi D =< \nu(D \cap K) x, y >$ is a measure. For any $J \in \mathcal{S}$, we have by Product Formula,

$$\xi J =< \nu(J \cap K)x, y >=< \mu(J \cap K)x, y >=< \mu J \mu K x, y >= \mu_{zx} J = \nu_{zx} J.$$

Thus $\xi = \nu_{zx}$ on $\mathbb{D}$, i.e. $< \nu(D \cap K)x, y >=< \nu D \mu K x, y >=< \nu K x, (\nu D)^* y >$. Again, both $< \nu(D \cap M)x, y >$ and $< \nu M x, (\nu D)^* y >$ are measures in $M \in \mathbb{D}$ and they are equal on $\mathcal{S}$ and hence on $\mathbb{D}$. Thus,

$$< \nu(D \cap M)x, y >=< \nu M x, (\nu D)^* y >=< \nu D \nu M x, y >.$$

Since $x, y \in H$ are arbitrary; we have the product formula: $\nu D \nu M = \nu(D \cap M)$. In particular, $(\nu D)^2 = \nu D$. Therefore νD is a projector. The uniqueness of ν follows from the uniqueness of ν_{xy}. □

26-2.11. **Theorem** Every spectral measure ν on $\mathbb{D}$ has a unique extension to a spectral measure π on the family $\mathbb{M}$ of all measurable sets.

Proof. Since each ν_x is a positive measure on $\mathbb{D}$, it can be extended to a positive measure π_x on $\mathbb{M}$ by $\pi_x M = \sup\{\nu_x D : M \supset D \in \mathbb{D}\}$ as in §20-2.2a. Since $\nu_x D \leq \|x\|^2$, we have $\pi_x M \leq \|x\|^2 < \infty$. Thus $\pi_{xy} = \frac{1}{4} i^n \sum_{n=0}^{3} \pi_{x + i^n y}$ is a complex measure on $\mathbb{M}$ where $i^2 = -1$. Next, let $x, y, z \in H$; $M \in \mathbb{M}$ and $\varepsilon > 0$ be given. There are $J_n, K_n, Q_n \in \mathbb{D}$ such that for $n = 0, 1, 2, 3$ we get

$$\pi_{x + i^n y} M \leq \nu_{x + i^n y} J_n + \varepsilon, \quad J_n \subset M \ ;$$

$$\pi_{x + i^n z} M \leq \nu_{x + i^n z} K_n + \varepsilon, \quad K_n \subset M \ ;$$

and
$$\pi_{x + i^n (y+z)} M \leq \nu_{x + i^n (y+z)} Q_n + +\varepsilon, \quad Q_n \subset M.$$

Then $D = \bigcup_{n=0}^{3} (J_n \cup K_n \cup Q_n)$ is a decent subset of M satisfying

$$0 \leq \pi_{x + i^n y} M \leq \nu_{x + i^n y} D + \varepsilon \ ;$$

$$0 \leq \pi_{x + i^n z} M \leq \nu_{x + i^n z} D + \varepsilon \ ;$$

and
$$0 \leq \pi_{x + i^n (y+z)} M \leq \nu_{x + i^n (y+z)} D + \varepsilon \ .$$

Observe that

$$\begin{aligned}
&|\pi_{x,y+z} M - \pi_{xy} M - \pi_{xz} M| \\
&= \left| \sum_{n=0}^{3} i^n \left\{ \pi_{x + i^n (y+z)} M - \pi_{x + i^n y} M - \pi_{x + i^n z} M \right\} \right| \\
&\leq \left| \sum_{n=0}^{3} i^n \left\{ \nu_{x + i^n (y+z)} D - \nu_{x + i^n y} D - \nu_{x + i^n z} D \right\} \right| \\
&\quad + \sum_{n=0}^{3} \left| \pi_{x + i^n (y+z)} M - \nu_{x + i^n (y+z)} D \right| \\
&\quad + \sum_{n=0}^{3} \left| \nu_{x + i^n y} D - \pi_{x + i^n y} M \right|
\end{aligned}$$

$$+\sum_{n=0}^{3}\left|\nu_{x+i^n z}D-\pi_{x+i^n z}M\right|$$
$$\leq |\nu_{x,y+z}D-\nu_{xy}D-\nu_{xz}D|+12\varepsilon=12\varepsilon.$$

Letting $\varepsilon \downarrow 0$, we have $\pi_{x,y+z}M = \pi_{xy}M + \pi_{xz}M$. Similarly, we can verify that $\pi_{xy}M$ is a sesquilinear form in (x,y). Since its associated quadratic form is bounded, there is an operator πM on H such that $<\pi Mx, y>=\pi_{xy}M$ for all $x, y \in H$. Because $<\pi Mx, x>=\pi_x M \geq 0$; πM is a positive operator. Next, we claim $\nu J\pi Nx = \pi(J\cap N)$ for all $J \in \mathbb{D}$, $N \in \mathbb{M}$. Indeed, let $\varepsilon > 0$ and $x \in H$ be given. Write $y = \nu Jx$. Choose $K \in \mathbb{D}$ such that $K \subset N$ and $\pi_{x+i^n y}N \leq \nu_{x+i^n y}K + \varepsilon$ for $n = 0, 1, 2, 3$. By §20-2.3, we may also assume $\pi_x(J\cap N) \leq \pi_x(J\cap K)+\varepsilon$. Hence

$$|\pi_{xy}N-\pi_x(J\cap N)| = \left|\sum_{n=0}^{3} i^n\pi_{x+i^n y}N-\pi_x(J\cap N)\right|$$
$$=\left|\sum_{n=0}^{3} i^n\pi_{x+i^n y}K-\pi_x(J\cap K)\right|$$
$$+\sum_{n=0}^{3}\left|\pi_{x+i^n y}N-\pi_{x+i^n y}K\right|$$
$$+|\pi_x(J\cap K)-\pi_x(J\cap N)|$$
$$=\left|\sum_{n=0}^{3} i^n\nu_{x+i^n y}K-\nu_x(J\cap K)\right|+5\varepsilon$$
$$=|\nu_{xy}K-\nu_x(J\cap K)|+5\varepsilon = |<\nu Kx,y>-<\nu(J\cap K)x,x>|+5\varepsilon$$
$$=|<\nu Kx,\nu Jx>-<\nu(J\cap K)x,x>|+5\varepsilon$$
$$=|<\nu J\nu Kx,x>-<\nu(J\cap K)x,x>|+5\varepsilon=5\varepsilon$$

because νJ is self-adjoint and the product formula holds for ν. Letting $\varepsilon \downarrow 0$, we have $\pi_{xy}N = \pi_x(J\cap N)$, that is $<\nu J\pi Nx, x>=<\pi(J\cap N)x, x>$ for all $x \in H$. Therefore $\nu J\pi Nx = \pi(J\cap N)$ as claim. Repeating the same process, we can prove that $\pi M\pi N = \pi(M\cap N)$ for all measurable sets M, N. In particular, πM is an idempotent. Consequently, πM is a projector. The uniqueness follows because the measures are finite. □

26-2.12. **Theorem** If an operator B commutes with μJ for every $J \in \mathcal{S}$, then B commutes with πM for every measurable set M.

Proof. For all $x, y \in H$ and $J \in \mathcal{S}$, we have $\mu_{x,B^*y}(J) = <\mu J\, x, B^*y> = <B\mu J\, x, y> = <(\mu J)Bx, y> = \mu_{Bx,y}(J)$. Thus $\mu_{x,B^*y} = \mu_{Bx,y}$ as scalar measures on $\mathcal{S}$. By uniqueness of extension from $\mathcal{S}$ over $\mathbb{D}$, we obtain $\nu_{x,B^*y} = \nu_{Bx,y}$. Now fix $x, y \in H$ and $M \in \mathbb{M}$. As in last proof, choose a decent subset D of M such that for $n = 0, 1, 2, 3$ we have

$$0 \le \pi_{x+i^nB^*y}M \le \nu_{x+i^nB^*y}D + \varepsilon$$

and $$0 \le \pi_{Bx+i^n(y+z)}M \le \nu_{Bx+i^n(y+z)}D + \varepsilon\,.$$

Observe that

$$\begin{aligned}
&|\pi_{x,B^*y}M - \pi_{Bx,y}M| \\
&= \left|\sum\nolimits_{n=0}^{3} i^n \left\{\pi_{x+i^nB^*y}M - \pi_{Bx+i^n(y+z)}M\right\}\right| \\
&\le \left|\sum\nolimits_{n=0}^{3} i^n \left\{\nu_{x+i^nB^*y}D - \nu_{Bx+i^n(y+z)}D\right\}\right| \\
&\qquad + \sum\nolimits_{n=0}^{3} \left|\pi_{x+i^nB^*y}M - \nu_{x+i^nB^*y}D\right| \\
&\qquad + \sum\nolimits_{n=0}^{3} \left|\nu_{Bx+i^n(y+z)}D - \pi_{Bx+i^n(y+z)}M\right| \\
&\le |\nu_{x,B^*y}D - \nu_{Bx,y}D| + 8\varepsilon = 8\varepsilon.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get

$$< B\pi Mx, y > = < \pi Mx, B^*y > = \pi_{x,B^*y}M = \pi_{Bx,y}M = < (\pi M)Bx, y >.$$

Since x, y are arbitrary, we have $B\pi M = (\pi M)B$. □

26-2.13. After this section, we identify ν, π as μ for convenience. The treatment below requires merely *bounded* complex measures on σ-algebras although we continue to use the term δ-spaces in order to be consistent within this book. It is a challenge to create new results by taking the full advantage of our machinery of vector measures on δ-rings.

26-3 Spectral Integration

26-3.1. Let μ be a spectral measure on a δ-space X into a Hilbert space H. Note that our definition of measurable sets and functions is independent of any measure. Because μ has been extended to all measurable sets, the integral of a simple function has been defined by §17-2.10 is an operator on H. A measurable function on X is *μ-integrable* if it is μ_{xy}-integrable for all $x, y \in H$. Let $L_1(\mu), L_1(\mu_{xy})$ denote the sets of all μ-integrable functions and μ_{xy}-integrable functions respectively. Clearly $L_1(\mu) = \cap\{L_1(\mu_{xy}) : x, y \in H\}$ is a conjugated complex algebra under pointwise operations.

26-3.2. **Theorem** Let f be a μ-integrable function. If f_n are simple functions with $f_n \to f$ and $|f_n| \uparrow |f|$, then $\int f_n d\mu \to \int f d\mu$ weakly to a unique operator $\int f d\mu$ on H such that $< \int f d\mu\; x, y >= \int f d\mu_{xy}$ for all $x, y \in H$. We shall prove later that $\int f_n d\mu \to \int f d\mu$ strongly in Spectral Dominated Convergence Theorem.

Proof. For all $x, y \in H$, $|f|$ is $|\mu_{xy}|$-integrable. By Dominated Convergence theorem, $< \int f_n d\mu\ x, y >= \int f_n d\mu_{xy} \to \int f d\mu_{xy}$. By §13-10.13, the weakly Cauchy sequence $\{\int f_n d\mu\}$ of operators converges weakly to some operator $\int f d\mu$, that is $< \int f_n d\mu\ x, y > \to < \int f d\mu\ x, y >$. Uniqueness of limits gives $< \int f d\mu\ x, y >= \int f d\mu_{xy}$ for all $x, y \in H$. Finally if A is any operator satisfying $< Ax, y >= \int f d\mu_{xy}$ for all $x, y \in H$. Then $< \int f d\mu\ x, y >=< Ax, y >$ for all $x, y \in H$. Consequently $\int f d\mu = A$. This proves the uniqueness too. □

26-3.3. **Theorem** If f is μ-integrable, then so is its conjugate f^-. Furthermore we have $\int f^- d\mu = (\int f d\mu)^*$.

Proof. It is an exercise to show that f^- is μ-integrable. For all $x, y \in H$,

$$< \int f^- d\mu\ x, y >= \int f^- d\mu_{xy} = (\int f d\mu_{xy}^-)^- = (\int f d\mu_{yx})^-$$
$$=< \int f d\mu\ y, x >^- =< x, \int f d\mu\ y >=< (\int f d\mu)^*\ x, y >.$$ □

26-3.4. **Corollary** If f is real μ-integrable, then $\int f d\mu$ is self-adjoint.

26-3.5. **Product Theorem** If f, g are μ-integrable, then so is the product fg. Furthermore we have $\int fg d\mu = (\int f d\mu)(\int g d\mu)$.

Proof. Let f_n, g_n be simple functions such that $f_m \to f$, $g_n \to g$, $|f_m| \uparrow |f|$ and $|g_n| \uparrow |g|$. By Step Mapping Theorem, write $f_m = \sum_{j=1}^p \alpha_i \rho_{M_i}$ and $g_n = \sum_{j=1}^p \beta_j \rho_{M_j}$ where $\alpha_j, \beta_k \in \mathbb{C}$ and M_j are disjoint measurable sets. Observe that

$$\left(\int f_m d\mu\right)\left(\int g_n d\mu\right) = \sum_{j,k} \alpha_j \beta_k \left(\int \rho_{M_j} d\mu\right)\left(\int \rho_{N_k} d\mu\right)$$
$$= \sum_{j,k} \alpha_j \beta_k \mu M_j \mu N_k = \sum_{j,k} \alpha_j \beta_k \mu(M_j \cap N_k)$$
$$= \sum_j \alpha_j \beta_j \mu M_j \qquad \text{because } M_j \text{ are disjoint}$$
$$= \int \sum_j \alpha_j \beta_j \rho_{M_j} d\mu = \int f_m g_n d\mu.$$

For all $x, y \in H$; letting $m \to \infty$ with fixed n, we have

$$\left\langle \left(\int f d\mu\right)\left(\int g_n d\mu\right) x, y \right\rangle = \lim_{m\to\infty} \left\langle \left(\int f_m d\mu\right)\left(\int g_n d\mu\right) x, y \right\rangle$$
$$= \lim_{m\to\infty} < \int f_m g_n d\mu) x, y >= \lim_{m\to\infty} \int f_m g_n d\mu_{xy} = \int f g_n d\mu_{xy}$$

because $|f_m g_n| \le |fg|$ and fg is μ_{xy}-integrable. From

$$< (\int f d\mu)(\int g d\mu) x, y >=< \int g d\mu\ x, z > \qquad \text{where } z = (\int f d\mu)^* y$$
$$= \int g d\mu_{xz} = \lim_{n\to\infty} \int g_n d\mu_{xz} = \lim_{n\to\infty} < \int g_n d\mu\ x, z >$$

$$= \lim_{n\to\infty} < \int g_n d\mu\ x, (\int f d\mu)^* y >= \lim_{n\to\infty} < (\int f d\mu) \int g_n d\mu\ x, y >$$
$$= \lim_{n\to\infty} < \int f g_n d\mu\ x, y >= \lim_{n\to\infty} \int f g_n d\mu_{xy} = \int f g d\mu_{xy}$$
$$=< \int f g d\mu\ x, y > \text{ for all } x, y \in H;$$

we have $(\int f d\mu)(\int g d\mu) = \int f g d\mu$ as required. □

26-3.6. **Corollary** If f, g are μ-integrable; then $\int f d\mu$, $\int g d\mu$ commute.

26-3.7. **Corollary** For every μ-integrable function, $\int f d\mu$ is normal.

26-3.8. **Corollary** $\|(\int f d\mu)x\|^2 = \int |f|^2 d\mu_x$ for every $f \in L_1(\mu)$ and $x \in H$.

Proof. $\|(\int f d\mu)x\|^2 = < (\int f d\mu)x, (\int f d\mu)x > = < (\int f d\mu)^-(\int f d\mu)x, x >$
$= < (\int f d\mu)^-(\int f d\mu)x, x > = < \int f^- f d\mu\ x, x > = \int |f|^2 d\mu_x.$ □

26-3.9. An operator is said to *commute* with μ on a family $\mathcal{K}$ of measurable sets if it commutes with μJ for every $J \in \mathcal{K}$. Two spectral measures *commute* on $\mathcal{K}$ if μJ commutes with νK for all $J, K \in \mathcal{K}$. When $\mathcal{K}$ is not mentioned explicitly, we understand that $\mathcal{K}$ is the family of all measurable sets. In §26-2.12, it suffices to test for a smaller class $\mathcal{S}$ of measurable sets.

26-3.10. **Theorem** If an operator B commutes with μ, then B commutes with $\int f d\mu$ for all μ-integrable function f.

Proof. Choose simple functions $f_n \to f$ with $|f_n| \uparrow |f|$. Write $f_n = \sum_{j=1}^k \alpha_j \rho_{M_j}$ where $\alpha_j \in \mathbb{C}$ and M_j are measurable sets. Then we have

$$B \int f_n d\mu = B \sum_{j=1}^k \alpha_j \mu M_j = \sum_{j=1}^k \alpha_j B \mu M_j$$
$$= \sum_{j=1}^k \alpha_j [(\mu M_j)B] = (\sum_{j=1}^k \alpha_j \mu M_j)B = (\int f_n d\mu)B.$$

Hence for all $x, y \in H$, we obtain

$$\int f_n d\mu_{x,B^*y} = < \int f_n d\mu\ x, B^* y > = < B \int f_n d\mu\ x, y >$$
$$= < (\int f_n d\mu)Bx, y >= \int f_n d\mu_{Bx,y}\ .$$

The scalar dominated convergence theorem shows $\int f d\mu_{x,B^*y} = \int f d\mu_{Bx,y}$. Thus we get

$$< B \int f d\mu\ x, y > = < \int f d\mu\ x, B^* y >$$
$$= \int f d\mu_{x,B^*y} = \int f d\mu_{Bx,y} = < (\int f d\mu)Bx, y >.$$

Therefore $B \int f d\mu = (\int f d\mu)B$. □

26-3.11. **Corollary** If two spectral measures μ, ν commutes on a semiring $\mathcal{S}$ over a set X, then for every μ-integrable function f and every ν-integrable function g we have $\int f d\mu \cdot \int g d\nu = \int g d\nu \cdot \int f d\mu$.

26-4 Null Sets of Spectral Measures

26-4.1. Let μ be a spectral measure on a δ-space X into a Hilbert space H. A measurable set is *μ-null* if it is μ_{xy}-null for all $x, y \in H$. As a result of $|\mu_{xy}| \le \frac{1}{2}(\mu_x + \mu_y)$, it suffices to be μ_x-null for all $x \in H$. Furthermore if $\mu J = 0$, then $\mu_{xy} J =< \mu J x, y >= 0$ and hence J is μ-null. Since countable unions of μ-null sets are μ-null, the Banach space $L_\infty(\mu)$ is well-defined by §§21-5.1,2,3,4. In particular, the *μ-essential sup-norm* of a measurable function f is given by $\|f\|_\infty = \inf_N \sup_{x \in X \setminus N} |f(x)|$ where N runs over all μ-null sets. The following results distinguish projector-valued spectral measures μ from the others. This might be the reason why some writers such as [Berberian-66] and [Meise] starts with bounded measurable functions.

26-4.2. **Theorem** $L_\infty(\mu) = L_1(\mu)$ and $\| \int f d\mu \| = \|f\|_\infty$ for every $f \in L_1(\mu)$.

Proof. Since every μ_{xy} is a bounded measure, every $f \in L_\infty$ is μ_{xy}-integrable. Hence $L_\infty(\mu) \subset L_1(\mu)$. Next observe that for all $x \in H$,

$$\| \int f d\mu \, x\|^2 = \int |f|^2 d\mu_x \le \int \|f\|_\infty^2 d\mu_x = \|f\|_\infty^2 \mu_x(X) \le \|f\|_\infty^2 \|x\|^2,$$

that is $\| \int f d\mu\| \le \|f\|_\infty$. Conversely, take any $f \in L_1(\mu)$. Let $u_j = \| \int f d\mu\| + \frac{1}{j}$, $M_j = \{t \in X : |f(t)| \ge u_j\}$ and $M = \{t \in X : |f(t)| > \| \int f d\mu\| \}$. Then for all $x \in H$, we have

$$\|\mu M_j x\|^2 = \int_{M_j} 1 d\mu_x \le \int_{M_j} u_j^{-2} |f|^2 d\mu_x \le u_j^{-2} \int |f|^2 d\mu_x$$
$$\le u_j^{-2} \| \int f d\mu \, x\|^2 \le u_j^{-2} \| \int f d\mu\|^2 \|x\|^2,$$

that is $\|\mu M_j\| \le u_j^{-1} \| \int f d\mu\| < 1$. As μM_j is a projector, we have $\mu M_j = 0$. Thus M_j is μ-null. Consequently $M = \bigcup_{j=1}^\infty M_j$ is μ-null. In other words, $f \in L_\infty(\mu)$ and $\|f\|_\infty \le \| \int f d\mu\|$. □

26-4.3. **Corollary** For all $f, g \in L_1(\mu)$, $\int f d\mu = \int g d\mu$ iff $f = g$, μ-ae.

Proof. It follows from $\|f - g\|_\infty = \| \int (f - g) \, d\mu\| = \| \int f d\mu - \int g d\mu\| = 0$. □

26-4.4. **Theorem** $\sup_{\|x\| \le 1} \int f d\mu_x = \inf_N \sup_{t \in X \notin N} f(t)$ for every $f \in L_1(\mu)$ where N runs over all μ-null sets. In particular we have $\|f\|_\infty = \sup_{\|x\| \le 1} \int |f| d\mu_x$.

Proof. Let $\theta = \inf_N \sup_{t \in X \setminus N} f(t)$. Similar to §21-5.3a, it is easy to show that $f \le \theta$, μ-ae. Take any $\|x\| \le 1$ in H. Since μ_x is a positive measure, we have $\int f d\mu_x \le \int \theta d\mu_x = \theta \mu_x X \le \theta \|x\|^2 \le \theta$, that is $\sup_{\|x\| \le 1} \int f d\mu_x \le \theta$. On the other hand, because $f \in L_\infty$, we have $g = \|f\|_\infty + f \ge 0$, μ-ae.

Without loss of generality, we may assume that $g \geq 0$ everywhere and hence $h = \sqrt{g} \in L_\infty(\mu) = L_1(\mu)$. Observe that

$$\sup_{\|x\|\leq 1} \int g d\mu_x = \sup_{\|x\|\leq 1} \int h^2 d\mu_x = \sup_{\|x\|\leq 1} \left\| \int h d\mu\, x \right\|^2 = \left\| \int h d\mu \right\|^2 = \|h\|_\infty^2$$

$$= \inf_N \sup_{t\in X\setminus N} h(t)^2 = \inf_N \sup_{t\in X\setminus N} g(t) = \|f\|_\infty + \inf_N \sup_{t\in X\setminus N} f(t) = \|f\|_\infty + \theta$$

where N runs over all μ-null sets. Also

$$\sup_{\|x\|\leq 1} \int g d\mu_x = \sup_{\|x\|\leq 1} \int (\ \|f\|_\infty + f\ ) d\mu_x = \sup_{\|x\|\leq 1} \left(\int \|f\|_\infty d\mu_x + \int f d\mu_x \right)$$

$$\leq \sup_{\|x\|\leq 1} \left(\|f\|_\infty \|x\|^2 + \int f d\mu_x \right) \leq \|f\|_\infty + \sup_{\|x\|\leq 1} \int f d\mu_x.$$

Hence $\theta \leq \sup_{\|x\|\leq 1} \int f d\mu_x$. This completes the proof. □

26-4.5. **Theorem** Let f, g be real μ-integrable functions. Then $f \leq g$, μ-ae iff $\int f d\mu \leq \int g d\mu$. In particular, if $f \geq 0$, μ-ae; then $\int f d\mu$ is a positive operator.

Proof. ($\Leftarrow$) By linearity, we may assume $g = 0$. For N runs over all μ-null sets, we have $\inf_N \sup_{t\in X\setminus N} f(t) = \sup_{\|x\|\leq 1} \int f d\mu_x = \sup_{\|x\|\leq 1} < \int f d\mu\, x, x > \leq 0$ from which we get $f \leq 0$, μ-ae.

($\Rightarrow$) By linearity, it suffices to prove that if $f \geq 0$, μ-ae, then $\int f d\mu \geq 0$. Choose simple functions $0 \leq f_n \uparrow f$. Then $\lim \int f_n d\mu \to \int f d\mu$ weakly. Write $f_n = \sum_{j=1}^m \alpha_j \rho_{M_j}$ where $\alpha_j \geq 0$ and M_j are measurable sets. Then for every $x \in H$ we have

$$< \int f d\mu\, x, x >= \lim < \int f_n d\mu\, x, x > = \lim \textstyle\sum_j \alpha_j < \int \rho_{M_j} d\mu\, x, x >$$

$$= \lim \textstyle\sum_j \alpha_j < \mu M_j x, x >= \lim \sum_j \alpha_j \|\mu M_j x\|^2 \geq 0.$$

Therefore $\int f d\mu \geq 0$. □

26-4.6. **Spectral Dominated Convergence Theorem** Let f_n, f be measurable functions on X. Suppose $f_n \to f$, μ-ae. If there is a μ-integrable function g with $|f_n| \leq g$, μ-ae for all n; then all f_n, f are μ-integrable and $\int f_n d\mu \to \int f d\mu$ strongly as $n \to \infty$. In particular, the convergence in §26-3.2 is strong.

Proof. For all $x, y \in H$; we have $f_n \to f$ and $|f_n| \leq g$, μ_{xy}-ae. By scalar dominated convergence theorem, all f_n, f are μ_{xy}-integrable and hence μ-integrable by definition. Next, from $|f_n - f|^2 \leq 4g^2$, μ_x-ae; we obtain

$$\left\| \left(\int f_n d\mu - \int f d\mu \right) x \right\|^2 = \int |f_n - f|^2 d\mu_x \to 0,$$

that is $\int f_n d\mu \to \int f d\mu$ strongly. □

26-4.7. **Corollary** If $J_n \uparrow K$ or $J_n \downarrow K$ as measurable sets, then $\mu J_n \to \mu K$ strongly.

26-4.8. **Spectral Integration Term by Term** Let $\{f_n\}$ be a sequence of measurable maps. If the upper function $\sum_{n=1}^{\infty} |f_n|$ is finite-valued μ-ae and is μ-integrable, then all f_n and $\sum_{n=1}^{\infty} f_n$ are μ-integrable and

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu , \qquad \text{strongly.}$$

Proof. From $|\sum_{n=1}^{m} \int f_n| \le \sum_{n=1}^{\infty} |f_n|$, μ-ae; we have

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{m\to\infty} \sum_{n=1}^{m} f_n d\mu = \lim_{m\to\infty} \int \sum_{n=1}^{m} f_n d\mu , \quad \text{strongly}$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} \int f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu. \qquad \square$$

26-4.9. **Corollary** μ is strongly countably additive on measurable sets.

Proof. Let $J = \bigcup_{n=1}^{\infty} K_n$ be a disjoint union of measurable sets. Then we have $\rho_J = \sum_{n=1}^{\infty} \rho_{K_n}$. Since μ is defined on all measurable sets, ρ_J is μ-integrable. The result follows immediately from last theorem. $\square$

26-4.10. A spectral measure μ is *normalized* if $\mu X = I$. From §26-1.11d, the spectral measure of a self-adjoint operator is normalized.

26-4.11. **Theorem** Let μ be normalized. Then $\int f d\mu$ is invertible iff $|f| \ge \theta > 0$, μ-ae for some θ. In this case, we have $(\int f d\mu)^{-1} = \int (1/f) d\mu$.

Proof. ($\Leftarrow$) Let N be a μ-null set such that $|f(t)| \ge \theta$ for all $t \in X \setminus N$. Then $g = f\rho_{X\setminus N} + \rho_N = f$, μ-ae. Hence $1/g$ interpreted as $1/f$ is defined everywhere and is measurable. Now $\int f d\mu \int (1/g) d\mu = \int g d\mu \int (1/g) d\mu = \int 1 d\mu = \mu X = I$ and similarly $\int (1/g) d\mu \int f d\mu = I$ show that $\int f d\mu$ is invertible with the required identity.
($\Rightarrow$) Assume that $A = \int f d\mu$ is invertible. For $Q = f^{-1}(0)$, $f_n = f + \frac{1}{n}[\text{sgn}(f) + \rho_Q]$ is measurable by §19-6.2. If $t \in Q$, then $f_n(t) = \frac{1}{n}$ and if $t \notin Q$, then $|f_n(t)| = (|f| + \frac{1}{n})|\text{sgn}(f)| \ge \frac{1}{n}$. Hence $A_n = \int f_n d\mu$ is invertible and $A_n^{-1} = \int (1/f_n) d\mu$. From $\|A_n - A\| = \|\int (f_n - f) d\mu\| = \|f_n - f\|_\infty \le \frac{1}{n} \to 0$, we have by §8-6.4 $\|A_n^{-1} - A^{-1}\| \to 0$. Thus $\|(1/f_m) - (1/f_n)\|_\infty = \|A_m^{-1} - A_n^{-1}\| \to 0$ as $m, n \to \infty$. Therefore $1/f_n \to g$ in $L_\infty(\mu)$. From $f_n \to f$, we obtain $g = 1/f$, μ-ae. Let $\alpha = \|g\|_\infty + 1$. There is a μ-null set N such that $|g| \le \alpha$ on $X \setminus N$, that is $|f| \ge (1/\alpha) > 0$ on $X \setminus N$. $\square$

26-4.12. **Theorem** Let μ be normalized. Then $A = \int f d\mu$ is unitary iff $|f| = 1$, μ-ae.

Proof. Observe $\int |f|^2 d\mu = \int f^- f d\mu = (\int f d\mu)^* \int f d\mu = A^*A$ and $I = \int 1 d\mu$. Hence A is unitary iff $A^*A = A^*A = I$ iff $\int f^- f d\mu = \int 1 d\mu$ iff $|f|^2 = 1$, μ-ae, i.e $|f| = 1$, μ-ae. □

26-5 Product Spectral Measures

26-5.1. Let $\mathcal{S}, \mathcal{T}$ be semirings of semi-intervals over $X = \mathbb{R}^u, Y = \mathbb{R}^v$ and μ, ν be projector-valued charges on $\mathcal{S}, \mathcal{T}$ into a Hilbert space H respectively. We assume that for all $J \in \mathcal{S}$ and $K \in \mathcal{T}$ we have $\mu J \nu K = \nu K \mu J$ which is denoted by $(\mu \times \nu)(J \times K)$. We use general notation because most of this section has been generalized to locally compact spaces.

26-5.2. **Lemma** $\mu \times \nu$ is a projector-valued charge on the semiring $\mathcal{S} \times \mathcal{T}$.

Proof. Let $J_0 \times K_0 = \bigcup_{i=1}^m J_i \times K_i$ be a disjoint union where $J_i \in \mathcal{S}$ and $K_i \in \mathcal{T}$ are all nonempty. By Semiring Formula, there are disjoint nonempty sets $M_j \in \mathcal{S}$, disjoint nonempty sets $N_k \in \mathcal{T}$ and α_{ij}, β_{ik} are 0 or 1 such that $\rho_{J_i} = \sum_{j=1}^p \alpha_{ij}\rho_{M_j}$ and $\rho_{K_i} = \sum_{k=1}^q \beta_{ik}\rho_{N_k}$. Observe that

$$\sum_{j=1}^p \sum_{k=1}^q \alpha_{0j}\beta_{0k}\rho_{M_j}\rho_{N_k} = (\sum_{j=1}^p \alpha_{0j}\rho_{M_j})(\sum_{k=1}^q \beta_{0k}\rho_{N_k})$$
$$= \rho_{J_0}\rho_{K_0} = \sum_{i=1}^m \rho_{J_i}\rho_{K_i} = \sum_{j=1}^p \sum_{k=1}^q \sum_{i=1}^m \alpha_{ij}\beta_{ik}\rho_{M_j}\rho_{N_k}.$$

Evaluating at any point in $M_j \times N_k$, we obtain $\alpha_{0j}\beta_{0k} = \sum_{i=1}^m \alpha_{ij}\beta_{ik}$. From

$$(\mu \times \nu)(J_0 \times K_0) = \mu J_0 \nu K_0 = \int \rho_{J_0} d\mu \int \rho_{K_0} d\nu$$
$$= (\int \sum_{j=1}^p \alpha_{0j}\rho_{M_j} d\mu)(\int \sum_{k=1}^q \beta_{0k}\rho_{N_k} d\nu)$$
$$= \sum_{j=1}^p \sum_{k=1}^q \alpha_{0j}\beta_{0k} \int \rho_{M_j} d\mu \int \rho_{N_k} d\nu$$
$$= \sum_{j=1}^p \sum_{k=1}^q \sum_{i=1}^m \alpha_{ij}\beta_{ik} \int \rho_{M_j} d\mu \int \rho_{N_k} d\nu$$
$$= \sum_{i=1}^m \int \sum_{j=1}^p \alpha_{ij}\rho_{M_j} d\mu \int \sum_{k=1}^q \beta_{ik}\rho_{N_k} d\nu$$
$$= \sum_{i=1}^m \int \rho_{J_i} d\mu \int \rho_{K_i} d\nu = \sum_{i=1}^m \mu J_i \nu K_i = \sum_{i=1}^m (\mu \times \nu)(J_i \times K_i)$$

$\mu \times \nu$ is additive on $\mathcal{S} \times \mathcal{T}$. Because $\mu J \nu K = \nu K \mu J$, it follows from §13-9.6 that $\mu \times \nu$ is projector-valued. □

26-5.3. For the rest of this section, we assume that μ, ν are spectral measures.

26-5.4. **Theorem** $\mu \times \nu$ is a spectral measure on $X \times Y = \mathbb{R}^{u+v}$.

Proof. To simplify the notation, let $\pi = \mu \times \nu$. For every $x \in H$, π_x is a positive charge on the semiring $\mathcal{S} \times \mathcal{T}$. It suffices to prove that π_x is countably additive. Let $J = \bigcup_{i=1}^\infty K_i$ be a disjoint union where J, K_i are nonempty sets in $\mathcal{S} \times \mathcal{T}$. By §17-2.12a, we have $\sum_{i=1}^\infty \pi_x K_i \le \pi_x J$. To prove the reserve inequality, let $e_u \in X$ and $e_v \in Y$ be the vectors with all coordinates equal to one. Draw pictures in order to visualize what are

$$J = (a, b] \times (\alpha, \beta],\ J^m = (a + \tfrac{1}{m}e_u, b] \times (\alpha + \tfrac{1}{m}e_v, \beta],$$
$$K_i = (a_i, b_i] \times (\alpha_i, \beta_i],\ K_i^m = (a_i, b_i + \tfrac{1}{m}e_u] \times (\alpha_i, \beta_i + \tfrac{1}{m}e_v].$$

Since $\mu(a + \frac{1}{m}e_u, b] \to \mu(a, b]$ and $\nu(\alpha + \frac{1}{m}e_u, \beta] \to \nu(\alpha, \beta]$ strongly, we obtain $\pi J^m \to \pi J$ strongly. Similarly, we also have $\pi K_i^m \to \pi K_i$ strongly as $m \to \infty$. For every $\varepsilon > 0$, there is m_0 such that $\|\pi J^m x - \pi J x\| \leq \varepsilon$ and $\|\pi K_i^m x - \pi K_i x\| \leq \varepsilon/2^i$ for all $m \geq m_0$. Hence

$$|\pi_x J^m - \pi_x J| = |< \pi J^m x - \pi J x, x > | \leq \|\pi J^m x - \pi J x\|\ \|x\| \leq \varepsilon\|x\|,$$

that is $\pi_x J \leq \pi_x J^m + \varepsilon\|x\|$ and similarly $\pi K_i^m \leq \pi K_i + \varepsilon\|x\|/2^i$. Next, clearly we have

$$[a + \tfrac{1}{m}e_u, b] \times [\alpha + \tfrac{1}{m}e_v, \beta] \subset \textstyle\bigcup_{i=1}^{\infty}(a_i, b_i + \tfrac{1}{m}e_u) \times (\alpha_i, \beta_i + \tfrac{1}{m}e_v).$$

By compactness, there is n such that $J^m \subset \bigcup_{i=1}^{n} K_i^m$. By §17-2.12b, we have

$$\pi_x J \leq \pi_x J^m + \varepsilon\|x\| \leq \textstyle\sum_{i=1}^{n} \pi_x K_i^m + \varepsilon\|x\|$$
$$\leq \textstyle\sum_{i=1}^{n} \pi_x K_i + 2\varepsilon\|x\| \leq \sum_{i=1}^{\infty} \pi_x K_i + 2\varepsilon\|x\|.$$

Letting $\varepsilon \downarrow 0$, we have $\pi_x J \leq \sum_{i=1}^{\infty} \pi_x K_i$. Therefore each π_x is a measure. By polarisation formula, all π_{xy} are measures on $\mathcal{S} \times \mathcal{T}$. □

26-5.5. **Exercise** Suppose that $\mu \times \nu$ is extended to a spectral measure $\mu \otimes \nu$ on the measurable sets of $X \times Y$. Prove that $(\mu \otimes \nu)(M \times N) = \mu M \nu N$ for all measurable sets M, N in X, Y respectively.

26-5.6. **Corollary** If ν is normalized on Y, then for every μ-integrable function f is $\mu \otimes \nu$-integrable with $\int f(s) d(\mu \otimes \nu)(s, t) = \int f(s) d\mu(s)$.

Proof. Let f_n be simple functions such that $f_n \to f$ and $|f_n| \uparrow |f|$. Write $f_n = \sum_{i=1}^{k} \alpha_i \rho_{J_i}$ where J_i are measurable subsets of X. Then for every $x \in H$, we have

$$< \int f_n(s) d(\mu \otimes \nu)(s, t)x, x >= \textstyle\sum_{i=1}^{k} \alpha_i < \int \rho_{J_i}(s) d(\mu \otimes \nu)(s, t)x, x >$$
$$= \textstyle\sum_{i=1}^{k} \alpha_i < \int \rho_{J_i \times Y}(s, t) d(\mu \otimes \nu)(s, t)x, x >$$
$$= \textstyle\sum_{i=1}^{k} \alpha_i < (\mu \otimes \nu)(J_i \times Y)x, x >= \sum_{i=1}^{k} \alpha_i < \mu J_i \nu Y x, x >$$
$$= \textstyle\sum_{i=1}^{k} \alpha_i < \mu J_i x, x >=< \int f_n(s) d\mu(s)x, x >.$$

Similarly, $\int |f_n| d(\mu \otimes \nu)_x = \int |f_n| d\mu_x \leq \int |f| d\mu_x < \infty$. By Monotone Convergence Theorem, f is $(\mu \otimes \nu)_x$-integrable for all x, i.e. $\mu \otimes \nu$-integrable. Letting $n \to \infty$, we obtain $< \int f(s) d(\mu \otimes \nu)(s, t)x, x >=< \int f(s) d\mu(s)x, x >$. Since $x \in H$ is arbitrary, the result follows. □

26-5.7. **Exercise** Prove that if μ, ν are normalized, then so is $\mu \otimes \nu$.

26-5.8. The complement of the union of all open μ-null sets is called the *support* of μ. It is denoted by $\text{supp}(\mu)$. Note that if $t \in \text{supp}(\mu)$, then every open ball $\mathbb{B}(t)$ is not μ-null.

26-5.9. **Theorem** The union N of all open μ-null sets is μ-null.

Proof. The open set N is the union of some sequence of compact sets K_n. The family $\mathcal{F}$ of all open μ-null sets is an open cover of K_n. There is a finite subset $\mathcal{G}_n$ of $\mathcal{F}$ such that $K_n \subset \cup\mathcal{G}_n$. Therefore N is contained in the countable union $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ of μ-null sets. Consequently, N is μ-null. □

26-5.10. **Exercise** Prove that if $f = g$, μ-ae on $\text{supp}(\mu)$, then $\int f d\mu = \int g d\mu$. Therefore we may work with functions defined only on $\text{supp}(\mu)$.

26-5.11. **Exercise** Prove that if μ has compact support, then every continuous function f is μ-integrable.

26-5.12. **Exercise** Prove that $\quad \text{supp}(\mu \otimes \nu) \subset \text{supp}(\mu) \times \text{supp}(\nu)$.

26-6 Spectral Measures of Normal Operators

26-6.1. Let μ be a normalized spectral measure with compact support on $X = \mathbb{K}$ into a Hilbert space H. In this case, μ is said to be *real* or *complex* depending on $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ respectively. Since every continuous function f on X is μ-integrable, $\int f d\mu$ is well-defined. In particular, $A = \int \lambda d\mu(\lambda)$ is a normal operator. In this case, μ is called the *spectral measure of* A. In this section, we study the relationship between A and μ. As a result of the following uniqueness theorem, the only real spectral measure of a self-adjoint operator is obtained from §26-1.

26-6.2. **Uniqueness Theorem** Let μ, ν be normalized spectral measures on $X = \mathbb{K}$ with compact support. If $\int \lambda d\mu(\lambda) = \int \lambda d\nu(\lambda)$, then $\mu = \nu$.

Proof. Let $K = \text{supp}(\mu) \cup \text{supp}(\nu)$ and let $\mathcal{F}$ be the vector space of continuous functions f on X such that $\int f d\mu = \int f d\nu$. Because both μ, ν are normalized, $\mathcal{F}$ contains all constant functions. Given that $h \in \mathcal{F}$ where $h(\lambda) = \lambda$, $\mathcal{F}$ separates points of K. Also $\mathcal{F}$ is a self-conjugated algebra by §§26-3.3,4,5. Hence $\mathcal{F}$ is dense in the Banach space $C_\infty(K)$ of continuous functions on K with the sup-norm. For every continuous function f on X, there are $f_n \in \mathcal{F}$ such that $f_n \to f$ uniformly on K. Hence $\| \int f_n d\mu - \int f d\mu \| \to 0$ and $\| \int f_n d\nu - \int f d\nu \| \to 0$.

For $f_n \in \mathfrak{F}$, we have $f \in \mathfrak{F}$. Next, for all $x, y \in H$, both μ_{xy}, ν_{xy} are measures on X such that $\int f d\mu_{xy} = \int f d\nu_{xy}$ for all continuous functions with compact support and hence $\mu_{xy} = \nu_{xy}$. Therefore we obtain $\mu = \nu$. □

26-6.3. **Theorem** If μ is the spectral measure of a normal operator A, then we have $\sigma(A) = \text{supp}(\mu)$.

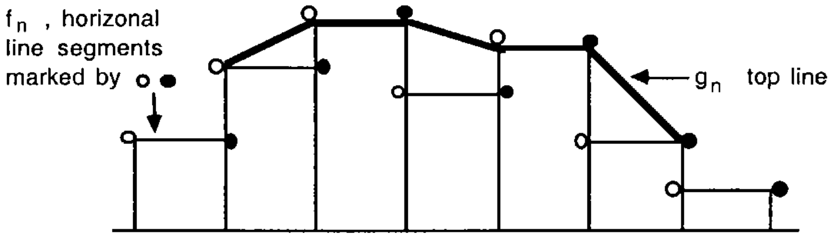
Proof. Take any $t \notin \text{supp}(\mu)$. There is $\mathbb{B}(t, r) \subset X \setminus \text{supp}(\mu)$. Then we have $\|Ax - tx\|^2 = \| \int (\lambda - t) d\mu(\lambda)\, x\|^2 = \int |\lambda - t|^2 d\mu_x(\lambda) \geq \int r^2 d\mu_x = r^2 \|x\|^2$. Hence $A - tI$ is bounded below. Conversely suppose $t \in \text{supp}(\mu)$. For every $r > 0$, we have $\mu J \neq 0$ where $J = \mathbb{B}(t, r)$. There is a unit vector $x \in (\mu J)(H)$, that is $x = \mu J x$. Hence $A - tI$ cannot be bounded below because

$$\|Ax - tx\|^2 = \| \int (\lambda - t) d\mu(\lambda) \int \rho_J(\lambda) d\mu(\lambda)\, x\|^2 = \| \int (\lambda - t) \rho_J(\lambda) d\mu(\lambda)\, x\|^2$$
$$= \int |\lambda - t|^2 \rho_J d\mu_x \leq \int r^2 d\mu_x = r^2 \|x\|^2.$$

Therefore $t \in \text{supp}(\mu)$ iff $A - tI$ cannot be bounded below iff t is an approximate eigenvalue by §14-2.2 iff $t \in \sigma(A)$ by §14-2.4. □

26-6.4. **Theorem** For every continuous function f on $\mathbb{R}$, we have $f(A) = \int f d\mu$ where $f(A)$ is defined by §14-6.5 and $\int f d\mu$ by §26-3.2. In particular, we have $A = \int t d\mu(t)$. Therefore, μ is the spectral measure of A.

Proof. Firstly, assume that f is a real function. Choose $a, b \in \mathbb{R}$ such that $a < \sigma(A) < b$. Subdivide $[a, b]$ with $a = c_0 < c_1 < \cdots < c_n = b$ where $c_j = a + j(b - a)/2^n$. Define $\alpha_j = \sup\{f(t) : c_{j-1} \leq t \leq c_j\}$, $\beta_j = \max\{\alpha_j, \alpha_{j+1}\}$ and $f_n = \sum_{j=1}^{2^n} \alpha_j \rho_{(c_{j-1}, c_j]}$.



As shown in the above picture, define g_n by joining the points (c_j, β_j) and extend it outside $[a, b]$ to a continuous function on $\mathbb{R}$. From $f_n \downarrow f$ on $\sigma(A)$, we have $\int f_n d\mu \to \int f d\mu$, strongly. Since $f \leq f_n \leq g_n$ on $\sigma(A)$, we have $f(A) \leq f_n(A) \leq g_n(A)$ by §26-1.9b. It follows from §13-10.4b that

$$\|f_n(A) - f(A)\| \leq \|g_n(A) - f(A)\| \to 0$$

because $g_n \to f$ uniformly on $\sigma(A)$. Therefore $f_n(A) \to f(A)$ uniformly and thus strongly. From §26-1.12a, we obtain

$$\int f_n d\mu = \sum_j \alpha_j \mu(c_{j-1}, c_j] = \sum_j \alpha_j \rho_{(c_{j-1}, c_j]}(A) = f_n(A).$$

The uniqueness of strong limits gives $\int f d\mu = f(A)$. Finally for complex functions, apply the above result to the real and imaginary parts; then assemble them to complete the proof. □

26-6.5. **Corollary** Every operator can be uniformly approximated by linear combination of projectors.

Proof. Firstly let A be a self-adjoint operator. For the special case $f(t) = t$, we have $f_n(A) \to f(A)$, that is $\int f_n d\mu = \sum_j \alpha_j P_j \to A$ uniformly where $P_j = \mu(c_{j-1}, c_j]$ are projectors. In general, every operator is the linear combination of its real and imaginary parts which are both self-adjoint, the result follows immediately. □

26-6.6. **Exercise** Prove that for every semi-continuous function f on $\mathbb{R}$, we have $f(A) = \int f d\mu$ where $f(A)$ is defined by §26-1.8 and $\int f d\mu$ by §26-3.2. In particular, we have $A = \int t d\mu(t)$.

26-6.7. **Theorem** Every normal operator A has a spectral measure μ on $\mathbb{C}$.

Proof. Let $A = B + iC$ be a normal operator where B, C are commuting self-adjoint operators and $i^2 = -1$. Let π, ν be the spectral measures of B, C respectively. Since B commutes with C, the operator B commutes with νK for all measurable set K in $\mathbb{R}$. This in turn implies πJ commutes with νK for every measurable set K in $\mathbb{R}$. Thus μ is a spectral measure on $\mathbb{R}^2$ which is identified as $\mathbb{C}$. Because both π, ν are normalized with compact support, so is μ. Consequently every continuous function f on $\mathbb{C}$ is μ-integrable. In particular, $\int \lambda d\mu(\lambda)$ is well-defined. Write $\lambda = s + it$ where s, t are real variables. The proof is completed by the calculation: $\int \lambda d\mu(\lambda) = \int s d(\pi \otimes \nu)(s,t) + i \int t d(\pi \otimes \nu)(s,t) = \int s d\pi(s) + i \int t d\nu(t) = B + iC = A$. By the way, if A is self-adjoint, then $C = 0$, $\nu = 0$ and $A = \int \lambda d\mu(\lambda) = \int s d\pi(s) = B$. We may identify μ and π. □

26-6.8. Let μ be the spectral measure of a normal operator A. For every μ-integrable function f, define $f(A) = \int f d\mu$. By §26-6.4, the new definition agrees real spectral measures of self-adjoint operators. Since μ is of compact support, every continuous function is integrable. In particular, every complex polynomial $g(\lambda) = \sum_{j,k=1}^m \alpha_{jk} \lambda^j \overline{\lambda}^k$ is integrable. Furthermore §§26-3.3,5; we have $g(A) = \int g d\mu = \sum_{j,k=1}^m \alpha_{jk} A^j (A^*)^k$. The following is an explicit formula to approximate $f(A)$.

26-6.9. **Theorem** Let f be a continuous function on $\mathbb{K}$. For every $\varepsilon > 0$, there is a complex polynomial g such that $\|f(A) - g(A)\| \leq \varepsilon$.

Proof. Since $\sigma(A) = \text{supp}(\mu)$ is compact, for every $\varepsilon > 0$, there is a complex polynomial g such that $|f(t) - g(t)| \leq \varepsilon$ for all $t \in \text{supp}(A)$. Hence we have $\| \int f d\mu - \int g d\mu\| = \|f - g\|_\infty \leq \varepsilon$. □

26-6.10. **Theorem** $\sigma[f(A)] = f[\sigma(A)]$ for every continuous function f on $\mathbb{C}$. By §26-6.4, this generalizes §14-6.11 from self-adjoint to normal operators.

Proof. Take $\lambda \in \sigma(A)$ but $f(\lambda) \notin \sigma[f(A)]$. Then $f(A) - f(\lambda)I$ is invertible. By §26-4.11, there is $\delta > 0$ such that $|g| \geq \delta$, μ-ae where $g(t) = f(t) - f(\lambda)$. There is μ-null set N with $|g| \geq \delta$ on $X \setminus N$. Since f is continuous, there is $r > 0$ such that $|g(t)| \leq \frac{1}{2}\delta$ for all $t \in \mathbb{B}(\lambda, r)$, i.e. $\mathbb{B}(\lambda, r) \subset N$. Thus $\mathbb{B}(\lambda, r)$ is a μ-null open set. The contradiction $\lambda \in \mathbb{B}(\lambda, r) \cap \sigma(A) = \mathbb{B}(\lambda, r) \cap \text{supp}(\mu) = \emptyset$ shows $f[\sigma(A)] \subset \sigma[f(A)]$. Conversely, take any $\lambda \notin f[\sigma(A)]$. Then $g(t) = f(t) - \lambda$ never vanishes on the compact set $\sigma(A) = \text{supp}(A)$. There is $\delta > 0$ such that $|g| \geq \delta$ on $\text{supp}(A)$. Then $f(A) - \lambda I = \int f d\mu - \lambda I = \int g d\mu$ is invertible, that is $\lambda \notin \sigma[f(A)]$. Therefore $\sigma[f(A)] \subset f[\sigma(A)]$. □

26-6.11. **Theorem** Let μ, ν be spectral measures on $X = \mathbb{K}$ of normal operators A, B respectively. Let U be a unitary operator such that $\nu J = U^*(\mu J)U$ for all semi-intervals J. Then a measurable function f is μ-integrable iff it is ν-integrable. Furthermore we have $\int f d\nu = U^*(\int f d\mu)U$. In particular, we have $B = U^* A U$.

Proof. By uniqueness of extension, we have $\nu M = U^*(\mu M)U$ for all measurable sets M. Choose simple functions $f_n \to f$ with $|f_n| \uparrow |f|$. Write $f_n = \sum_{j=1}^k \alpha_j \rho_{M_j}$ where $\alpha_j \in \mathbb{C}$ and M_j are disjoint measurable sets. Suppose that f is μ-integrable. Take any $x \in H$ and let $y = Ux$. Then we have

$$\int |f_n| d\nu_x = \sum_{j=1}^k |\alpha_j| < \nu M_k x, x >= \sum_{j=1}^k |\alpha_j| < U^*(\mu M_k)Ux, x >$$

$$= \sum_{j=1}^k |\alpha_j| < \mu M_k y, y >= \int |f_n| d\mu_y \leq \int |f| d\mu_y < \infty.$$

By Monotone Convergence Theorem, $|f|$ and hence f are ν_x-integrable. The converse follows by symmetry. Letting $n \to \infty$ in $\int f_n d\nu_x = \int f_n d\mu_y$, we obtain $\int f d\nu_x = \int f d\mu_y$, that is

$$< (\int f d\nu)x, x >=< (\int f d\mu)y, y >=< (\int f d\mu)Ux, Ux >=< U^*(\int f d\mu)Ux, x >.$$

Since x is arbitrary, we get $\int f d\nu = U^*(\int f d\mu)U$ as required. □

26-6.12. **Theorem** Let μ, ν be spectral measures on of normal operators A, B

respectively. Let U be a unitary operator such that $B = U^*AU$. Then we have $\nu M = U^*(\mu M)U$ for all measurable sets M.

Proof. Let $\pi M = U^*(\mu M)U$ for all measurable sets M. Then π is a spectral measure of $\int \lambda d\pi(\lambda) = U^*[\int \lambda d\mu(\lambda)]U = U^*AU = B$. By uniqueness of spectral measure of B, we have $\nu = \pi$. □

26-6.13. **Theorem** If an operator B commutes with A, then B commutes with $f(A)$ for every μ-integrable function f.

Proof. Consider the special case that $B = U$ is unitary. From $UA = AU$, we have $A = U^*AU$. It follows $\mu M = U^*\mu MU$ for all measurable sets M. Consequently $f(A) = \int f d\mu = U^*(\int f d\mu)U = U^*f(A)U$, that is $Uf(A) = f(A)U$. In general, the family $\mathcal{F} = \{A, A^*\}$ is self-adjoint and $B \in \mathcal{F}'$ by Fuglede's Theorem. From §14-6.18, B is a linear combination of unitary operators in $\mathcal{F}'$. The proof is completed by the first case. □

26-6.14. **Theorem** λ is an eigenvalue of A iff $\mu\{\lambda\} \neq 0$. Furthermore the eigenspace corresponding to λ is the range of $\mu\{\lambda\}$, i.e. $ker(A - \lambda I) = \mu\{\lambda\}(H)$.

Proof. ($\Rightarrow$) Let x be a nonzero eigenvector corresponding to the eigenvalue λ. For each n, let $Q_n = \{t \in X : |t - \lambda| \geq \frac{1}{n}\}$ and $Q = X \setminus \{\lambda\}$. Define $f(t) = t - \lambda$ and

$$f_n(t) = \begin{cases} 1/(t - \lambda) & \text{if } t \in Q_n \,; \\ 0 & \text{if } t \notin Q_n \,. \end{cases}$$

The bounded measurable functions and the continuous function f are all μ-integrable. Observe that

$$\begin{aligned} \mu Q_n x &= \int \rho_{Q_n} d\mu \; x = \int f_n f d\mu \; x \\ &= \int f_n d\mu \int (t - \lambda) d\mu(t) \; x = (\int f_n d\mu)(A - \lambda I)x = 0. \end{aligned}$$

Since $Q_n \uparrow Q$, we have

$$\|\mu Q x\|^2 = <\mu Q x, x> = \mu_x Q = \lim \mu_x Q_n = \lim <\mu Q_n x, x> = \lim \|\mu Q_n x\|^2 = 0.$$

Hence $\mu\{\lambda\}x = \mu X x - \mu Q x = Ix = x \neq 0$. Therefore we have $\mu\{\lambda\} \neq 0$ and $ker(A - \lambda I) \subset \mu\{\lambda\}(H)$.

($\Leftarrow$) Pick any nonzero $x \in \mu\{\lambda\}(H)$. Because $\mu\{\lambda\}$ is a projector, we have

$$\begin{aligned} Ax &= [\int t d\mu(t)] \; \mu\{\lambda\}x = [\int t d\mu(t)] \; \int \rho_{\{\lambda\}} d\mu \; x = \int t \rho_{\{\lambda\}}(t) d\mu(t) \; x \\ &= \int \lambda \rho_{\{\lambda\}}(t) d\mu(t) \; x = \lambda \int \rho_{\{\lambda\}}(t) d\mu(t) \; x = \lambda \mu\{\lambda\}x = \lambda x. \end{aligned}$$

Hence λ is an eigenvalue with the corresponding eigenvector x. Therefore $\mu\{\lambda\}(H) \subset ker(A - \lambda I)$. □

26-6.15. **Corollary** Every isolated point of the spectrum $\sigma(A)$ of a normal operator A is an eigenvalue.

Proof. Being an isolated point, there is an open set Q such that $Q \cap \sigma(A) = \{\lambda\}$. If $\mu Q = 0$, then $\emptyset = Q \cap \text{supp}(A) = Q \cap \sigma(A)$ is a contradiction. Therefore $\mu\{\lambda\} = \mu[Q \cap \text{supp}(\mu)] = \mu Q \neq 0$. Consequently, λ is an eigenvalue. □

26-6.16. **Exercise** Let $A = diag(2, 2, 3, 3, 3, 0)$ be a diagonal matrix. Express $\mu\{2\}$ as a projector-matrix. Show that $2\rho_{\{2\}} + 3\rho_{\{3\}}$ equals $f(t) = t$, μ-ae. Find $\int \rho_{\{2\}} d\mu$ and interpret $\int \lambda d\mu(\lambda)$ in terms of matrices.

26-99. **References and Further Readings** : Berberian-66, Garnir-74, Helmberg, Lugovaya and Blank.

Chapter 27

Locally Compact Spaces

27-1 Regular Measures

27-1.1. Throughout this chapter, elementary properties of locally compact spaces will be assumed. Both $\mathbb{R}^n$ with usual topology and an infinite discrete metric space should be used as typical examples. In this section we shall start with the general framework of decent sets and measurable sets. Note that measurable sets in this book have nothing to do with any measure. To control the measures in terms of topology, we require the regularity which will guarantee that the continuous functions with compact support are dense in L_p for every $1 \leq p < \infty$.

27-1.2. Let X be a separated locally compact space and F, E, FE Banach spaces with an admissible bilinear map $F \times E \to FE$. Let $\mathbb{D}$ be the δ-ring generated by the family of all compact subsets of X. Hence every decent set is relatively compact by §18-1.6b. Also a set M is measurable iff for each compact set V, $M \cap V$ is a decent set by §19-1.3. A measure μ on the δ-ring of decent sets of X is also called a measure *on* X.

27-1.3. **Lemma** The decent family $\mathbb{D}$ is also the δ-ring generated by open relatively compact sets. A set M is measurable iff $M \cap V$ is a decent set for each open relatively compact set V.

Proof. Let V be an open relatively compact set. Then both the closure $\overline{V}$ and the boundary ∂V are compact sets and hence $V = \overline{V} \setminus \partial V$ is a decent sets. On the other hand, let A be any compact set. There is an open relatively compact neighborhood V of A. Then $V \setminus A$ is an open relatively compact set. Hence $A = V \setminus (V \setminus A)$ belongs to the δ-ring generated by open relatively compact sets. The last statement follows by §19-1.3. □

27-1.4. **Theorem** Open sets, closed sets and compact sets are measurable. Open relatively compact sets are decent sets. Every continuous map is measurable.

Proof. Let M be an open set. If V is any open relatively compact set, then so is $M \cap V$. Hence M is measurable. Since complements of measurable sets are measurable, all close sets are measurable. Because compact sets in separated topological spaces are closed, they are also measurable. An open relatively compact set V is a measurable subset of the decent set $\overline{V}$ and hence V is a decent set. The proof of last statement is similar to §23-2.4. □

27-1.5. **Exercise** The *long line* X consists of all points in $\mathbb{R}^2$. Let $\mathcal{T}$ be the family of subsets V satisfying the following conditions: for every $(a, b) \in V$ there is $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \times \{b\}$ is contained in V. Prove that $\mathcal{T}$ is a topology on X. Show that X becomes a separated locally compact space. Verify that the set $Q \times \mathbb{R}$ is a measurable set where Q denote the set of all rational numbers. Does the set $Q \times \mathbb{R}$ belong to the σ-algebra generated by open sets?

27-1.6. **Theorem** Let μ be a vector measure on X into E. Every continuous map $g : X \to F$ is locally integrable. Every continuous map $h : X \to F$ with compact support belongs to $L_p(X, F)$ for all $1 \le p \le \infty$.

Proof. Every decent set A has compact closure $\overline{A}$ on which the continuous map g is bounded, say $|g\rho_A| \le t\rho_A$ for some $0 \le t < \infty$. Since the decent function $t\rho_A$ is $|\mu|$-integrable, so is $g\rho_A$. Hence g is μ-integrable on A. In particular for $1 \le p < \infty$, $|h|^p$ is integrable on its support and hence it is integrable on X, i.e. $h \in L_p(X, F)$. Finally, since h is bounded, it also belongs to $L_\infty(X, F)$. □

27-1.7. A vector measure μ on X into E is said to be *regular* if

(a) μ is *outer regular* at every decent set A, that is for every $\varepsilon > 0$, there is an open relatively compact set V such that $A \subset V$ and $|\mu|(V \setminus A) \le \varepsilon$; and

(b) μ is *inner regular* at every open relatively compact set V, that is for every $\varepsilon > 0$, there is a compact set A such that $A \subset V$ and $|\mu|(V \setminus A) \le \varepsilon$.

27-1.8. **Exercise** Prove that linear combinations of regular measures are regular. The variation of a regular measure is regular.

27-1.9. **Exercise** Let μ be a regular vector measure on X. Prove that for every open relatively compact set V and for every $\varepsilon > 0$, there is an open set W such that $\overline{W}$ is a compact subset of V and $|\mu|(V \setminus \overline{W}) \le \varepsilon$.

27-1.10. **Compact Regularity Theorem** Let μ be a regular vector measure on X. Then for every decent set A and for every $\varepsilon > 0$, there is a compact subset B of A such that $|\mu|(A \setminus B) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. There is an open relatively compact set V containing A such that $|\mu|(V \setminus A) \leq \varepsilon$. Again, there is an open relatively compact set W containing $V \setminus A$ and satisfying $|\mu|\{W \setminus (V \setminus A)\} \leq \varepsilon$. Finally, there is a compact subset Q of V such that $|\mu|(V \setminus Q) \leq \varepsilon$. Now $B = Q \setminus W$ is compact. Furthermore, $B \subset V \setminus W \subset V \setminus (V \setminus A) = A$. The proof is complete by the following calculation:

$$\begin{aligned} |\mu|(A \setminus B) &\leq |\mu|(V \setminus B) \leq |\mu|(V \setminus [Q \setminus W]) \leq |\mu|(V \setminus Q) + |\mu|W \\ &\leq \varepsilon + |\mu|(W \setminus [V \setminus A]) + |\mu|(V \setminus A) \leq 3\varepsilon. \end{aligned}$$ □

27-1.11. **Theorem** Let μ be a regular vector measure on X into E. Then the set $K(X, F)$ of all continuous maps with compact support is dense in $L_p(X, F)$ for every $1 \leq p < \infty$. Furthermore if $0 \leq f \in L_p^+$, then for every $\varepsilon > 0$ there is a continuous function $g \geq 0$ with compact support such that $\|f - g\|_p \leq \varepsilon$.

Proof. Let A be a decent set and $\varepsilon > 0$ be given. There is an open relatively compact set V such that $A \subset V$ and $|\mu|(V \setminus A) \leq \varepsilon$. There is also a compact subset Q of A such that $|\mu|(A \setminus Q) \leq \varepsilon$. Take any continuous function g satisfying $\rho_Q \leq g \leq \rho_V$. Then we have $g(X) \subset [0, 1]$ and

$$|g - \rho_A| = \max(g, \rho_A) - \min(g, \rho_A) \leq \rho_V - \rho_Q = \rho_{V \setminus Q}.$$

Hence

$$\int |g - \rho_A|^p d|\mu| \leq \int \rho_{V \setminus Q} d|\mu| \leq |\mu|(V - Q) \leq |\mu|(V \setminus A) + |\mu|(A \setminus Q) \leq 2\varepsilon.$$

Therefore $\|g - \rho_A\|_p \leq (2\varepsilon)^{1/p} \to 0$ as $\varepsilon \downarrow 0$. Next consider the general case. Let $f \in L_p(X, F)$ and $\varepsilon > 0$ be given. By Density Theorem, there is a decent map g such that $\|f - g\|_p \leq \varepsilon$. Write $g = \sum_{j=1}^n \alpha_j \rho_{A_j}$ where A_j are disjoint decent sets and $\alpha_j \in F$. For each j, there is a continuous function $h_j : X \to [0, 1]$ such that $\|h_j - \rho_{A_j}\| \leq \dfrac{\varepsilon}{n(1 + \|\alpha_j\|)}$. Now $h = \sum_{j=1}^n \alpha_j h_j$ is a continuous map with compact support satisfying

$$\|g - h\|_p \leq \textstyle\sum_{j=1}^n \|\alpha_j(\rho_{A_j} - h_j)\|_p = \sum_{j=1}^n \|\alpha_j\| \; \|\rho_{A_j} - h_j\|_p \leq \varepsilon$$

that is $\|f - h\|_p \leq 2\varepsilon$. Finally, if $f \geq 0$, then we may choose $g \geq 0$. Since A_j's are disjoint, all $\alpha_j \geq 0$ and hence $h \geq 0$. □

27-1.12. **Theorem** Let μ be a positive measure on X. Then the integration $\int g d\mu$ is a positive linear form in g on the complex vector lattice $K(X)$. In next section, we shall recover μ from its integration.

27-1.13. **Exercise** Let μ be a regular vector measure on X. Prove the following statements.
(a) A measurable set N is null if $N \cap A$ is null for every compact set A.
(b) For every integrable set M and every $\varepsilon > 0$, there is a compact subset A of M such that $|\mu|(M \setminus A) \leq \varepsilon$.
(c) For every μ-σ-finite set H, there exist compact sets $A_m \subset A_{m+1}$ and a μ-null set N such that $H = \bigcup_{m=1}^{\infty} A_m \cup N$.

27-1.98. Let X be an infinite dimensional normed space. Then it is not locally compact. The empty set is the only open relatively compact set. Furthermore, the zero-function is the only continuous function with compact support. In this case, the whole treatment becomes trivial but meaningless. New definition of decent sets on general topological spaces is required. It is well known that nontrivial translation invariant measure on the σ-algebra generated by open sets on general infinite dimensional Banach spaces does not exist. This is why we promote almost periodic functions at the end of this book and include our own work in that direction drawing the attention of community at large.

27-2 Construction from Positive Linear Forms

27-2.1. Suppose that Riemann integral of continuous functions on closed bounded intervals of $\mathbb{R}$ has been constructed. It is trivially extended to a positive linear form I on the vector lattice of continuous functions with compact support on $\mathbb{R}$. Assuming this background, how do we construct the Lebesgue measure? This section will identify each positive linear form I with a regular positive measure.

27-2.2. Throughout this section, let I be a positive linear form on the complex vector lattice $K(X)$ of continuous functions with compact support on a separated locally compact space X. Let $K^+(X)$ denote the set of positive functions $f \geq 0$ in $K(X)$. For convenience, write $M \leq f$ if $\rho_M \leq f$ where M is a subset of X and f a real function on X. In this case, we say that M is dominated by f. Similar notations: $f \leq M, f \geq M, M \geq f$ are also understood. For every open relatively compact set V, define $\lambda V = \sup\{I(f) : V \geq f \in K^+(X)\}$. Clearly $\lambda\emptyset = 0$. Since every relatively compact set V is dominated by some $h \in K^+(X)$, we have $0 \leq \lambda V \leq I(h) < \infty$. An outer measure λ^* on X from which a positive measure on λ^*-measurable sets is obtained. It is important to realize that λ^*-measurable sets defined in terms of outer measure may not be measurable in terms of decent sets.

27-2.3. **Lemma** For every open relatively compact set V, we have $\lambda^* V = \lambda V$.

Proof. Take any $f \leq V$ in $K^+(X)$. Then $A = \text{supp}(f)$ is compact. Suppose $\{W_j\}$ is a sequential cover of V by open relatively compact sets. Then it is also a cover of A. There is a finite subcover $\{W_j : j \leq n\}$ of the compact set A. Let $\{\pi_j : j \leq n\}$ be a partition of unity on A subordinated to $\{W_j : j \leq n\}$. Then we have $\pi_j \leq W_j$, for each $j \leq n$. So,

$$I(f) \leq I\left(\sum_{j=1}^{n} \pi_j\right) \leq \sum_{j=1}^{n} I(\pi_j) \leq \sum_{j=1}^{n} \lambda W_j \leq \sum_{j=1}^{\infty} \lambda W_j.$$

Taking infimum over $\{W_j\}$, we have $I(f) \leq \lambda^* V$. Thus $\lambda V = \sup I(f) \leq \lambda^* V$. Since $\lambda^* V \leq \lambda V$ is always true, the proof is complete. □

27-2.4. **Lemma** For every relatively compact set M, we have $\lambda^* M = \inf \lambda U$ where U runs over all open relatively compact sets containing M.

Proof. Since M is relatively compact, it is contained in some open relatively compact set W. For every $\varepsilon > 0$, there are open relatively compact sets V_j such that $M \subset \bigcup_{j=1}^{\infty} V_j$ and $\sum_{j=1}^{\infty} \lambda V_j \leq \lambda^* M + \varepsilon$. Observe that $W \cap (\bigcup_{j=1}^{\infty} V_j)$ is an open relatively compact set containing M. Observe that

$$\begin{aligned} \inf \lambda U &\leq \lambda\left[W \cap \left(\bigcup_{j=1}^{\infty} V_j\right)\right] = \lambda^*\left[W \cap \left(\bigcup_{j=1}^{\infty} V_j\right)\right] \\ &\leq \sum_{j=1}^{\infty} \lambda^* V_j \leq \sum_{j=1}^{\infty} \lambda V_j \leq \lambda^* M + \varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have $\inf \lambda U \leq \lambda^* M$. The reverse inequality follows from monotonicity of μ^*. □

27-2.5. **Lemma** For every open relatively compact set V, we have $\lambda V = \sup \lambda^* B$ where B runs over all compact subsets of V.

Proof. For every $\varepsilon > 0$, there is $f \in K^+(X)$ satisfying $f \leq V$ and $\lambda V - \varepsilon \leq I(f)$. Since the compact set $A = \text{supp}(f)$ is contained in the open set V, there is an open set W such that $A \subset W \subset \overline{W} \subset V$. There is $g \in K^+(X)$ such that $g(A) = 1$ and $\text{supp}(g) \subset W$. Since $f \leq g$, we obtain $\lambda V - \varepsilon \leq I(f) \leq I(g) \leq \lambda W$. Taking the infimum over W, we get $\lambda V - \varepsilon \leq \lambda^* A \leq \sup\ \lambda^* B$. Letting $\varepsilon \downarrow 0$, we have $\lambda V \leq \sup \lambda^* B$. The reverse inequality is obvious. □

27-2.6. **Lemma** If A, B are disjoint compact sets, then we have

$$\lambda^*(A \cup B) = \lambda^* A + \lambda^* B.$$

Proof. Let V, W be open neighborhoods of A, B respectively such that their closures $\overline{V}, \overline{W}$ are disjoint compact sets. For every $\varepsilon > 0$, there is an open relatively compact set $U \supset A \cup B$ such that $\lambda U \leq \lambda^*(A \cup B) + \varepsilon$. There are $f, g \in K^+(X)$ such that $f \leq U \cap V$, $g \leq U \cap W$, $\lambda(U \cap V) \leq I(f) + \varepsilon$ and $\lambda(U \cap W) \leq I(g) + \varepsilon$. Since V, W are disjoint, $f + g \leq U$. Therefore

$$\begin{aligned} \lambda^* A + \lambda^* B &\leq \lambda(U \cap V) + \lambda(U \cap W) \leq I(f) + I(g) + 2\varepsilon \\ &\leq I(f+g) + 2\varepsilon \leq \lambda U + 2\varepsilon \leq \lambda^*(A \cup B) + 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have $\lambda^* A + \lambda^* B \leq \lambda^*(A \cup B)$. The subadditivity of outer measure completes the proof. □

27-2.7. **Lemma** If a compact set A is disjoint from an open relatively compact set V, then we have $\lambda^*(A \cup V) = \lambda^* A + \lambda^* V$.

Proof. Take any compact subset B of V. Since A, V are disjoint, so are A, B. Hence $\lambda^* A + \lambda^* B \leq \lambda^*(A \cup B) \leq \lambda^*(A \cup V)$. Taking supremum over all compact subsets B of V, we have $\lambda^* A + \lambda^* V \leq \lambda^*(A \cup V)$. The reverse inequality is obvious. □

27-2.8. **Theorem** Every open set V is λ^*-measurable.

Proof. Take any open relatively compact set W. For every compact subset A of $V \cap W$, it is disjoint from the open relatively compact set $W \setminus A$. Hence we have $\lambda^* A + \lambda^*(W \setminus V) \leq \lambda^* A + \lambda^*(W \setminus A) = \lambda^* W = \lambda W$. Therefore, $\lambda^*(W \cap V) + \lambda^*(W \setminus V) \leq \lambda W$. The result follows by §18-2.6. □

27-2.9. In particular, every open relatively compact set is λ^*-measurable. Consequently, all decent sets are λ^*-measurable. The restriction of the outer measure λ^* defines a positive measure μ on the family $\mathbb{D}$ of decent sets. It is called the *measure induced by* the linear form I.

27-2.10. **Lemma** Let A be a compact set and let $f \in K^+(X)$ be given.
(a) If $A \leq f$ then $\mu A \leq I(f)$.
(b) If $A \geq f$ then $\mu A \geq I(f)$.

Proof. (a) For every $0 < \varepsilon < 1$, $V = \{x \in X : f(x) > 1 - \varepsilon\}$ is an open set containing A. For any $g \in K^+(X)$ with $g \leq V$, we have $g \leq f/(1-\varepsilon)$ and hence $\mu A \leq \mu V = \sup I(g) \leq I(f)/(1-\varepsilon)$. The result follows as $\varepsilon \downarrow 0$.
(b) Take any open relatively compact neighborhood V of A. Then $f \leq V$. Hence $I(f) \leq \lambda V$. Since V is arbitrary, we get $I(f) \leq \lambda^* A = \mu A$. □

27-2.11. **Riesz Representation Theorem** For every positive linear form I on $K(X)$, there is a regular positive measure μ on the decent sets of X such that for every $f \in K(X)$, we have $I(f) = \int f d\mu$.

Proof. Let μ be constructed by previous lemmas of this section. Firstly, assume $f \geq 0$. Let $\varepsilon > 0$ be given. Since f is bounded, there is some integer n such that $|f| < n\varepsilon$. For each integer $1 \leq j \leq n$, define

$$g_j(x) = \begin{cases} 0, & \text{if } f(x) \leq (j-1)\varepsilon \ ; \\ f(x) - (j-1)\varepsilon, & \text{if } (j-1)\varepsilon < f(x) \leq j\varepsilon \ ; \\ \varepsilon, & \text{if } j\varepsilon < f(x). \end{cases}$$

Then $g_j \in K^+(X)$. Clearly both $A_0 = \text{supp}(f)$ and $A_j = \{x \in X : f(x) \geq j\varepsilon\}$ are compact sets satisfying $A_j \leq (1/\varepsilon)g_j \leq A_{j-1}$. Since $f = \sum_{j=1}^n g_j$, integrating with respect to μ gives $\sum_{j=1}^n \varepsilon\mu A_j \leq \int f d\mu \leq \sum_{j=1}^n \varepsilon\mu A_{j-1}$. By last lemma, we have $\varepsilon\mu A_j \leq I(g_j) \leq \varepsilon\mu A_{j-1}$, and hence $\sum_{j=1}^n \varepsilon\mu A_j \leq I(f) \leq \sum_{j=1}^n \varepsilon\mu A_{j-1}$. Consequently, $\left|I(f) - \int f d\mu\right| \leq \sum_{j=1}^n \varepsilon\mu A_{j-1} - \sum_{j=1}^n \varepsilon\mu A_j = \varepsilon\mu A_0$ because $A_n = \emptyset$. Letting $\varepsilon \downarrow 0$, we have $I(f) = \int f d\mu, \forall\ f \in K^+(X)$. By linearity, we have $I(f) = \int f d\mu$, $\forall\ f \in K(X)$. The regularity of μ follows immediately from §§27-2.4,5. □

27-3 Representations of Order-Bounded Linear Forms

27-3.1. Let X be a separated locally compact space and $K(X)$ the set of all continuous functions with compact support. In this section, we shall identify the order-bounded linear forms on $K(X)$ with regular complex measures. This identification actually preserves lattice operations. For every $f \in K(X)$, let $\|f\| = \sup\{|f(x)| : x \in X\}$.

27-3.2. **Lemma** $K(X)$ is a breakable vector lattice. Consequently, the dual space of order bounded linear forms is also a vector lattice.

Proof. Clearly it is a vector lattice under pointwise operations. Let $|f| \leq g+h$ where $f, g, h \in K(X)$ with $g, h \geq 0$. Write $|f| = g_1 + h_1$ where $0 \leq g_1 \leq g$, and $0 \leq h_1 \leq h$ are in $K(X)$. Define $g_2 = g_1\text{sgn}(f)$ and $h_2 = h_1\text{sgn}(f)$. Take any point $a \in X$. Since $\text{sgn}(f) = f/|f|$ is continuous at each point $a \in X$ satisfying $f(a) \neq 0$, so is g_2. Now suppose $f(a) = 0$. For every $\varepsilon > 0$, there is a neighborhood V of a such that for every $x \in V$ we have $|f(x)| \leq \varepsilon$. Because $|g_2| \leq g_1 \leq |f|$, we have $|g_2(x) - g_2(a)| \leq |f(x)| \leq \varepsilon$ for all $x \in V$. Hence g_2

is continuous at every point $a \in X$. Since $|f|$ is of compact support, so is g_2. Therefore g_2, h_2 belong to $K(X)$. Clearly $|g_2| \le g_1 \le g; |h_2| \le h_1 \le h$ and $f = g_2 + h_2$. Consequently, $K(X)$ is breakable. □

27-3.3. **Lemma** Let I be a linear form on $K(X)$. Then I is order-bounded iff for every compact set A, there is $t > 0$ such that for all $g \in K(X)$ with $\text{supp}(g) \subset A$, we have $|I(g)| \le t\|g\|$.

Proof. ($\Rightarrow$) Assume that I is order-bounded and let $|I|$ denote its variation. Take any compact set A. Let $f \in K^+(X)$ satisfy $f(A) = 1$. Now for any $0 \ne g \in K(X)$ with $\text{supp}(g) \subset A$, we have $|g| \le \|g\| f$ and hence

$$|I(g)| \le |I|(|g|) \le |I|(\|g\|f) = t\|g\| \text{ where } t = |I|(f).$$

($\Leftarrow$) Let $f \in K^+(X)$ be given. Then $A = \text{supp}(f)$ is compact. Choose $t > 0$ by given condition. Then for any $g \in K(X)$ with $|g| \le f$, we have $\text{supp}(g) \subset A$. Hence $|I(g)| \le t\|g\| \le t\|f\| < \infty$. Therefore I is order-bounded. □

27-3.4. **Theorem** Let μ be a regular complex measure on X. For every $f \in K(X)$, let $I(f) = \int f d\mu$. Then I is an order-bounded linear form on $K(X)$. Furthermore, we have $|I|(f) = \int f d|\mu|$ for all $f \in K(X)$.

Proof. For any $|g| \le f$ in $K(X)$, we have $|I(g)| = \left|\int g d\mu\right| \le \int |g|\, d|\mu| \le \int f d|\mu| < \infty$. Hence I is order-bounded and $|I|(f) = \sup |I(g)| \le \int f d|\mu|$. Conversely, take any $f \in K^+(X)$ and any $\varepsilon > 0$. Clearly $g = (f - \varepsilon) \vee 0 \in K^+(X)$ and $f \le g + \varepsilon \rho_B$ where $B = \text{supp}(g)$. There is a simple function $0 \le s \le g$ such that $\int g d|\mu| \le \int s d|\mu| + \varepsilon$. Write $s = \sum_{j=1}^k \alpha_j \rho_{M_j}$ where M_j are disjoint measurable sets and $\alpha_j > 0$ for all j. Since $M_j \subset B$, it is a decent set. By inner regularity, there is a compact set $A_j \subset M_j$ such that $|\mu|(M_j \setminus A_j) \le \varepsilon/(k\alpha_j)$. Now $t = \sum_{j=1}^k \alpha_j \rho_{A_j}$ is also a decent function satisfying $t \le g$ and $\int g d|\mu| \le \int t d|\mu| + 2\varepsilon$. By outer regularity, there is an open relatively compact set V_j containing A_j such that $|\mu|(V_j \setminus A_j) \le \varepsilon/(k\alpha_j)$. Observe that for every $x \in A_j$, $\alpha_j \le g(x) \le f(x) - \varepsilon$. Hence we have $A_j \subset \{x \in X : f(x) > \alpha_j\}$. Thus we may assume that for every $x \in V_j$ we get $f(x) > \alpha_j$. Since A_j are disjoint compact sets, we may also assume that V_j are disjoint by local compactness of X. Let $h_j \in K(X)$ satisfy $A_j \le h_j \le V_j$. Then $h = \sum_{j=1}^k \alpha_j h_j \in K^+(X)$. For any $x \in V_j$ we obtain $h(x) = \alpha_j h_j(x) \le \alpha_j < f(x)$. Therefore $|h| \le f$. Furthermore,

$$\int g d|\mu| \le \int t d|\mu| + 2\varepsilon \le \left|\int h d\mu\right| + \sum_{j=1}^k \alpha_j \int (h_j - \rho_{A_j}) d|\mu| + 2\varepsilon$$

$$\leq |I(h)| + \sum_{j=1}^{k} \alpha_j \int (\rho_{V_j} - \rho_{A_j}) d|\mu| + 2\varepsilon$$

$$\leq |I(h)| + \sum_{j=1}^{k} \alpha_j |\mu|(V_j \setminus A_j) + 2\varepsilon \leq |I(h)| + 3\varepsilon \leq |I|(f) + 3\varepsilon,$$

that is $$\int f d|\mu| \leq \int g d|\mu| + \varepsilon|\mu|(B) \leq |I|(f) + 3\varepsilon + \varepsilon|\mu|(B).$$

Letting $\varepsilon \downarrow 0$, we have $|I|(f) = \int f d|\mu|$ for all $f \in K^+(X)$. The proof follows by linearity in f. □

27-3.5. **Complex Representation Theorem** For every order-bounded linear form I on $K(X)$, there is a unique regular complex measure μ on the decent sets of X such that for every $f \in K(X)$, we have $I(f) = \int f d\mu$. This correspondence preserves the lattice-operations. Furthermore, if I is positive (respectively real), then so is μ.

Proof. Write I as a linear combination of positive linear forms. As a result, the existence follows immediately from Riesz Representation Theorem. The preservation of lattice-operations was proved in last theorem. For uniqueness, let μ, λ be regular complex measures such that $\int f d(\mu - \lambda) = 0$ for all $f \in K(X)$ which is dense in L_1 for $\mu - \lambda$. Hence $\int f d(\mu - \lambda) = 0$ for all decent functions f. Therefore $\mu M = \lambda M$ for every decent set M, that is $\mu = \lambda$. □

27-3.6. **Corollary** Let X be a separated compact space and $C_\infty(X)$ the Banach space of all continuous functions on X. For every regular measure μ on X, let I_μ be the corresponding integration. Then $\mu \to I_\mu$ is an isometry from the Banach space of regular measures onto the dual space of $C_\infty(X)$.

Proof. By §27-3.3, the dual space $C'_\infty(X)$ of $C_\infty(X)$ is identical to order-bounded linear forms with $\|I\| = |f|(1)$. On the other hand, regular measures μ on compact space X are bounded with $\|\mu\| = |\mu|(X)$. The isometry follows as a special case of §27-3.4. □

27-99. **References and Further Readings** : Dinculeanu-74, Nachbin, Brooks and Uglanov.

Chapter 28

Almost Periodic Functions on Groups

28-1 Almost Periodicity

28-1.1. Starting with a simple example, we introduce group representations which motivates the definition of almost periodicity in terms of ε-covers. Basic properties of almost periodic functions are developed in this section.

28-1.2. **Example** The sum of two periodic functions need not be periodic.

Solution. Clearly both $\sin x$ and $\sin \pi x$ are periodic functions. Suppose to the contrary that their sum $f(x) = \sin x + \sin \pi x$ is periodic with period $p > 0$, that is $\sin(x+p) + \sin \pi(x+p) = \sin x + \sin \pi x$ for all x. Letting $x = 0$, we get $\sin p + \sin \pi p = 0$. Differentiating twice and letting $x = 0$, we have $-\sin p - \pi^2 \sin \pi p = 0$. Hence $\sin p = \sin \pi p = 0$. From $\sin \pi p = 0$, p is an integer. From $\sin p = 0$, we obtain $p = n\pi$ for some integer n. Since $p > 0$, we have $\pi = p/n$ but π is irrational. Therefore f cannot be periodic. □

28-1.3. Observe that sine and cosine functions are linear combinations of exponential functions of the form $\rho : \mathbb{R} \to GL(1)$ where $\rho(x) = e^{i\theta x}$ and $GL(1) = \{x \in \mathbb{C} : |z| = 1\}$. Clearly we have $\rho(x+y) = \rho(x) + \rho(y)$, $|\rho(x)| \leq 1$ and $\rho(x)$ is an invertible matrix of order one for all $x, y \in \mathbb{R}$. Note that harmonic analysis including special functions has been developed in the context of group representations. We vote for matrix representations for their simplicity.

28-1.4. Let G be a (multiplicative) group. The identity element of every group is denoted by the same symbol e unless it is specified otherwise. The general linear group $GL(s)$ is the multiplicative group of all invertible matrices of order s with complex entries. A homomorphism from G into $GL(s)$ is called a (matrix) ***representation*** of order s. A representation $D : G \to GL(s)$ is ***bounded*** if all entry functions D_{ij} are bounded on G. The function ρ in last paragraph is an example of a bounded representation of the additive group $\mathbb{R}$.

28-1.5. Let $f : G \to \mathbb{K}$ be a function where the scalar field $\mathbb{K}$ can be real or complex. For any $\varepsilon > 0$, a finite cover $\{A_i : 1 \leq i \leq n\}$ of G is called a ***left***

ε*-cover* for a function f if $|f(ax) - f(bx)| \le \varepsilon$ for each i and for all $a, b \in A_i$; $x \in G$. A function f is said to be *left almost periodic* on G if every $\varepsilon > 0$ have a left ε-cover of G for f. As usual, write $\|f\|_\infty = \sup\{|f(x)| : x \in G\}$.

28-1.6. **Lemma** Every entry function of a bounded representation $D = [D_{ij}]$ is left almost periodic.

Proof. Since D is bounded, define $\alpha = \max\{\|D_{ij}\|_\infty : 1 \le i, j \le s\}$ where D_{ij} are the entry functions of D. For every $\varepsilon > 0$, choose open intervals B_k of length $\le \varepsilon$ such that $[-\alpha, \alpha] \subset \cup\{B_k : 1 \le k \le m\}$. Then for every $1 \le i, j \le s$, we have $G \subset \cup\{A_{ijk} : 1 \le k \le m\}$ where $A_{ijk} = D_{ij}^{-1}(B_k)$ and s is the order of D. Thus $G \subset \cap_{ij} \cup_k A_{ijk} = \cup\{\cap_{i,j=1}^s A_{ij,k(i,j)} : 1 \le k(i,j) \le m\}$. Take any $a, b \in A_{ij,k(i,j)}$. Then both $D_{ij}(a)$ and $D_{ij}(b)$ belong to $B_{k(i,j)}$, that is $|D_{ij}(a) - D_{ij}(b)| \le \varepsilon$ for all i, j. Observe that for any $x \in G$, we have $D(ax) = D(a)D(x)$, i.e. $[D_{ij}(ax)] = [D_{ij}(a)]\ [D_{ij}(ax)]$, or $D_{ij}(ax) = \sum_{k=1}^s D_{ik}(a)D_{kj}(x)$. This standard calculation will not shown explicitly any more. From

$$\begin{aligned}|D_{ij}(ax) - D_{ij}(bx)| &= |\textstyle\sum_{k=1}^s D_{ik}(a)D_{kj}(x) - \sum_{k=1}^s D_{ik}(b)D_{kj}(x)| \\ &\le \textstyle\sum_{k=1}^s |D_{ik}(a) - D_{ik}(b)|\ |D_{kj}(x)| \le s\alpha\varepsilon,\end{aligned}$$

the family $\{\cap_{i,j=1}^s A_{ij,k(i,j)} : 1 \le i, j \le s, 1 \le k(i,j) \le m\}$ is a left $(s\alpha\varepsilon)$-cover of G for D_{ij}. Therefore D_{ij} is left almost periodic on G. □

28-1.7. In particular, the function ρ above is left almost periodic. A (two sided) ε-cover for a function $f : G \to \mathbb{K}$ is a finite cover $\{A_i : 1 \le i \le n\}$ of G such that $|f(yax) - f(ybx)| \le \varepsilon$ for every i and for all $a, b \in A_i$ and $x, y \in G$. A function $f : G \to \mathbb{K}$ is said to be (two sided) *almost periodic* on G if every $\varepsilon > 0$ have a two sided ε-cover of G for f. Similarly, right ε-cover and right almost periodic function are defined. Their properties will be assumed. Almost periodic functions are also called *ap-functions* for convenience.

28-1.8. **Lemma** Every left ap-function f is uniformly bounded.

Proof. Let $\{A_i : i \le n\}$ be a left ε-cover for f. Choose $b_i \in A_i$ for each i. Take any $a \in G$. Then $a \in A_i$ for some i. Hence $|f(a) - f(b_i)| \le \varepsilon$, or

$$|f(a)| \le |f(b_i)| + \varepsilon \le \textstyle\sum_{j=1}^n |f(b_j)| + \varepsilon.$$

Since $a \in G$ is arbitrary, f is uniformly bounded. □

28-1.9. As a subset of the Banach space $B(G)$ of all bounded functions on G, the uniform norm of every left periodic function is defined. The left translate $L_a f : G \to \mathbb{K}$ of f by $a \in G$ is defined by $L_a f(x) = f(ax)$, that is to replace the argument x by ax. Similarly, define $R_a f(y) = f(ya)$ for all $x \in G$. Clearly we have $L_{ab} = L_b L_a$ and $R_{ab} = R_a R_b$. In fact, $L_b L_a f(x) = L_b(L_a f)(x) = L_a f(bx) = f(abx) = L_{ab} f(x)$. Also $R_a R_b f(y) = R_a(R_b f)(y) = R_b f(ya) = f(yab) = R_{ab} f(y)$.

28-1.10. **Lemma** Let f be a bounded function on G. Then f is left almost periodic iff $L(f) = \{L_a f : a \in G\}$ is relatively compact in $B(G)$. For right and two sided almost periodicity, work with $R(f) = \{R_a f : a \in G\}$ and $T(f) = \{L_a R_b f : a \in G\}$ respectively.

Proof. Suppose f is left almost periodic. Choose any left ε-cover $\{A_i : i \leq n\}$ and any $c_i \in A_i$ for each i. Take any $a, x \in G$. Then $a \in A_i$ for some i. Thus $|f(ax) - f(c_i x)| \leq \varepsilon$, i.e. $|L_a f(x) - L_{c_i} f(x)| \leq \varepsilon$, or $L_a f \in \mathbb{B}(L_{c_i} f, \varepsilon)$. Therefore $L(f) \subset \cup_{i=1}^{n} \mathbb{B}(L_{c_i} f, \varepsilon)$. Consequently, $L(f)$ is precompact in $B(G)$. Since $B(G)$ is complete, $L(f)$ is relatively compact. Conversely, assume that $L(f)$ is relatively compact. By precompactness of $L(f)$, for every $\varepsilon > 0$, there are $g_1, \cdots, g_n \in B(G)$ such that $L(f) \subset \bigcup_{i=1}^{n} \mathbb{B}(g_i, \varepsilon)$. Define $A_i = \{a \in G : \|L_a f - g_i\|_\infty \leq \varepsilon\}$. It follows immediately that $\{A_i : i \leq n\}$ is a cover of G. For any $a, b \in A_i$ and $x \in G$, we have $\|L_a f - g_i\|_\infty \leq \varepsilon$ and $\|L_b f - g_i\|_\infty \leq \varepsilon$, or $\|L_a f - L_b f\|_\infty \leq 2\varepsilon$, i.e. $|f(ax) - f(bx)| = |L_a f(x) - L_b f(x)| \leq 2\varepsilon$. Hence $\{A_i : i \leq n\}$ is a left (2ε)-cover. Therefore f is left almost periodic. The other parts follow in a similar way. □

28-1.11. **Theorem** The set $A(G)$ of all left ap-functions on G is a closed subalgebra of $B(G)$. Consequently $A(G)$ is itself a Banach space under the sup-norm. Let f be a left ap-function. Then the left translate of a left ap-function is left almost periodic. The complex conjugate f^-, the inversion f^v and the hermitian f^* defined by $f^-(x) = f(x)^-$, $f^v(x) = f(x^{-1})$ and $f^*(x) = f(x^{-1})^-$ respectively are all left almost periodic.

Proof. The first statement follows immediately because the linear combinations, the product and the closure of precompact sets in $B(G)$ are precompact. The second statement follows from $L(L_b f) = L(f)$. Since the map $f \to f^- : B(G) \to B(G)$ is continuous, the image $L(f^-)$ of the relatively compact set $L(f)$ is relatively compact and hence f^- is left almost periodic. Similar argument completes the proof. □

28-1.12. **Exercise** Prove the following statement directly without using any property of compactness.

(a) Linear combinations and products of left ap-functions are left ap-functions.

(b) Uniform limits of left ap-functions are left almost periodic.

(c) The translates, conjugates, inversions, hermitians and absolute values of left ap-functions are left almost periodic.

28-1.13. **Theorem** A function f is left almost periodic iff it is two sided almost periodic. Similar result holds for right periodicity. As a result, there is no need to distinguish left, right and two sided periodicity.

Proof. Suppose that $\{A_i : i \leq n\}$ is a two sided ε-cover of G for f. Then for all $a, b \in A_i$ and all $x, y \in G$ we have $|f(yax) - f(ybx)| \leq \varepsilon$. In particular, for $y = e$ the identity element, we have $|f(ax) - f(bx)| \leq \varepsilon$. Hence $\{A_i : i \leq n\}$ is also a left ε-cover $\{A_i : i \leq n\}$. Consequently, f is left almost periodic. Conversely, suppose that f is a left ap-function. Let $\{A_i : i \leq n\}$ be a left ε-cover for f. Choose any $c_i \in A_i$. Since $L_{c_i} f$ is left almost periodic, let $\{B_{ij} : j \leq m_i\}$ be its left ε-cover. Define $\mathcal{K} = \{B_{1j(1)} \cap \cdots \cap B_{nj(n)} : j(i) \leq m_i\}$. Since $G = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m_i} B_{ij} = \cup\mathcal{K}$, $\mathcal{K}$ is a finite cover of G. Let $a, b \in B_{1j(1)} \cap \cdots \cap B_{nj(n)}$ and $x, y \in G$ be given. Then $y \in A_i$ for some i. By $y, a \in A_i$, we get $|f(yax) - f(c_i ax)| \leq \varepsilon$ and $|f(ybx) - f(c_i bx)| \leq \varepsilon$. Because $a, b \in B_{ij(i)}$, we obtain $|L_{c_i} f(ax) - L_{c_i} f(bx)| \leq \varepsilon$. Hence we have
$|f(yax) - f(ybx)| \leq |f(yax) - f(c_i ax)| + |f(c_i ax) - f(c_i bx)| + |f(c_i bx) - f(ybx)| \leq \varepsilon$.
Therefore f is two sided almost periodic. □

28-1.14. **Theorem** Let f be an ap-function on G. If φ is a continuous function on a compact set K containing the range of f, then the composite φf is almost periodic. In particular, $|f|^p$ is almost periodic for all p with $1 \leq p < \infty$.

Proof. For the last statement, the closed ball $\overline{\mathbb{B}}(0, \|f\|_\infty)$ is a compact set containing $f(G)$. To prove the first statement, let $\varepsilon > 0$ be given. By uniform continuity of φ on K, there is $\delta > 0$ such that $\alpha, \beta \in K$ and $|\alpha - \beta| \leq \delta$ implies $|\varphi(\alpha) - \varphi(\beta)| \leq \varepsilon$. Choose a δ-cover $\{A_i : i \leq n\}$ for f. For all $a, b \in A_i$ and $x \in G$, we have $f(ax), f(bx) \in K$ and $|f(ax) - f(bx)| \leq \delta$. Hence $|\varphi f(ax) - \varphi f(bx)| \leq \varepsilon$. Therefore $\{A_i : i \leq n\}$ is also a left ε-cover for φf. Consequently, φf is almost periodic on G. □

28-1.15. **Exercise** Prove that if $f \in A(G)$ with $|f| \geq r > 0$, then $1/f \in A(G)$.

28-1.16. **Corollary** $A(G)$ is a complex vector lattice. See §16-2.2.

28-1.17. **Theorem** Let G, H be groups and $\varphi : H \to G$ a homomorphism. If f is an ap-function on G, then the function $f\varphi$ is an ap-function on H.

Proof. In fact, if $\{A_i : i \leq n\}$ is a left ε-cover of G for f, then $\{\varphi^{-1}(A_i) : i \leq n\}$ is a left ε-cover of H for $f\varphi$. □

28-2 Mean-Values

28-2.1. The objective of this section is to construct a translation invariant continuous linear form on the vector space of ap-functions on a group. We need two combinatorial lemmas to accomplish this. The mean-value of a function f is the average of f over the whole group. For convenience, all indices start with 1.

28-2.2. **Lemma** Let $\{A_i : i \leq n\}$ be a family of subsets of a set G. If for every $1 \leq r \leq n$, the union of any r sets of $\{A_i : i \leq n\}$ contains at least r elements; then there are n distinct elements $x_1, \cdots, x_n \in G$ such that $x_i \in A_i$ for each i.

Proof. If $n = 1$, it is trivial. For every $B \subset G$, let νB denote the number of elements in B. Firstly, suppose there are indices $1 \leq k_1 < k_2 < \cdots < k_r \leq n$ such that $\nu \bigcup_{j=1}^{r} A_{k_j} = r < n$. Without loss of generality, we may assume that $k_j = j$ for all $j \leq r$. By induction, there are distinct elements $x_j \in A_j$ for each $j \leq r$. Define $B_i = A_i \setminus \{x_1, \cdots, x_r\}$ for each i with $r + 1 \leq i \leq n$. Then for $1 \leq p \leq n - r$ and $r + 1 \leq k_1 < k_2 < \cdots < k_p \leq n$, we have

$$\begin{aligned}
&\nu(B_{k_1} \cup B_{k_2} \cup \cdots \cup B_{k_p}) \\
&= \nu(B_{k_1} \cup B_{k_2} \cup \cdots \cup B_{k_p}) + \nu(A_1 \cup A_2 \cup \cdots \cup A_r) - r \\
&\geq \nu(B_{k_1} \cup B_{k_2} \cup \cdots \cup B_{k_p} \cup A_1 \cup A_2 \cup \cdots \cup A_r) - r \\
&= \nu(A_1 \cup A_2 \cup \cdots \cup A_r \cup A_{k_1} \cup A_{k_2} \cup \cdots \cup A_{k_p}) - r \\
&\geq (r + p) - r = p.
\end{aligned}$$

Thus $B_{r+1}, B_{r+2}, \cdots, B_n$ satisfy the inductive assumption. There are distinct elements $x_i \in B_i$ for $r + 1 \leq i \leq n$. Consequently, $x_1, x_2, \cdots, x_n$ are distinct elements satisfying $x_i \in A_i$ for each $1 \leq i \leq n$. Secondly, assume that for all k_i with $1 \leq k_1 < k_2 < \cdots < k_r < n$, we have $\nu \bigcup_{j=1}^{r} A_{k_j} \geq r+1$. Pick any $x_1 \in A_1$. Define $B_i = A_i \setminus \{x_1\}$ for all $2 \leq i \leq n$. If $2 \leq k_1 < k_2 < \cdots < k_r \leq n$, then we get $\nu(B_{k_1} \cup B_{k_2} \cup \cdots \cup B_{k_r}) = \nu(A_{k_1} \cup A_{k_2} \cup \cdots \cup A_{k_r}) - 1 \geq (r + 1) - 1 = r$. Hence by induction on $B_2, B_3, \cdots, B_n$, there are distinct elements $x_i \in B_i$ for all $2 \leq i \leq n$. Consequently, $x_1, x_2, \cdots, x_n$ are distinct elements satisfying $x_i \in A_i$ for each $1 \leq i \leq n$. □

28-2.3. **Lemma** Let $\{A_i : i \leq n\}$ and $\{B_i : i \leq n\}$ be two families of subsets of a set G. If for each $1 \leq r \leq n$, the union of r sets of $\{A_i : i \leq n\}$ meets at least r sets of $\{B_i : i \leq n\}$, then there is a permutation p of $N = \{1, 2, \cdots, n\}$ such that $A_i \cap B_{p(i)} \neq \emptyset$ for all $1 \leq i \leq n$.

Proof. Let $D_i = \{j \in N : A_i \cap B_j \neq \emptyset\}$ for each $1 \leq i \leq n$. Suppose $1 \leq r \leq n$ and $1 \leq k_1 < k_2 < \cdots < k_r \leq n$. For convenience, we may assume that each $k_i = i$. Observe that $j \in \bigcup_{i=1}^r D_i$ iff for some i, $j \in D_i$, i.e. $A_i \cap B_j \neq \emptyset$ iff $(\bigcup_{i=1}^r A_i) \cap B_j \neq \emptyset$. The given condition assures that $\bigcup_{i=1}^r D_i$ contains at least r elements. Hence the union of r sets of $D_1, D_2, \cdots, D_n$ contains at least r elements. By last lemma, choose distinct elements $p(i) \in D_i$ for each $1 \leq i \leq n$. Hence $A_i \cap B_{p(i)} \neq \emptyset$. Since $p(1), p(2), \cdots, p(n)$ are distinct elements of N, p must be a permutation. □

28-2.4. Let f be an ap-function on a group G. An ε-cover $\{A_i : i \leq n\}$ of G for f is *good* if the number n is the smallest among all ε-covers for f.

28-2.5. **Lemma** If $\{A_i : i \leq n\}$ and $\{B_i : i \leq n\}$ are good ε-covers for f, then there is a permutation p of $\{1, 2, \cdots, n\}$ such that $A_i \cap B_{p(i)} \neq \emptyset$.

Proof. It suffices to prove that the union of any r sets of $\{A_i : i \leq n\}$ meets at least r sets of $\{B_i : i \leq n\}$. Suppose it is false. Without loss of generality, we may assume that $A_1 \cup A_2 \cup \cdots \cup A_r$ meets only $B_1, B_2, \cdots, B_s$ with $1 \leq s < r \leq n$. Since $\{B_i : i \leq n\}$ is a cover of G, we have $A_1 \cup A_2 \cup \cdots \cup A_r \subset B_1 \cup B_2 \cup \cdots \cup B_s$. Hence $\{B_1, B_2, \cdots, B_s, A_{r+1}, A_{r+2}, \cdots, A_n\}$ is also a cover of G. Clearly it is an ε-cover containing at most $r - 1$ sets. This contradicts the minimality of n. □

28-2.6. **Lemma** Let $\{A_i : i \leq n\}$ be a good ε-cover for f and let $a_i \in A_i$ for each $i \leq n$. Then $\left|\frac{1}{n}\sum_{i=1}^n f(a_i) - \frac{1}{n}\sum_{i=1}^n f(ya_ix)\right| \leq 2\varepsilon$ for all $x, y \in G$.

Proof. Fix $x, y \in G$. Clearly $\{yA_ix : i \leq n\}$ is also a good ε-cover. There is a permutation p of $\{1, 2, \cdots, n\}$ such that for each $i \leq n$ there is some $b_i \in A_i \cap yA_{p(i)}x$. Thus $|f(a_i) - f(b_i)| \leq \varepsilon$ and $|f[ya_{p(i)}x] - f(b_i)| \leq \varepsilon$. Hence $|f(a_i) - f[ya_{p(i)}x]| \leq 2\varepsilon$. The result follows by summing over $i = 1$ to n and dividing by n. □

28-2.7. **Lemma** Let $\{A_i : i \leq n\}$ be a good ε-cover and $\{B_j : j \leq m\}$ a good δ-cover for f. Then $\left|\frac{1}{n}\sum_{i=1}^n f(a_i) - \frac{1}{m}\sum_{j=1}^m f(b_j)\right| \leq 2(\varepsilon + \delta)$ for all $a_i \in A_i$ and $b_j \in B_j$.

Proof. Letting $y = e$ and $x = b_j$ in last lemma, we get

$$\left|\frac{1}{n}\sum_{i=1}^n f(a_i) - \frac{1}{n}\sum_{i=1}^n f(a_ib_j)\right| \leq 2\varepsilon.$$

Summing over $j = 1$ to m and dividing by m, we have

$$\left|\frac{1}{n}\sum\nolimits_{i=1}^{n} f(a_i) - \frac{1}{mn}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{m} f(a_i b_j)\right| \le 2\varepsilon.$$

By symmetry, we obtain

$$\left|\frac{1}{m}\sum\nolimits_{j=1}^{m} f(b_j) - \frac{1}{mn}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{m} f(a_i b_j)\right| \le 2\delta.$$

Their sum completes the proof. □

28-2.8. **Theorem** Every ap-function f on a group G has a number $\mu(f) \in \mathbb{K}$ satisfying the following condition: for every $\varepsilon > 0$, there are $a_1, \cdots, a_n \in G$ and $\alpha_1, \cdots, \alpha_n \ge 0$ such that $\alpha_1 + \cdots + \alpha_n = 1$ and

$$\left|\mu(f) - \sum\nolimits_{i=1}^{n} \alpha_i f(y a_i x)\right| \le \varepsilon, \qquad \forall\, x, y \in G.$$

The number $\mu(f)$ is called the ***mean-value*** of f and is also denoted by $\int f d_m$ or $\int f(t) d_m t$. Note that the subscript m distinguishes mean-values from integrals although we use the integral sign for both of them. Also t in $\int f(t) d_m t$ is a dummy variable.

Proof. For each integer $k \ge 1$, let $\{A_1^k, \cdots, A_{n(k)}^k\}$ be a good $(1/k)$-cover for f. Choose any $a_i^k \in A_i^k$ for each k and $1 \le i \le n(k)$. Define $\mu_k = \frac{1}{n(k)}\sum\nolimits_{i=1}^{n(k)} f(a_i^k)$. Then we have

$$|\mu_k - \mu_p| = \left|\frac{1}{n(k)}\sum_{i=1}^{n(k)} f(a_i^k) - \frac{1}{n(p)}\sum_{i=1}^{n(p)} f(a_i^p)\right| \le 2\left(\frac{1}{k} + \frac{1}{p}\right).$$

Thus $\{\mu_k\}$ is a Cauchy sequence. Let $\mu(f) = \lim \mu_k$. Now take any $\varepsilon > 0$. Choose an integer $p > (1/\varepsilon)$ such that $|\mu(f) - \mu_k| \le \varepsilon$ for all $k \ge p$. From Hence we get

$$\left|\mu_k - \frac{1}{n(k)}\sum\nolimits_{i=1}^{n(k)} f(y a_i^k x)\right| \le \frac{2}{k} \le 2\varepsilon$$

for all $x, y \in G$ and $k \ge p$. Therefore

$$\left|\mu(f) - \frac{1}{n(k)}\sum\nolimits_{i=1}^{n(k)} f(y a_i^k x)\right| \le 3\varepsilon.$$

This proves the existence. For the uniqueness, let u, v be a complex numbers satisfying the following condition: for every $\varepsilon > 0$ there are $a_1, \cdots, a_p, b_1, \cdots, b_q$ in G and $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q \ge 0$ such that

$$\alpha_1 + \cdots + \alpha_p = \beta_1 + \cdots + \beta_q = 1,$$

$$|u - \textstyle\sum_{i=1}^{p} \alpha_i f(y a_i x)| \le \varepsilon$$

and $$|v - \textstyle\sum_{j=1}^{q} \beta_i f(y b_i x)| \le \varepsilon \text{ for all } x, y \in G.$$

Setting $y = e$ and $x = b_j$ in the first inequality, we get $|u - \sum_{i=1}^{p} \alpha_i f(a_i b_j)| \le \varepsilon$. Multiplying β_j and summing for $j = 1$ to q, we obtain

$$\left|u - \sum_{j=1}^{q} \sum_{i=1}^{p} \alpha_i \beta_j f(a_i b_j)\right| \le \varepsilon.$$

Similarly we have

$$\left|v - \sum_{j=1}^{q} \sum_{i=1}^{p} \alpha_i \beta_j f(a_i b_j)\right| \le \varepsilon.$$

Therefore $|u - v| \le 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $u = v$. This proves the uniqueness of mean-value. □

28-2.9. The map $\mu : A(G) \to \mathbb{K}$ given by $f \to \int f d_m$ is called the ***mean-value form*** on the group G. Reader should review positive linear forms, §16-4.4.

28-2.10. **Theorem** The *mean-value form* $\mu : A(G) \to \mathbb{K}$ given by $f \to \int f d_m$ is a linear form on $A(G)$. Furthermore the following conditions hold.

(a) Translation invariant, i.e. $\int L_a f d_m = \int R_a f d_m = \int f d_m$.

(b) Normalized, i.e. $\int f d_m = 1$ where $f = 1$ is a constant function.

(c) Positive, i.e. $\int f d_m \ge 0$ for $f \ge 0$. In particular, if f is real, then $\int f d_m$ is also real. If $f \le g$, then $\int f d_m \le \int g d_m$.

(d) Nondegenerate, i.e. $\int f d_m > 0$ for $0 \le f \ne 0$.

(e) $|\int f d_m| \le \int |f| \, d_m$.

Proof. To prove that μ is linear, take any $f, g \in A(G)$ and $\alpha, \beta \in \mathbb{K}$. For any $\varepsilon > 0$ there are $a_1, \cdots, a_p, b_1, \cdots, b_q \in G$ and $\alpha_i, \beta_j \ge 0$ such that

$$\alpha_1 + \cdots + \alpha_p = \beta_1 + \cdots + \beta_q = 1$$

and for all $x, y \in G$,

$$\left|\int f d_m - \sum_{i=1}^{p} \alpha_i f(y a_i x)\right| \le \varepsilon \qquad \#1$$

and

$$\left|\int g d_m - \sum_{j=1}^{q} \beta_j f(y b_j x)\right| \le \varepsilon. \qquad \#2$$

Replacing x by $b_j x$ in #1, we have

$$\left|\int f d_m - \sum_{i=1}^{p} \alpha_i f(y a_i b_j x)\right| \le \varepsilon, \qquad \forall\ x, y \in G.$$

Multiplying β_j and summing over j, we get

$$\left|\int f d_m - \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j f(y a_i b_j x)\right| \le \varepsilon.$$

Similarly,

$$\left|\int g d_m - \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j f(y a_i b_j x)\right| \le \varepsilon, \qquad \forall\ x, y \in G.$$

Hence

$$\left|\alpha \int f d_m + \beta \int g d_m - \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j (\alpha f + \beta g)(y a_i b_j x)\right| \le \varepsilon$$

for all $x, y \in G$. The uniqueness of mean-value gives the linearity of μ:

$$\int(\alpha f + \beta g)d_m = \alpha \int f d_m + \beta \int g d_m.$$

(a) Replacing y by ay in #1, we have

$$\left|\int f d_m - \sum_{i=1}^p \alpha_i L_a f(y a_i x)\right| = \left|\int f d_m - \sum_{i=1}^p \alpha_i f(a y a_i x)\right| \leq \varepsilon.$$

The uniqueness of mean-value gives $\int L_a f d_m = \int f d_m$. Similarly replacement of x by xa in #2 gives $\int R_a f d_m = \int f d_m$.

(b) By #1, for $f = 1$ we have $\left|\int f d_m - 1\right| = \left|\int f d_m - \sum_{i=1}^p \alpha_i f(y a_i x)\right| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result follows.

(c) If $f \geq 0$, then #1 entails $-\varepsilon \leq \int f d_m - \sum_{i=1}^p \alpha_i f(y a_i x) \leq \int f d_m$. Letting $\varepsilon \downarrow 0$, we have $\int f d_m \geq 0$. The last statement follows from §16-4.5a.

(d) Suppose that $f(c) > 0$ for some $c \in G$. Let $\varepsilon = \frac{1}{2} f(c)$ and $\{A_i : i \leq p\}$ an ε-cover for f. Choose any $a_i \in A_i$ for each i. Then every $x \in G$ is contained in some A_j. Hence for every $y \in G$, $|f(yx) - f(y a_j)| \leq \varepsilon$. Picking $y = c a_j^{-1}$, we get $|f(c a_j^{-1} x) - f(c)| \leq \varepsilon$, or $-\varepsilon \leq f(c a_j^{-1} x) - f(c)$, i.e.

$$\varepsilon = f(c) - \varepsilon \leq f(c a_j^{-1} x) \leq \sum_{i=1}^p f(c a_i^{-1} x).$$

Taking the mean-value, we obtain

$$\varepsilon \leq \sum_{i=1}^p \int f(c a_i^{-1} x) d_m x = \sum_{i=1}^p \int f(x) d_m x = p \int f(x) d_m x$$

by translation invariance. Therefore $\int f d_m \geq \varepsilon / p > 0$.

(e) It follows immediately from §16-4.5c. □

28-2.11. **Theorem** The mean-value form is inversion invariant, that is for all $f \in A(G)$ we have $\int f(x^{-1}) d_m x = \int f(x) d_m x$.

Proof. For every $\varepsilon > 0$ choose $a_i \in G$ and $\alpha_i \geq 0$ such that $\alpha_1 + \cdots + \alpha_n = 1$ and $\left|\int f d_m - \sum_{i=1}^n \alpha_i f(x^{-1} a_i y^{-1})\right| \leq \varepsilon$ for all $x, y \in G$. Then we obtain $\left|\int f d_m - \sum_{i=1}^n \alpha_i f^v(y a_i^{-1} x)\right| \leq \varepsilon$ for all $x, y \in G$. Therefore by uniqueness of mean-value, we have $\int f^v d_m = \int f d_m$. □

28-2.12. **Theorem** Every left invariant subspace M of $A(G)$ has a unique left translation invariant normalized positive linear form.

Proof. The restriction of mean-value form to M provides the existence. For uniqueness, suppose that μ is any left translation invariant normalized positive linear form. Take any real ap-function f in M. For every $\varepsilon > 0$ there are $a_i \in G$ and $\alpha_i \geq 0$ such that $\alpha_1 + \cdots + \alpha_n = 1$ and $\left|\int f d_m - \sum_{i=1}^n \alpha_i f(y a_i x)\right| \leq \varepsilon$ for all $x, y \in G$. Letting $y = e$, we have $\int f d_m - \varepsilon \leq \sum_{i=1}^n \alpha_i L_{a_i} f(x) \leq \int f d_m + \varepsilon$ for all $x \in G$. Applying μ, we obtain $\displaystyle\int f d_m - \varepsilon \leq \sum_{i=1}^n \alpha_i \mu(f) = \mu(f) \leq \int f d_m + \varepsilon$.

Letting $\varepsilon \downarrow 0$, we have $\mu(f) = \int f d_m$ for every real function in M. The complex case follows from linearity. □

28-3 Convolutions

28-3.1. **Example** Let $G = \{x_1, x_2, \cdots, x_n\}$ be a finite group and $A_i = \{x_i\}$. Every function $f : G \to \mathbb{K}$ is almost periodic because $\{A_i\}$ is an ε-cover for f. Let $\mu(f) = \frac{1}{n}\sum_{j=1}^n f(x_j)$. Since μ is left translation invariant normalized positive linear form, it is the mean-value form. For every $a \in G$, define $f_a(x) = \delta_{ax}$ which is 1 if $x = a$ and 0 otherwise. Then $f = \sum_{a\in G} f(a)f_a$. Identify f_a with a, functions on G can be identified as the formal sums $u = \sum_{a\in G} u_a a$ and $v = \sum_{b\in G} v_b b$ where $u_a, v_b \in \mathbb{K}$. The average of their formal product is given by

$$u \times v = \frac{1}{n}\sum_{a\in G}\sum_{b\in G}(u_a v_b)(ab) = \frac{1}{n}\sum_{c\in G}\sum_{ab=c}(u_a v_b)c = \sum_{c\in G}\left(\frac{1}{n}\sum_{b\in G} u_{cb^{-1}} v_b\right) c.$$

Converting back to function notation, we have $(f \times g)(c) = \frac{1}{n}\sum_{b\in G} f(cb^{-1})g(b)$. This motivates the definition of convolution in terms of group algebra.

28-3.2. **Theorem** Let G, H be two groups and f an ap-function on the product group $G \times H$.

(a) For each $x \in G$, $y \to f(x,y)$ is an ap-function on H.

(b) The function $x \to \int f(x,y)d_m y$ is almost periodic on G.

(c) $\int f(x,y)d_m(x,y) = \int \left[\int f(x,y)dy\right] d_m x$. For convenience, it is also denoted by $\int d_m x \int f(x,y)d_m y$. Therefore we are allowed to change the order of taking mean-values.

Proof. (a) Let $\{A_i : i \le n\}$ be an ε-cover of $G \times H$ for f. Fix $x \in G$. Let $N_i = \{y \in H : (x,y) \in A_i\}$. Clearly $\{N_i : i \le n\}$ is a cover of H. Suppose $a, b \in N_i$ and $z \in H$. Then $(x,a), (x,b) \in A_i$ and $(e,z) \in G \times H$. Hence

$$|f(x,az) - f(x,bz)| = |f\{(x,a)(e,z)\} - f\{(x,b)(e,z)\}| \le \varepsilon. \qquad \#1$$

Thus $\{N_i : i \le n\}$ is an ε-cover of H for $y \to f(x,y)$. Consequently, $f(x,y)$ is almost periodic in $y \in H$.

(b) Next fix az and bz but allow x to vary. Then $f(x,az), f(x,bz)$ are almost periodic in $x \in G$. Taking mean-value with respect to x in #1, we have

$$\left|\int f(x,az)d_m x - \int f(x,bz)d_m x\right| \le \int |f(x,az) - f(x,bz)|\, d_m x \le \varepsilon.$$

Hence $\{N_i : i \le n\}$ is also a left ε-cover of H for the function $y \to \int f(x,y)d_m x$. Therefore $\int f(x,y)d_m x$ is almost periodic in $y \in H$.
(c) Define $\mu(f) = \int\int f(x,y)d_m x d_m y$ for every ap-function f on $G \times H$. Clearly μ is a translation invariant normalized positive linear form on $A(G \times H)$. By uniqueness of mean-values, we have $\int f(x,y)d_m(x,y) = \int\int f(x,y)d_m x d_m y$. The proof is completed by symmetry between x, y. □

28-3.3. **Lemma** If f is an ap-function on $G \times H$, then the function $(x,y) \to f(x^{-1}, y)$ is also almost periodic on $G \times H$.

Proof. Define $g(x,y) = f(x^{-1}, y)$ for all $(x,y) \in G \times H$. Let $\{A_i : i \le n\}$ be an ε-cover of $G \times H$ for f. Consider $B_i = \{(x,y) \in G \times H : (x^{-1}, y) \in A_i\}$. Clearly $\{B_i : i \le n\}$ is a cover of $G \times H$. Take any $(a,c),(b,d) \in B_i$ and $(x,y) \in G \times H$. Since $(a^{-1}, c), (b^{-1}, d) \in A_i$, we get

$$| \ g[(a,c)(x,y)] - g[(b,d)(x,y)] \ | = |f(x^{-1}a^{-1}, cy) - f(x^{-1}b^{-1}, dy)|$$

$$= | \ f[(x^{-1}, e_2)(a^{-1}, c)(e_1, y)] - f[(x^{-1}, e_2)(b^{-1}, d)(e_1, y)] \ | \le \varepsilon$$

where e_1, e_2 are the identity element of G, H respectively. Therefore $\{B_i : i \le n\}$ is a left ε-cover of $G \times H$ for g. Consequently g is almost periodic on $G \times H$. □

28-3.4. **Lemma** If h is an almost periodic on H, then the function f on $G \times H$ given by $f(x,y) = h(y)$ is almost periodic.

Proof. Let $\{B_i : i \le n\}$ be an ε-cover of H for h. Clearly $\{A_i : i \le n\}$ is a cover of $G \times H$ where $A_i = G \times B_i$ for each i. Take any $(a,c),(b,d) \in A_i$ and $(x,y) \in G \times H$. Since $b, d \in B_i$; we have

$$| \ f[(a,c)(x,y)] - f[(b,d)(x,y)] \ | = |h(cy) - h(dy)| \le \varepsilon.$$

Therefore $\{A_i : i \le n\}$ is an ε-cover of $G \times H$ for f. Consequently f is almost periodic on $G \times H$. □

28-3.5. **Lemma** If f is a real ap-function on $G \times H$, then the functions g, h on G defined by $g(x) = \sup\{f(x,y) : y \in H\}$ and $h(x) = \inf\{f(x,y) : y \in H\}$ are almost periodic.

Proof. Let $\{A_i : i \le n\}$ be an ε-cover of $G \times H$ for f. Define $B_i = \{x \in G : (x, e_2) \in A_i\}$. Clearly $\{B_i : i \le n\}$ is a cover of G. For all $a, b \in B_i$, we have $| \ f[(a,e_2)(x,y)] - f[(b,e_2)(x,y)] \ | \le \varepsilon$ for all $(x,y) \in G \times H$ i.e. $f(ax,y) - \varepsilon \le f(bx,y) \le f(ax,y) + \varepsilon$. Taking supremum over $y \in H$, we get $g(ax) - \varepsilon \le g(bx) \le g(ax) + \varepsilon$, or $|g(ax) - g(bx)| \le \varepsilon$. Therefore $\{B_i : i \le n\}$

is an ε-cover of G for g. Consequently, g is almost periodic on G. Since $h(x) = -\sup\{-f(x,y) : y \in H\}$, the function h is also almost periodic. □

28-3.6. **Lemma** Let f be an ap-function on G. Then the functions taking $(x,y) \in G \times G$ to $f(xy)$, $f(x^{-1}y)$, $f(xy^{-1})$, $f(x^{-1}y^{-1})$, $f(yx)$, $f(y^{-1}x)$, $f(yx^{-1})$, $f(y^{-1}x^{-1})$ respectively are all almost periodic on $G \times G$.

Proof. Define $g(x,y) = f(xy)$ for all $x, y \in G$. To show that g is almost periodic on G, let $\{A_i : i \leq n\}$ be an ε-cover of G for f. Define $B_{ij} = A_i \times A_j$. Clearly $\{B_{ij} : i, j \leq n\}$ is a cover of $G \times G$. For all $(a_i, a_j), (b_i, b_j) \in B_{ij}$ and $(x,y) \in G \times G$, we have

$$\begin{aligned} &| \; g[(a_i,a_j)(x,y)] - g[(b_i,b_j)(x,y)] \; | = |f(a_ix \; a_jy) - f(b_ix \; b_jy)| \\ &\leq |f(a_ixa_jy) - f(a_ixb_jy)| + |f(a_ixb_jy) - f(b_ixb_jy)| \leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Therefore g is almost periodic on $G \times G$. Next, applying §28-3.3to g, $f(x^{-1}y)$ is almost periodic in $(x,y) \in G \times G$. Similarly, $f(x^{-1}y^{-1})$ is almost periodic in $(x,y) \in G \times G$. Since the inversion of f is almost periodic on G, $f(yx) = f(\{x^{-1}y^{-1}\}^{-1})$ is almost periodic in $(x,y) \in G \times G$. Repeated application of §28-3.3completes the proof. □

28-3.7. **Theorem** Let f, g be ap-function on G. For every $x \in G$ we have

$$\begin{aligned} &\int f(xy^{-1})g(y)d_my = \int f(xy)g(y^{-1})d_my \\ &= \int f(y)g(y^{-1}x)d_my = \int f(y^{-1})g(yx)d_my. \end{aligned}$$

This common value is denoted by $f \times g(x)$. The ap-function $f \times g$ on G is called the *convolution* of f, g.

Proof. We have proved that all functions involved are almost periodic. Replacing y by y^{-1} we have $\int f(xy^{-1})g(y)d_my = \int f(xy)g(y^{-1})d_my$. Replacing y by $x^{-1}y$ we get $\int f(xy)g(y^{-1})d_my = \int f(y)g(y^{-1}x)d_my$. Replacing y by y^{-1} we obtain $\int f(y)g(y^{-1}x)d_my = \int f(y^{-1})g(yx)d_my$. Finally, $f(xy^{-1})$ and $g(y)$ are almost periodic in $(x,y) \in G \times G$, so is their product $f(xy^{-1})g(y)$. Hence its mean-value for $y \in G$ is almost periodic in $x \in G$. Therefore $f \times g$ is almost periodic on G. □

28-3.8. **Theorem** Let $f_1, f_2, \cdots, f_p$ be ap-functions on G. Then for every $\varepsilon > 0$ there are $\alpha_i \geq 0$ and $a_i \in G$ such that $\alpha_1 + \cdots + \alpha_n = 1$ and

$$\left|\int f_k d_m - \sum_{i=1}^n \alpha_i f_k(xa_iy)\right| \leq \varepsilon \qquad \forall \; x, y \in G, \; \forall \; 1 \leq k \leq p.$$

Proof. It is trivial for $p = 1$. Inductively, suppose that g is an ap-function in addition to $f_1, f_2, \cdots, f_p$. For every $\varepsilon > 0$ there are $\alpha_i, \beta_j \geq 0$ and $a_i, b_j \in G$ such that $\alpha_1 + \cdots + \alpha_u = \beta_1 + \cdots + \beta_v = 1$ and for all $x, y \in G$; $1 \leq k \leq p$;

$$\left|\int f_k d_m - \sum_{i=1}^{u} \alpha_i f_k(xa_i y)\right| \le \varepsilon \quad \text{and} \quad \left|\int g d_m - \sum_{j=1}^{v} \beta_j g(xb_j y)\right| \le \varepsilon.$$

Replacing y by $b_j y$ in the first inequality, multiplying by β_j and summing over $j = 1$ to v, we have $\left|\int f_k d_m - \sum_{i=1}^{u}\sum_{j=1}^{v} \alpha_i\beta_j f_k(xa_i b_j y)\right| \le \varepsilon$. Similarly replacing x by xa_i, multiplying by α_i and summing over $i = 1$ to u, we obtain $\left|\int g d_m - \sum_{i=1}^{u}\sum_{j=1}^{v} \alpha_i\beta_j g(xa_i b_j y)\right| \le \varepsilon$. The proof is completed by $\sum_{i=1}^{u}\sum_{j=1}^{v} \alpha_i\beta_j = (\sum_{i=1}^{u}\alpha_i)(\sum_{j=1}^{v}\beta_j) = 1$. □

28-3.9. **Theorem** Convolutions can be approximated by linear combinations of translates of their factors. More precisely, for all $f, g \in A(G)$ and for every $\varepsilon > 0$, there are $\alpha_i, \beta_j \in \mathbb{K}$ and $a_i, b_j \in G$ such that for all $x \in G$ we have

$$|f \times g(x) - \sum_{i=1}^{n} \alpha_i f(xa_i)| \le \varepsilon \quad \text{and} \quad \left|f \times g(x) - \sum_{j=1}^{k} \beta_j g(b_j x)\right| \le \varepsilon.$$

Proof. Choose $\delta > 0$ satisfying $\delta\|g\|_\infty \le \varepsilon$. Let $\{A_i : i \le n\}$ be a δ-cover for f and pick any $a_i \in A_i$ for each $i \le n$. Fix $x \in G$. Select A_i containing x. Then $|f(xy^{-1}) - f(a_i y^{-1})| \le \delta$ for all y. Hence

$$\begin{aligned}&\left|\int f(xy^{-1})g(y)d_m y - \int f(a_i y^{-1})g(y)d_m y\right| \\ &\le \int |f(xy^{-1}) - f(a_i y^{-1})| \, \|g\|_\infty d_m y \le \int \delta\|g\|_\infty d_m y = \delta\|g\|_\infty \le \varepsilon.\end{aligned}$$

There are $\alpha_j \ge 0, b_j \in G$ such that $\alpha_1 + \cdots + \alpha_p = 1$ and for each $i \le n$ we get

$$\left|\int f(a_i y^{-1})g(y)d_m y - \sum_{j=1}^{p} \alpha_j f(a_i b_j^{-1})g(b_j)\right| \le \varepsilon.$$

Because $x, a_i \in A_i$, we obtain $|f(a_i b_j^{-1}) - f(xb_j^{-1})| \le \delta$ and thus

$$\left|\sum_{j=1}^{p} \alpha_j f(a_i b_j^{-1})g(b_j) - \sum_{j=1}^{p} \alpha_j f(xb_j^{-1})g(b_j)\right| \le \delta\|g\|_\infty \le \varepsilon.$$

Combining all inequalities we obtain

$$\left|f \times g(x) - \sum_{j=1}^{p}[\alpha_j g(b_j)]f(xb_j^{-1})\right| \le 3\varepsilon.$$

Because it is independent of the choice of a_i, it works for all $x \in G$. The first inequality is proved by obvious modification of symbols. The second follows in a similar manner. □

28-3.10. **Theorem** Let f be an ap-function on G. Then for every $\varepsilon > 0$ there are ap-functions $g, h \ge 0$ such that $\int g d_m = \int h d_m = 1$ and $\|f - f \times g\|_\infty \le \varepsilon$, $\|f - h \times f\|_\infty \le \varepsilon$.

Proof. For every $y \in G$, define $\varphi(y) = 0 \vee [\varepsilon - \sup\{|f(x) - f(xy^{-1})| : x \in G\}]$. Clearly $|f(x) - f(xy^{-1})| \, \varphi(y) \le \varepsilon\varphi(y)$ for all $x, y \in G$. Since $\varphi(e) = \varepsilon > 0$, we have $\int \varphi(y)d_m y > 0$ and $g(y) = \varphi(y)/\int \varphi d_m$ is well-defined. Hence we obtain $0 \le g \in A(G)$, $\int g d_m = 1$ and $|f(x) - f(xy^{-1})| \, g(y) \le \varepsilon g(y)$. Therefore

$$|f(x) - f \times g(x)| = \left|\int f(x)g(y)d_m y - \int f(xy^{-1})g(y)d_m y\right|$$
$$\leq \int |f(x) - f(xy^{-1})|\, g(y)d_m y \leq \int \varepsilon g(y)d_m y \leq \varepsilon$$

i.e. $\|f - f \times g\|_\infty \leq \varepsilon$. The second part follows in a similar way. □

28-3.11. **Theorem** $A(G)$ is an associative algebra under convolution.

Proof. Let f, g, h be ap-functions on G. Then for all $x \in G$,

$$[f \times (g \times h)](x) = \int f(xy^{-1})(g \times h)(y)d_m y$$
$$= \int\int f(xy^{-1})g(yz^{-1})h(z)d_m z d_m y = \int \left[\int f(xy^{-1})g(yz^{-1})d_m y\right] h(z)d_m z$$
$$= \int \left[\int f(xz^{-1}u^{-1})g(u)du\right] h(z)d_m z, \qquad yz^{-1} = u, \quad y^{-1} = z^{-1}u^{-1}$$
$$= \int (f \times g)(xz^{-1})h(z)d_m z = [(f \times g) \times h](x).$$

Therefore the convolution is associative. It is routine to verify that the convolution is distributive and $\alpha(f \times g) = (\alpha f) \times g = f \times (\alpha g)$ for all $\alpha \in \mathbb{K}$. □

28-3.12. **Theorem** For all $f, g \in A(G)$ and all $a \in G$; we have

$$L_a(f \times g) = (L_a f) \times g,\ R_a(f \times g) = f \times R_a g \text{ and } (R_a f) \times g = f \times L_a g.$$

Proof. The first and the last equality are verified as follow:

$$L_a(f \times g)(x) = (f \times g)(ax)$$
$$= \int f(axy^{-1})g(y)d_m y = \int (L_a f)(xy^{-1})g(y)d_m y = (L_a f) \times g(x)$$

and

$$(R_a f) \times g(x) = \int (R_a f)(y)g(y^{-1}x)d_m y = \int f(ya)g(y^{-1}x)d_m y$$
$$= \int f(z)g(az^{-1}x)d_m y = \int f(z)(L_a g)(z^{-1}x)d_m y = (f \times L_a g)(x).$$ □

28-3.13. **Theorem** If x is in the center of G i.e. $xy = yx$ for all $y \in G$, then $(f \times g)(x) = (g \times f)(x)$. In particular, if G is abelian, then $A(G)$ is commutative.

Proof. $(f \times g)(x) = \int f(xy)g(y^{-1})d_m y = \int g(y^{-1})f(yx)d_m y = (g \times f)(x)$. □

28-3.14. **Theorem** For all $f, g \in A(G)$, the following statements hold.

(a) $f^* \in A(G)$ and $\|f^*\|_\infty = \|f\|_\infty$ where $f^*(x) = f(x^{-1})^-$.

(b) $f^{**} = f$, $(f + g)^* = f^* + g^*$ and $(\lambda f)^* = \lambda^- f^*$ for all $\lambda \in \mathbb{K}$.

(c) $(f \times g)^* = g^* \times f^*$.

Proof. To verify (c), observe that

$$(f \times g)^*(x) = \overline{(f \times g)(x^{-1})} = \overline{\int f(x^{-1}y^{-1})g(y)d_m y}$$
$$= \int \overline{f}(x^{-1}y^{-1})\, \overline{g}(y)d_m y = \int g^*(y^{-1})f^*(yx)d_m y = (g^* \times f^*)(x).$$ □

28-4 Eigen Expansion

28-4.1. The technique of integral equations are employed to approximate ap-functions. This prepares the ground for further approximation by trigonometric polynomials.

28-4.2. Let $A(G)$ be the set of all ap-functions on a group G; $1 \le p, q \le \infty$ conjugate indices satisfying $\frac{1}{p} + \frac{1}{q} = 1$. We have proved that $A(G)$ is a Banach space under the sup-norm $\|f\|_\infty = \sup\{|f(x)| : x \in G\}$. For every $1 \le p < \infty$, since $|f|^p$ is almost periodic, $\|f\|_p = \left[\int |f(x)|^p d_m x\right]^{1/p}$ is well-defined. Write $A_p(G)$ to emphasize the p-norm.

28-4.3. **Theorem** (a) $A_p(G)$ is a normed space.
(b) $A_2(G)$ is an inner product space under $< f, g >= \int f(x)\overline{g(x)}\, d_m x$. Since we use the square norm most of the time, write $\|f\| = \|f\|_2$ for all $f \in A(G)$.
(c) $\left|\int f(x)g(x)d_m x\right| \le \|f\|_p \|g\|_q$ for all $f, g \in A(g)$.

Proof. Suppose $0 \ne f \in A(G)$. Then $0 \le |f|^p \ne 0$ and hence $\int |f|^p d_m > 0$ by §28-2.10d. It follows $\|f\|_p \ne 0$. It is routine to complete the proof as in measure theory. □

28-4.4. **Theorem** Let $f_n, f, g_n, g \in A(G)$.
(a) $\|L_a f\| = \|R_a f\| = \|f^*\|_p = \|f\|_p \le \|f\|_\infty$ for every $a \in G$.
(b) $\|f \times g\|_\infty \le \|f\|_p\, \|g\|_q$, $\left|\int f(x) d_m x\right| \le \|f\|_p$ and $\|f \times g\| \le \|f\|\, \|g\|$.
(c) If $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_q \to 0$, then $\|f_n \times g_n - f \times g\|_\infty \to 0$ as $n \to \infty$.

Proof. (a) It follows immediately from definition.
(b)
$$|(f \times g)(x)| = \left|\int f(xy^{-1})g(y)\, d_m y\right|$$
$$\le \left[\int |f(xy^{-1})|^p d_m y\right]^{1/p} \left[\int |g(y)|^q d_m y\right]^{1/q} = \|f\|_p\, \|g\|_q\,.$$
Letting $g = 1$, we get
$$\left|\int f(x) d_m x\right| = \left|\int f(x) g(x^{-1}e) d_m x\right| = |(f \times g)(e)| \le \|f\|_p \|g\|_q = \|f\|_p.$$
Next, $\|f \times g\| = \|f \times g\|_2 \le \|f \times g\|_\infty \le \|f\|_2 \|g\|_2 = \|f\|\, \|g\|$.
(c) It is a standard proof that the product is continuous. □

28-4.5. **Theorem** Let $f, g \in A(G)$. Write $\|f\| = \|f\|_2$.
(a) $\|f\|^2 = \|f^*\|^2 = (f \times f^*)(e) = (f^* \times f)(e)$ where e is the identity element of G. In particular, if $f^* = f$, then $\|f\|^2 = (f \times f)(e)$.
(b) $< f, g >=< g^*, f^* >$ and $\|f \times g\| \le \|f\|\, \|g\|$.
(c) $(f \times g)(x) =< L_x f, g^* >=< R_x g, f^* >=< f, (L_{x^{-1}} g)^* >=< g, (R_{x^{-1}} f)^* >$.

Proof. Observe that $\|f \times g\| \le \|f \times g\|_\infty \le \|f\|\ \|g\|$ and that

$$< L_x f, g^* >= \int L_x f(y) g^{*-}(y) d_m y = \int f(xy) g(y^{-1}) d_m y = (f \times g)(x),$$

$$< f, (L_{x^{-1}} g)^* >= \int f(y)(L_{x^{-1}} g)^v(y) d_m y = \int f(y) g[(x^{-1}y)^{-1}] d_m y = (f \times g)(x),$$

$$< g, (R_{x^{-1}} f)^* >= \int g(y)(R_{x^{-1}} f)^v(y) d_m y = \int f[(yx^{-1})^{-1}] g(y) d_m y = (f \times g)(x).$$

It is routine computation to complete the proof. □

28-4.6. **Lemma** Let f be an ap-function on G.

(a) The common value of the following is denoted by Γ_k.

$$(f^* \times f)^k(e) = (f \times f^*)^k(e)$$

$= \|f^* \times f \times f^* \times \cdots\|^2$, k alternative factors ended with either f^* or f

$= \|f \times f^* \times f \times \cdots\|^2$, k alternative factors ended with either f or f^*

(b) $\Gamma_k^2 \le \Gamma_{k-1}\Gamma_{k+1}$ and $\Gamma_{j+k} \le \Gamma_j \Gamma_k$.

Proof. (a) Because e is in the center of G, we have

$$(f \times f^*)^k(e) = f \times [(f^* \times f)^{k-1} \times f^*](e) = [(f^* \times f)^{k-1} \times f^*] \times f(e)$$

$$= (f^* \times f)^k(e) = (f^* \times f \times \cdots \times f^* \times f)(e)$$

$= (f^* \times f \times \cdots) \times (\cdots \times f^* \times f)(e)$, k factors in each part

$$= (f^* \times f \times \cdots) \times (f^* \times f \times \cdots)^*(e) = \|f^* \times f \times \cdots\|^2$$

Similarly $(f \times f^*)^k(e) = \|f \times f^* \times \cdots\|^2$.

(b) Observe that

$\Gamma_k \le \|f \times f^* \times \cdots \times f \times f^*\|_\infty$, $2k$ factors

$$\le \|f \times f^* \times \cdots\| \cdot \|\cdots \times f \times f^*\| = \sqrt{\Gamma_{k+1}\Gamma_{k-1}}$$

first with $(k+1)$ factors, second $(k-1)$ factors. Also

$\Gamma_{j+k} = \|f^* \times f \times \cdots\|^2$, $j+k$ alternative factors

$\le \|f^* \times f \times \cdots\|^2 \ \ \|\cdots\|^2$, first j and second k alternative factors

$= \Gamma_j \Gamma_k$. □

28-4.7. **Lemma** Let $f \neq 0$ be an ap-function on G.

(a) $\Gamma_k > 0$ for all $k \ge 1$.

(b) $\lambda = \lim_{k\to\infty} \Gamma_{k+1}/\Gamma_k$ exists, $0 < \lambda \le \|f\|^2$ and $\|f^* \times f\|^2 \le \lambda \|f\|^2$.

(c) $\nu = \lim_{k\to\infty} \Gamma_k/\lambda^k$ exists and $\nu \ge 1$.

Proof. (a) Suppose $\Gamma_k = 0$ for some $k \ge 3$. Then $0 \le \Gamma_{k-1}^2 \le \Gamma_{k-2}\Gamma_k = 0$ implies $\Gamma_{k-1} = 0$. Repeating the same argument, we obtain $\Gamma_2 = \|f^* \times f\|^2 = 0$, i.e. $f^* \times f = 0$. Hence $(f^* \times f)(e) = \|f\|^2 = 0$, or $f = 0$. This proves (a).

(b) From $\Gamma_k^2 \le \Gamma_{k-1}\Gamma_{k+1}$, we have $0 < \Gamma_k/\Gamma_{k-1} \le \Gamma_{k+1}/\Gamma_k \le \Gamma_1$. It follows that $0 < \Gamma_{k+1}/\Gamma_k \uparrow \lambda \le \Gamma_1$ for some limit λ, that is $\lambda \le \Gamma_1 = \|f\|^2$. When $k = 1$, we get $\|f^* \times f\|^2 = \Gamma_2 \le \lambda\Gamma_1 = \|f\|^2$.
(c) Since $\Gamma_{k+1}/\Gamma_k \le \lambda$, we have $\Gamma_k/\lambda^k \ge \Gamma_{k+1}/\lambda^{k+1} > 0$. It follows that $\Gamma_k/\lambda^k \downarrow \nu$ for some limit ν. Observe that

$$\Gamma_k \ge \frac{\Gamma_{j+k}}{\Gamma_j} = \frac{\Gamma_{j+k}}{\Gamma_{j+k-1}}\,\frac{\Gamma_{j+k-1}}{\Gamma_{j+k-2}}\,\frac{\Gamma_{j+k-2}}{\Gamma_{j+k-3}}\cdots\,\frac{\Gamma_{j+1}}{\Gamma_j} \ge \left(\frac{\Gamma_{j+k}}{\Gamma_{j+k-1}}\right)^k.$$

For $j \to \infty$, we get $\Gamma_k \ge \lambda^k$, or $\Gamma_k/\lambda^k \ge 1$. Letting $k \to \infty$ we have $\nu \ge 1$. □

28-4.8. An ap-function φ is called a *projector* if $\varphi^* = \varphi = \varphi \times \varphi$. The map $f \to f^*$ is called the *involution* on $A(G)$. Interpret $f^* \times f$ as the operator $\varphi \to (f^* \times f) \times \varphi$ on $A(G)$. Next lemma gives an eigenvalue with a corresponding eigenvector and next theorem expands $f^* \times f$ in terms of eigenvalues and eigenvectors. It prepares the ground for approximation by trigonometric polynomials later.

28-4.9. **Lemma** Let $f \neq 0$ be an ap-function on G.
(a) The sequence $\{(f^* \times f)^k/\lambda^k : k \ge 1\}$ converges uniformly to a projector φ.
(b) $f^* \times f \times \varphi = \lambda\varphi = \varphi \times f^* \times f$ and $\|\varphi\|^2 = \nu$.

Proof. For all $j, k \ge 1$, we have

$$\begin{aligned}
&\left\|\frac{(f^* \times f)^{j+1}}{\lambda^{j+1}} - \frac{(f^* \times f)^{k+1}}{\lambda^{k+1}}\right\|_\infty^2 \\
&= \frac{1}{\lambda^2}\left\|f^* \times \left[\frac{(f^* \times f)^j}{\lambda^j} - \frac{(f^* \times f)^k}{\lambda^k}\right] \times f\right\|_\infty^2 \\
&\le \frac{1}{\lambda^2}\left\|f^* \times \left\{\frac{(f^* \times f)^j}{\lambda^j} - \frac{(f^* \times f)^k}{\lambda^k}\right\}\right\|^2 \|f\|^2 \\
&\le \frac{1}{\lambda^2}\|f^*\|^2 \left\|\frac{(f^* \times f)^j}{\lambda^j} - \frac{(f^* \times f)^k}{\lambda^k}\right\|^2 \|f\|^2 \\
&\le \frac{\Gamma_1^2}{\lambda^2}\left[\frac{(f^* \times f)^j}{\lambda^j} - \frac{(f^* \times f)^k}{\lambda^k}\right]^2 (e) \\
&= \frac{\Gamma_1^2}{\lambda^2}\left[\frac{(f^* \times f)^{2j}(e)}{\lambda^{2j}} - 2\,\frac{(f^* \times f)^{j+k}(e)}{\lambda^{j+k}} + \frac{(f^* \times f)^{2k}(e)}{\lambda^{2k}}\right] \\
&= \frac{\Gamma_1^2}{\lambda^2}\left(\frac{\Gamma_{2j}}{\lambda^{2j}} - 2\,\frac{\Gamma_{j+k}}{\lambda^{j+k}} + \frac{\Gamma_{2k}}{\lambda^{2k}}\right) \to 0, \qquad \text{as} \quad j, k \to \infty.
\end{aligned}$$

Therefore the Cauchy sequence $\{(f^* \times f)^k/\lambda^k : k \ge 1\}$ converges uniformly on G to some ap-function φ. Since convolution is continuous in $A_2(G)$, we have

$$\varphi \times \varphi = \left[\lim_{k\to\infty} \frac{(f^* \times f)^k}{\lambda^k}\right] \times \left[\lim_{k\to\infty} \frac{(f^* \times f)^k}{\lambda^k}\right]$$
$$= \lim_{k\to\infty} \left[\frac{(f^* \times f)^k}{\lambda^k} \times \frac{(f^* \times f)^k}{\lambda^k}\right] = \lim_{k\to\infty} \frac{(f^* \times f)^{2k}}{\lambda^{2k}} = \varphi.$$

Since involution is continuous in $A_2(G)$, we get

$$\varphi^* = \left[\lim_{k\to\infty} \frac{(f^* \times f)^k}{\lambda^k}\right]^* = \lim_{k\to\infty} \left[\frac{(f^* \times f)^k}{\lambda^k}\right]^* = \lim_{k\to\infty} \frac{(f^* \times f)^k}{\lambda^k} = \varphi.$$

Next observe that

$$\varphi = \lim_{k\to\infty} \frac{(f^* \times f)^{k+1}}{\lambda^{k+1}} = \frac{f^* \times f}{\lambda} \times \left[\lim_{k\to\infty} \frac{(f^* \times f)^{k+1}}{\lambda^{k+1}}\right] = \frac{f^* \times f}{\lambda} \times \varphi$$

that is $f^* \times f \times \varphi = \lambda\varphi$. Taking the hermitian we have $\varphi \times f^* \times f = \lambda\varphi$. Finally,

$$\|\varphi\|^2 = \lim_{k\to\infty} \left\|\frac{(f^* \times f)^k}{\lambda^k}\right\|^2 = \lim_{k\to\infty} \frac{\Gamma_{2k}}{\lambda^{2k}} = \nu. \qquad \square$$

28-4.10. **Theorem** For every ap-function $f \neq 0$ on G, there is a sequence, finite or infinite, of real numbers $\lambda_1 > \lambda_2 > \cdots \downarrow 0$ and a sequence $\{\varphi_n : n \geq 1\}$ of projectors in $A(G)$ such that

(a) $\|\varphi_n\| \geq 1$, $\varphi_n = \varphi_n^* = \varphi_n \times \varphi_n$,

(b) $f_n^* \times f_n \times \varphi_n = \lambda_n\varphi_n = \varphi_n \times f_n^* \times f_n$,

(c) $\varphi_j \times \varphi_k = 0$ for all $j \neq k$,

(d) $f^* \times f(x) = \sum_{n\geq 1} \lambda_n\varphi_n(x)$ uniformly on G,

(e) $\|f\|^2 = \sum_{n\geq 1} \lambda_n\|\varphi_n\|^2$.

Proof. Let $f_1 = f$. Choose $\lambda_1 > 0$ such that $(f_1^* \times f_1)^k/\lambda_1^k \to \varphi_1$ in $A_\infty(G)$ as $k \to \infty$, $\|\varphi_1\| \geq 1$, $\varphi_1 = \varphi_1^* = \varphi_1 \times \varphi_1$ and $f_1^* \times f_1 \times \varphi_1 = \lambda_1\varphi_1 = \varphi_1 \times f_1^* \times f_1$. Inductively, let $f_{n+1} = f_n - f_n \times \varphi_n$. If $f_{n+1} = 0$, stop otherwise choose $\lambda_{n+1} > 0$ such that $(f_{n+1}^* \times f_{n+1})^k/\lambda_{n+1}^k \to \varphi_{n+1}$ in $A_\infty(G)$ as $k \to \infty$, $\|\varphi_{n+1}\| \geq 1$, $\varphi_{n+1} = \varphi_{n+1}^* = \varphi_{n+1} \times \varphi_{n+1}$ and $f_{n+1}^* \times f_{n+1} \times \varphi_{n+1} = \lambda_{n+1}\varphi_{n+1} = \varphi_{n+1} \times f_{n+1}^* \times f_{n+1}$. We claim $(f_{n+1}^* \times f_{n+1})^k = (f_n^* \times f_n)^k - \lambda_n^k\varphi_n$. For $k = 1$,

$$f_{n+1}^* \times f_{n+1} = (f_n - f_n \times \varphi_n)^* \times (f_n - f_n \times \varphi_n)$$
$$= (f_n^* - \varphi_n \times f_n^*) \times (f_n - f_n \times \varphi_n)$$
$$= f_n^* \times f_n - \varphi_n \times f_n^* \times f_n - f_n^* \times f_n \times \varphi_n + \varphi_n \times f_n^* \times f_n \times \varphi_n$$
$$= f_n^* \times f_n - \lambda_n\varphi_n.$$

and in general by induction,

$$(f_{n+1}^* \times f_{n+1})^{k+1} = (f_n^* \times f_n - \lambda_n \varphi_n)^{k+1}$$
$$= [(f_n^* \times f_n)^k - \lambda_n^k \varphi_n] \times (f_n^* \times f_n - \lambda_n \varphi_n)$$
$$= (f_n^* \times f_n)^{k+1} - \lambda_n^k \varphi_n \times f_n^* \times f_n - (f_n^* \times f_n)^k \times \lambda_n \varphi_n + \lambda_n^k \varphi_n \times \lambda_n \varphi_n$$
$$= (f_n^* \times f_n)^{k+1} - \lambda_n^{k+1} \varphi_n.$$

It follows $\left(\dfrac{\lambda_{n+1}}{\lambda_n}\right)^k \left\| \dfrac{(f_{n+1}^* \times f_{n+1})^k}{\lambda_{n+1}^k} \right\| = \left\| \dfrac{(f_n^* \times f_n)^k}{\lambda_n^k} - \varphi_n \right\| \to 0$ as $k \to \infty$.

Since $\|(f_{n+1}^* \times f_{n+1})^k / \lambda_{n+1}^k\| \to \|\varphi_{n+1}\| \geq 1$ as $k \to \infty$, we have $\lambda_{n+1}/\lambda_n < 1$, i.e. $\lambda_{n+1} < \lambda_n$. Next summing up the following

$$f_{n+1}^* \times f_{n+1} = f_n^* \times f_n - \lambda_n \varphi_n$$
$$f_n^* \times f_n = f_{n-1}^* \times f_{n-1} - \lambda_{n-1} \varphi_{n-1}$$
$$\cdots = \cdots$$
$$f_2^* \times f_2 = f_1^* \times f_1 - \lambda_1 \varphi_1$$

we obtain $f_{n+1}^* \times f_{n+1} = f^* \times f - \sum_{k=1}^n \lambda_k \varphi_k$. Next we claim that for all $k \geq 1$ we have $f_{n+k}^* \times f_{n+k} \times \varphi_n = 0$ and $\varphi_{n+k} \times \varphi_n = 0$. In fact for $k = 1$, we get

$$f_{n+1}^* \times f_{n+1} \times \varphi_n = (f_n^* \times f_n - \lambda_n \varphi_n) \times \varphi_n$$
$$= f_n^* \times f_n \times \varphi_n - \lambda_n \varphi_n \times \varphi_n = \lambda_n \varphi_n - \lambda_n \varphi_n = 0$$

and $\lambda_{n+1} \varphi_{n+1} \times \varphi_n = (\varphi_{n+1} \times f_{n+1}^* \times f_{n+1}) \times \varphi_n = \varphi_{n+1} \times (f_{n+1}^* \times f_{n+1} \times \varphi_n) = 0$. or $\varphi_{n+1} \times \varphi_n = 0$ since $\lambda_{n+1} > 0$. Next for arbitrary $k > 1$, we have

$$f_{n+k+1}^* \times f_{n+k+1} \times \varphi_n = (f_{n+k}^* \times f_{n+k} - \lambda_{n+k} \varphi_{n+k}) \times \varphi_n$$
$$= f_{n+k}^* \times f_{n+k} \times \varphi_n - \lambda_{n+k} \varphi_{n+k} \times \varphi_n = 0, \qquad \text{by induction ;}$$

and also

$$\lambda_{n+k+1} \varphi_{n+k+1} \times \varphi_n = (\varphi_{n+k} \times f_{n+k+1}^* \times f_{n+k+1}) \times \varphi_n$$
$$= \varphi_{n+k} \times (f_{n+k+1}^* \times f_{n+k+1} \times \varphi_n) = 0.$$

Therefore we obtain

$$f^* \times f \times \varphi_n = \left(f_{n+1}^* \times f_{n+1} + \textstyle\sum_{k=1}^n \lambda_k \varphi_k\right) \times \varphi_n$$
$$= f_{n+1}^* \times f_{n+1} \times \varphi_n + \lambda_n \varphi_n \times \varphi_n + \left(\textstyle\sum_{k=1}^{n-1} \lambda_k \varphi_k\right) \times \varphi_n$$
$$= 0 + \lambda_n \varphi_n + 0 = \lambda_n \varphi_n.$$

Taking the involution, we have $\varphi_n \times f^* \times f = \lambda_n \varphi_n$. From

$$\|f\|^2 = (f^* \times f)(e) = (f_{n+1}^* \times f_{n+1})(e) + \textstyle\sum_{k=1}^n \lambda_k \varphi_k(e)$$
$$= \|f_{n+1}\|^2 + \textstyle\sum_{k=1}^n \lambda_k \|\varphi_k\|^2 \geq \sum_{k=1}^n \lambda_k$$

we get $\lambda_k \downarrow 0$. By §28-4.7b, $\|f_n^* \times f_n\|^2 \leq \lambda_n \|f_n\|^2 \leq \lambda_n \|f\|^2 \to 0$. Hence $f_n^* \times f_n \to 0$ in $A_2(G)$. On the other hand since

$$\sum_{k=1}^{n} \|\lambda_k \varphi_k\|_\infty \le \sum_{k=1}^{n} \lambda_k \|\varphi_k \times \varphi_k\|_\infty \le \sum_{k=1}^{n} \lambda_k \|\varphi_k\|^2 \le \|f\|^2,$$

the series $h = \sum_{k=1}^{\infty} \lambda_k \varphi_k$ converges to some ap-function $h \in A_\infty(G)$. Hence $f_{n+1}^* \times f_{n+1} = f^* \times f - \sum_{k=1}^{n} \lambda_k \varphi_k$ converges to $f^* \times f - h$ in $A_\infty(G)$ and thus also in $A_2(G)$. Since $\|f_n \times f_n\|^2 \to 0$, we have $f^* \times f - h = 0$. Therefore $f^* \times f = \sum_{k=1}^{\infty} \lambda_k \varphi_k$ in $A_\infty(G)$. The proof is competed as follow:

$$\|f\|^2 = (f^* \times f)(e) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(e) = \sum_{k=1}^{\infty} \lambda_k \varphi_k \times \varphi_k(e) = \sum_{k=1}^{\infty} \lambda_k \|\varphi_k\|^2. \quad \square$$

28-4.11. **Theorem** For every $f \in A(G)$ and every $\varepsilon > 0$, there are projectors ψ, ξ in $A(G)$ such that $\|f - f \times \psi\| \le \varepsilon$ and $\|f - \xi \times f\| \le \varepsilon$.

Proof. Choose $0 < \lambda_n \downarrow 0$ and projectors $\varphi_n \in A(G)$ according to the last theorem. Then for some n, we have $\|f\|^2 - \sum_{k=1}^{n} \lambda_k \|\varphi_k\|^2 \le \varepsilon^2$. Since $\varphi_j \times \varphi_k = 0$ for $j \ne k$, their sum $\psi = \sum_{k=1}^{n} \varphi_k$ is also a projector in $A(G)$. Furthermore,

$$\begin{aligned} &\|f - f \times \psi\|^2 = (f - f \times \psi)^* \times (f - f \times \psi)(e) \\ &= f^* \times f(e) - (\psi \times f^* \times f)(e) - (f^* \times f \times \psi)(e) + (\psi \times f^* \times f \times \psi)(e) \\ &= \|f\|^2 - \textstyle\sum_{k=1}^{n} \lambda_k \|\varphi_k\|^2 \le \varepsilon^2. \end{aligned}$$

This proves the first inequality. Taking the hermitian we get $\|f^* - \psi \times f^*\| \le \varepsilon$. Since f^* is also arbitrary in $A(G)$, the proof is completed. $\square$

28-4.12. **Theorem** Let f be an ap-function in $A(G)$. For every $\varepsilon > 0$, there are $g, \varphi \in A(G)$ such that $g \ge 0$, $\int g d_m = 1$, $\varphi^* = \varphi = \varphi \times \varphi$ and $\|f - f \times g \times \varphi\|_\infty \le \varepsilon$. For every $\varepsilon > 0$, there are $h, \xi \in A(G)$ such that $h \ge 0$, $\int h d_m = 1$, $\psi^* = \psi = \psi \times \psi$ and $\|f - \psi \times h \times f\|_\infty \le \varepsilon$.

Proof. If $f = 0$, then $g = \varphi = 1$ satisfy the requirement. Assume $f \ne 0$. Choose $g \ge 0$ such that $\int g d_m = 1$ and $\|f - f \times g\|_\infty \le \frac{1}{2}\varepsilon$. There is a projector $\varphi \in A(G)$ such that $\|g - g \times \varphi\| \le \frac{1}{2}\varepsilon / \|f\|$. Therefore we have

$$\begin{aligned} &\|f - f \times g \times \varphi\|_\infty \le \|f - f \times g\|_\infty + \|f \times g - f \times g \times \varphi\|_\infty \\ &\le \tfrac{1}{2}\varepsilon + \|f\| \; \|g - g \times \varphi\| \le \tfrac{1}{2}\varepsilon + \|f\| \; \tfrac{1}{2}\varepsilon / \|f\| = \varepsilon. \end{aligned}$$

Replacing f by f^* and taking involution, the proof is completed. $\square$

28-4.13. Alternative method of integral equations and Hilbert spaces can be found in [Naimark, p189].

28-99. **References and Further Readings** : vonNeumann, Maak, Corduneanu, Amerio, Chulaevsky and Pankov.

Chapter 29

Group Representations

29-1 Matrix Representations

29-1.1. We initiated the study of almost periodic functions in terms of linear combinations of entry functions of group representations of which the basic theory is developed in this chapter. We restrict ourselves to finite dimensional representations for its connection to almost periodic functions.

29-1.2. The *trace* of a square matrix is the sum of its diagonal entries. Note that $tr(AB) = tr(BA)$ for all conformable square matrices A, B. The set $mat(s,t)$ of all complex matrices of size $s \times t$ is an inner product space under $< A, B >= tr(B^*A) = \sum_{i=1}^{s} \sum_{j=1}^{t} \alpha_{ij}\beta_{ij}^{-} =< A^t, B^t >$ for all $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ in $mat(s,t)$ where the *hermitian* $B^* = B^{t-}$ is the complex conjugate of the transpose of B.

29-1.3. **Lemma** (a) $\|AB\| \leq \|A\| \ \|B\|$ and $|tr(AB)| \leq \|A\| \ \|B\|$ for all conformable matrices A, B.
(b) If A is unitary of order s, then $\|A\| = s$.
(c) If A, B are similar matrices, then $\|A\| = \|B\|$.
(d) $\|A^*\| = \|A\|$ in general.

29-1.4. Let G be a group. A map $E = [E_{ij}] : G \to mat(s,t)$ is *almost periodic* if all its entry functions E_{ij} are almost periodic. Almost periodic maps are also called *ap-maps* for convenience. The mean-value of an ap-map E is defined by $\int E(x)d_mx = \left[\int E_{ij}(x)d_mx\right]$. Properties of mean-values of ap-maps follow immediately from their entry functions. Note that $E^*(x) = E(x^{-1})^{t-} = E(x^{-1})^*$ but we do not need this except when E is a function.

29-1.5. **Lemma** Let $D(x), E(x)$ be ap-maps on G.
(a) $\int E(ax)d_mx = \int E(x)d_mx = \int E(x^{-1})d_mx$.
(b) $\int E(x)^- d_mx = [\int E(x)d_mx]^-$ and $\int E(x)^t d_mx = [\int E(x)d_mx]^t$.
(c) $\int tr\ E(x)d_mx = tr \int E(x)d_mx$.
(d) For conformable matrices B, C we have $\int BE(x)C\ d_mx = B[\int E(x)d_mx]C$.

(e) The function $x \to \|E(x)\|^2$ is almost periodic and we have

$$\| \int E(x) d_m x\|^2 \le \int \|E(x)\|^2 d_m x.$$

(f) For every conformable matrix B the following generalizes the convolution of scalar functions:

$$\int D(xy^{-1}) B E(y) d_m y = \int D(xy) B E(y^{-1}) d_m y$$
$$= \int D(y) B E(y^{-1}x) d_m y = \int D(y^{-1}) B E(yx) d_m y.$$

29-1.6. The general linear group $GL(s)$ is the multiplicative group of all invertible matrices of order s. A homomorphism from G into $GL(s)$ is called a *representation*. A representation $D : G \to GL(s)$ is ***bounded*** if all entry functions D_{ij} are bounded on G and ***unitary*** if $D(x)$ is a unitary matrix for every $x \in G$. The group of all unitary matrices of order s is denoted by $U(s)$. Two representations D, E are ***equivalent***, denoted by $D \sim E$ if there is an invertible matrix P such that $D(x) = P^{-1}E(x)P$ for all $x \in G$. In this case, the equivalence is said to be under the ***intertwining matrix*** P.

29-1.7. **Theorem** Let D be a representation of a group G. Then the following statements are equivalent.

(a) D is equivalent to a unitary representation.

(b) D is a bounded representation.

(c) D is an ap-map.

Proof. $(a \Rightarrow b)$ Suppose D is equivalent to a unitary representation E under an intertwining matrix P of order s, i.e. $D(x) = P^{-1}E(x)P$ for all $x \in G$. Then

$$\|D(x)\| \le \|P^{-1}\| \; \|E(x)\| \; \|P\| \le s\|P^{-1}\| \; \|P\|$$

shows that D is bounded on G.

$(b \Rightarrow c)$ It has been proved in §28-1.6.

$(c \Rightarrow a)$ Suppose that $D(x)$ is an ap-map. Let $A = \int D(x)^* D(x) d_m x$. Then $A^* = A$. Since D is a representation of G, $D(e)$ is the identity matrix. For any nonzero column vector $v \in \mathbb{C}^s$, we have $\|D(e)v\|^2 = \|v\|^2 > 0$. Thus

$$v^* A v = \int v^* D(x)^* D(x) v d_m x = \int \|D(x)v\|^2 d_m x > 0.$$

Hence A is a positive definite matrix. Therefore $A = P^*P$ for some invertible matrix P. By translation invariance, we have for all $y \in G$,

$$A = \int D(x)^* D(x) d_m x = \int D(xy)^* D(xy) d_m x$$
$$= \int D(y)^* D(x)^* D(x) D(y) d_m x = D(y)^* \left[\int D(x)^* D(x) d_m x\right] D(y)$$
$$= D(y)^* A D(y),$$

that is $P^*P = D(y)^*P^*PD(y)$. Now for every $y \in G$, define $E(y) = PD(y)P^{-1}$. Then E is a representation of G. Finally E is a unitary representation because

$$\begin{aligned} E(y)^*E(y) &= [PD(y)P^{-1}]^*PD(y)P^{-1} \\ &= P^{-1*}[D(y)^*P^*PD(y)]P^{-1} = P^{-1*}P^*PP^{-1} = I. \end{aligned}$$

□

29-1.8. A representation D of G is *reducible* if there is an invertible matrix P such that $P^{-1}D(x)P = \begin{bmatrix} A(x) & 0 \\ C(x) & B(x) \end{bmatrix}$ for all $x \in G$ where A, B are square matrix maps of order at least one. A representation is *irreducible* if it is not reducible. Clearly if A, B are representations of G, then $D(x) = A(x) \oplus B(x)$ for all $x \in G$ is a representation which is called the *direct sum* of A, B. In this case, we write $D = A \oplus B$.

29-1.9. **Theorem** If a representation $D(x) = \begin{bmatrix} A(x) & 0 \\ C(x) & B(x) \end{bmatrix}$ of G is reducible, then it is equivalent to $A \oplus B$.

Proof. Equating the corresponding blocks from

$$\begin{bmatrix} A(xy) & 0 \\ C(xy) & B(xy) \end{bmatrix} = D(xy) = D(x)D(y) = \begin{bmatrix} A(x) & 0 \\ C(x) & B(x) \end{bmatrix} \begin{bmatrix} A(y) & 0 \\ C(y) & B(y) \end{bmatrix},$$

we get $A(xy) = A(x)A(y)$, $C(xy) = C(x)A(y) + B(x)C(y)$ and $B(xy) = B(x)B(y)$. Let $N = \int C(y)A(y^{-1})d_my$ and $P = \begin{bmatrix} I & 0 \\ N & I \end{bmatrix}$. From

$$\begin{aligned} NA(x) &= \int C(y)A(y^{-1})A(x)d_my = \int C(y)A(y^{-1}x)d_my \\ &= \int C(xy)A(y^{-1})d_my = \int [C(x)A(y) + B(x)C(y)]A(y^{-1})d_my \\ &= \int C(x)d_my + B(x)\int C(y)A(y^{-1})d_my = C(x) + B(x)N, \end{aligned}$$

we have $D(x)P = P[A(x) \oplus B(x)]$, i.e. $P^{-1}D(x)P = A(x) \oplus B(x)$ for all $x \in G$. Therefore D is equivalent to $A \oplus B$. □

29-1.10. **Exercise** Prove that if A, B are bounded representations of G, then $A \oplus B$ and $B \oplus A$ are equivalent bounded representations. Also show that if A, B are equivalent bounded representations, then $A \oplus B$ is equivalent to $A \oplus A$. That means we can move the diagonal blocks to get the next theorem.

29-1.11. **Theorem** (a) Every bounded representation D of a group is equivalent to the direct sum of the form $\oplus_{w=1}^{n} m_w D^w$ where $D^1, D^2, \cdots, D^n$ are inequivalent irreducible unitary representations. In this case, m_w is called the *multiplicity* of D^w in D.

(b) If $D \sim \oplus_{k=1}^{t} r_k E^k$ where $E^1, E^2, \cdots, E^p$ are inequivalent irreducible bounded representations, then $n = t$ and there is a permutation p on $\{1, 2, \cdots, n\}$ such that $r_{p(w)} = m_w$ and $E^{p(w)} \sim D^w$ for every $1 \le w \le n$.

29-1.12. **Schur's Lemma** Let D, E be irreducible representations of G of degrees s, t respectively. If A is an $s \times t$ matrix satisfying $D(x)A = AE(x)$ for all $x \in G$, then either $A = 0$ or A is an invertible matrix. In this case, we must have $s = t$.

Proof. Let r be the rank of A. There are invertible matrices P, Q of order s, t respectively such that $A = P \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} Q$. Then for every $x \in G$, we have $P^{-1}D(x)P \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix} QE(x)Q^{-1}$. Suppose that

$$P^{-1}D(x)P = \begin{bmatrix} D_{11}(x) & D_{12}(x) \\ D_{21}(x) & D_{22}(x) \end{bmatrix} \quad \text{and} \quad Q^{-1}E(x)Q = \begin{bmatrix} E_{11}(x) & E_{12}(x) \\ E_{21}(x) & E_{22}(x) \end{bmatrix}$$

where $D_{22}(x)$ and $E_{11}(x)$ are square matrices of order r. Simple substitution gives $\begin{bmatrix} D_{12}(x) & 0 \\ D_{22}(x) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ E_{11}(x) & E_{12}(x) \end{bmatrix}$, i.e. $D_{12}(x) = 0$ and $E_{12}(x) = 0$. Since D, E are irreducible, we have either $r = 0$ or $r = s = t$. If $r = 0$, then $A = 0$. If $r = s = t$, then $A = PQ$ is invertible. □

29-1.13. **Theorem** Let D be an irreducible representation of G of degree s. If A is a matrix satisfying $D(x)A = AD(x)$ for all $x \in G$, then $A = \lambda I$ for some $\lambda \in \mathbb{C}$. If $D(x)D(y) = D(y)D(x)$ for all $x, y \in G$, then D is of degree one.

Proof. Let λ be an eigenvalue of A. Then $D(x)(A - \lambda I) = (A - \lambda I)D(x)$ for all $x \in G$. Since $A - \lambda I$ is singular, we have $A - \lambda I = 0$ as required. Next, take any $y \in G$. Since $D(x)D(y) = D(y)D(x)$ for all $x \in G$, we have $D(y) = \lambda(y)I$ for some $\lambda(y) \in \mathbb{C}$. Since D is irreducible, the order of D is one. The last statement is obvious. □

29-1.14. **Corollary** Every irreducible representation of an abelian group is of degree one.

29-1.15. **Theorem** Equivalent irreducible unitary representations D, E are unitarily equivalent, that is, there is some unitary matrix P such that $D(x) = P^{-1}E(x)P$ for all $x \in G$.

Proof. Let A be an invertible matrix such that $D(x) = A^{-1}E(x)A$ for all $x \in G$. Taking the hermitian, we have $D(x)^* = A^*E(x)^*(A^*)^{-1}$. Since D, E

are unitary representations, we get $D(x^{-1}) = A^*E(x^{-1})(A^*)^{-1}$ for all $x \in G$. Hence $D(x) = A^*E(x)A^{-1*}$, or $E(x) = (A^*)^{-1}D(x)A^*$ for all $x \in G$. Thus

$$D(x) = A^{-1}E(x)A = A^{-1}(A^*)^{-1}D(x)A^*A = (A^*A)^{-1}D(x)A^*A.$$

It follows that $A^*A = \lambda I$ for some $\lambda \in \mathbb{C}$. Because A^*A is positive definite and invertible, $\lambda = tr(A^*A)/s > 0$. Define $P = \frac{1}{\sqrt{\lambda}}A$. Then we get $P^*P = (\frac{1}{\sqrt{\lambda}}A^*)(\frac{1}{\sqrt{\lambda}}A) = I$. Therefore P is unitary. Furthermore,

$$P^{-1}E(x)P = \left(\tfrac{1}{\sqrt{\lambda}}A\right)^{-1} E(x)\tfrac{1}{\sqrt{\lambda}}A = \sqrt{\lambda}A^{-1}E(x)\tfrac{1}{\sqrt{\lambda}}A = D(x), \qquad \forall\, x \in G.$$

Consequently, D, E are unitarily equivalent. □

29-1.16. **Lemma** If D, E are bounded representations of G, then for every conformable matrix B and all $x \in G$ we have

$$D(x) \int D(y^{-1})BE(y)d_my = \left[\int D(y^{-1})BE(y)d_my\right] E(x).$$

Proof. Changing variables $y = zx$ as in convolution, we have

$$D(x) \int D(y^{-1})BE(y)d_my = \int D(xy^{-1})BE(y)d_my$$
$$= \int D(z^{-1})BE(zx)d_mz = \left[\int D(z^{-1})BE(z)d_mz\right] E(x).$$ □

29-1.17. **Theorem** If D, E are inequivalent irreducible bounded representations, then for all conformable matrix B we have $\int D(y^{-1})BE(y)d_my = 0$.

Proof. If $\int D(y^{-1})BE(y)d_my = 0$ is nonzero, then it is invertible by Schur's Lemma and consequently D, E are equivalent. □

29-1.18. **Theorem** If D is an irreducible bounded representation of degree s, then for every conformable matrix B we have $\displaystyle\int D(y^{-1})BD(y)d_my = \frac{tr(B)}{s}I$.

Proof. Since D is irreducible, the identity

$$D(x) \int D(y^{-1})BD(y)d_my = \left[\int D(y^{-1})BD(y)d_my\right] D(x)$$

shows $\int D(y^{-1})BD(y)d_my = \lambda I$ for some $\lambda \in \mathbb{C}$. Taking the trace, we have

$$\lambda s = tr \int D(y^{-1})BD(y)d_my = \int tr[D(y^{-1})BD(y)]d_my$$
$$= \int tr[D(y)D(y^{-1})B]d_my = \int tr(B)d_my = tr(B).$$

Therefore we have $\int D(y^{-1})BD(y)d_my = \lambda I = \frac{tr(B)}{s}I$ as required. □

29-1.19. **Corollary** Let D, E be inequivalent irreducible bounded representations of G of degrees s, t respectively. Suppose that D_{ij}, E_{kn} are entry functions of D, E respectively.

(a) $D_{ij} \times E_{kn} = 0$ and $D_{ij} \times D_{kn} = \frac{1}{s}\delta_{jk}D_{in}$.

(b) The functions D_{ij}, E_{kn} for $1 \le i,j \le s$, $1 \le k,n \le t$ are linearly independent.

Proof. (a) For fixed j,k, let $B = [\delta_{pj}\delta_{kq}]$ be the matrix of size $s \times t$ with all zero entries except the (j,k)-th which is one. Since D, E are inequivalent irreducible, we have $\int D(xy^{-1})BE(y)d_m y = D(x)\int D(y^{-1})BE(y)d_m y = 0$ for all $x \in G$. The (i,n)-th entry of this matrix is

$$0 = \int \sum_{p=1}^{s} \sum_{q=1}^{t} D_{ip}(xy^{-1})(\delta_{pj}\delta_{kq})E_{qn}(y)d_m y$$
$$= \int D_{ij}(xy^{-1})E_{kn}(y)d_m y = D_{ij} \times E_{kn}(x), \qquad \forall\, x \in G,$$

that is $D_{ij} \times E_{kn} = 0$. Next, observe that

$$\int D(xy^{-1})BD(y)d_m y = D(x)\int D(y^{-1})BD(y)d_m y = \tfrac{1}{s}tr(B)D(x), \qquad \forall\, x \in G.$$

Equating the (i,n)-th entries of both sides, we have

$$D_{ij} \times D_{kn}(x) = \tfrac{1}{s}tr(B)D_{in}(x) = \tfrac{1}{s}\delta_{jk}D_{in}(x), \qquad \forall\, x \in G,$$

that is $D_{ij} \times D_{kn} = \frac{1}{s}\delta_{jk}D_{in}$.

(b) Assume that $\sum_{ij}\alpha_{ij}D_{ij} + \sum_{kn}\beta_{kn}E_{kn} = 0$ for some $\alpha_{ij}, \beta_{kn} \in \mathbb{C}$. Then

$$D_{1p} \times \left(\sum_{ij}\alpha_{ij}D_{ij} + \sum_{kn}\beta_{kn}E_{kn}\right) \times D_{q1} = 0$$

i.e.

$$\sum_{ij}\alpha_{ij}\delta_{pi}\delta_{jq}D_{11} + 0 = 0,$$

or $\alpha_{pq} = \alpha_{pq}D_{11}(e) = 0$. Hence all $\alpha_{ij} = 0$. Similarly, all $\beta_{kn} = 0$. Consequently all D_{ij} and E_{kn} are linearly independent. □

29-1.20. **Corollary** Let D, E be inequivalent irreducible representations of G. If both D, E are unitary, then

(a) $< D_{ij}, E_{kn} >= \int D_{ij}(x)\overline{E_{kn}(x)}d_m x = 0$,

(b) $< D_{ij}, D_{kn} >= \int D_{ij}(x)\overline{D_{kn}(x)}d_m x = \frac{1}{s}\delta_{ik}\delta_{jn}$.

Proof. (a)

$$< D_{ij}, E_{kn} >= \int D_{ij}(x)\overline{E_{kn}(x)}d_m x$$
$$= \int D_{ij}(x)E_{nk}(x^{-1})d_m x = (D_{ij} \times E_{nk})(e) = 0.$$

(b) $< D_{ij}, D_{kn} >= (D_{ij} \times D_{nk})(e) = \frac{1}{s}\delta_{in}D_{ik}$. □

29-2 Characterization of Projectors

29-2.1. Through eigen expansion, we constructed a nonzero projector φ associated with every nonzero ap-function. In this section, we show that the solution space S_φ of $\varphi \times f = f$ in f is finite dimensional and right translation invariant. It is then used to characterize projectors.

29-2.2. Let $A(G)$ denote the space of ap-functions on a group G. A vector subspace M of $A(G)$ is called a ***closed right invariant ideal*** if M is closed in $A_\infty(G)$ and M is right translation invariant, i.e. for every $a \in G$ we have $R_a M \subset M$. Similarly we define closed left and two-sided invariant ideals. Two-sided ideals are simply called ***ideals***. A subset M of $A(G)$ is ***hermitian*** or ***self-adjoint*** if $f^* \in M$ for all $f \in M$. Clearly, a hermitian closed right or left invariant ideal is a closed two-sided invariant ideal because $(R_{a^{-1}} f^*)^* = L_a f$.

29-2.3. **Theorem** If M is a closed right invariant ideal of $A(G)$, then M is a right ideal of the convolution algebra, i.e. $M \times A(G) \subset M$. Furthermore if $f \in M$, then the function g given by $g(x) = \int f(xy) d_m y$ belongs to M. Similar result holds for left and two-sided ideals.

Proof. Let $f \in M$ and $h \in A(G)$. Then $f \times h$ is the uniform limit of linear combinations of right translations of f. Hence $f \times h \in M$. In particular for $h = 1$, we have $g = f \times h \in M$. □

29-2.4. **Theorem** Let M be a finite dimensional closed right invariant ideal of the inner product space $A_2(G)$ and $H = [h_1, \cdots, h_s]$ an ordered orthonormal basis of M. For every $a \in G$, let $D(a) = [D_{ij}(a)]$ be the matrix representation of the right translate $R_a : M_H \to M_H$. Then D is a representation of G of degree s. It is called the ***representation associated with*** H. Furthermore for all $a, x \in G$ we have $h_j(xa) = \sum_{i=1}^{s} h_i(x) D_{ij}(a)$.

Proof. Clearly R_a is a linear operator on M. Since for all $f, g \in M$,

$$< R_a f, R_a g >= \int f(xa)\overline{g(xa)} d_m x = \int g(x)\overline{g(x)} d_m x =< f, g > ;$$

R_a is an isometry. Hence each $D(a)$ is a unitary matrix. Furthermore, because $D(ab) = [R_{ab}] = [R_a R_b] = [R_a]\ [R_b] = D(a)D(b)$, D is a representation of G. Since $D(a)$ is the matrix representation of R_a, by §7-5.8 we have $R_a H = HD(a)$, that is $[h_1(xa), \cdots, h_s(xa)] = [h_1(x), \cdots, h_s(x)]\ [D_{ij}(a)]$. Equating the j-th column, the result follows. □

29-2.5. **Theorem** Let H, K be ordered orthonormal bases of a closed right invariant ideal M of $A(G)$. Then the unitary representations D, E associated with H, K respectively are equivalent. A representation associated with some orthonormal basis of M is said to be ***associated*** with M.

Proof. Let P be the transition matrix from H to K. Then for each $a \in G$ we have $D(a) = P^{-1} E(a) P$ and P is a unitary matrix. Therefore D, E are unitarily equivalent. □

29-2.6. **Theorem** Let φ be a projector in $A(G)$. The *right solution space* of φ defined by $S_\varphi = \{f \in A(G) : \varphi \times f = f\}$ is a finite dimensional closed right invariant ideal of $A(G)$.

Proof. Let $\xi(f) = \varphi \times f - f$ for every $f \in A(G)$. Since

$$\|\xi(f)\|_\infty \le \|\varphi \times f\|_\infty + \|f\|_\infty \le \|\varphi\|_\infty \|f\|_\infty + \|f\|_\infty = (\|\varphi\|_\infty + 1)\|f\|_\infty,$$

the linear map $\xi : A_\infty(G) \to A_\infty(G)$ is continuous. Therefore its kernel S_φ is closed vector subspace of $A_\infty(G)$. Let $H = [h_1, \cdots, h_s]$ be any orthonormal set in S_φ. Define $\pi(x,y) = \varphi(xy^{-1}) - \sum_{i=1}^s h_i(x)\overline{h_i(y)}$ for all $x, y \in G$. Then π is an ap-function on $G \times G$. Observe that

$$\begin{aligned}
0 &\le \int\int |\pi(x,y)|^2 d_m x d_m y \\
&= \int\int \left[\varphi(xy^{-1}) - \textstyle\sum_{i=1}^s h_i(x)\overline{h_i(y)}\right] \overline{\left[\varphi(xy^{-1}) - \textstyle\sum_{j=1}^s h_j(x)\overline{h_j(y)}\right]} d_m x d_m y \\
&= \int\int |\varphi(xy^{-1})|^2 d_m x d_m y - \textstyle\sum_{i=1}^s \int\int h_i(x)\overline{h_i(y)}\ \overline{\varphi(xy^{-1})} d_m x d_m y \\
&\qquad - \textstyle\sum_{j=1}^s \int\int \overline{h_j(x)} h_j(y)\varphi(xy^{-1}) d_m x d_m y \\
&\qquad + \textstyle\sum_{i,j=1}^s \int\int h_i(x)\overline{h_i(y)}\ \overline{h_j(x)} h_j(y) d_m x d_m y \\
&= \|\varphi\|^2 - \textstyle\sum_{i=1}^s \int h_i(x)\overline{(\varphi \times h_i)(x)} d_m x - \textstyle\sum_{j=1}^s \int \overline{h_j(x)}(\varphi \times h_j)(x) d_m x \\
&\qquad + \textstyle\sum_{i,j=1}^s \left(\int h_i(x)\overline{h_j(x)} d_m x\right)\left(\int h_j(y)\overline{h_i(y)} d_m x\right) \\
&= \|\varphi\|^2 - \textstyle\sum_{i=1}^s \int h_i(x)\overline{h_i(x)} d_m x - \textstyle\sum_{j=1}^s \int \overline{h_j(x)} h_j(x) d_m x + \textstyle\sum_{i,j=1}^s \delta_{ij}\delta_{ji} \\
&= \|\varphi\|^2 - s,
\end{aligned}$$

or $s \le \|\varphi\|^2$. Therefore S_φ is finite dimensional. For every $a \in G$ and $f \in S_\varphi$, we have

$$\begin{aligned}
\varphi \times (R_a f)(x) &= \int \varphi(xy^{-1}) R_a f(y) d_m y = \int \varphi(xy^{-1}) f(ya) d_m y \\
&= \int \varphi(xay^{-1}) f(y) d_m y = (\varphi \times f)(xa) = R_a(\varphi \times f)(x) = R_a f(x).
\end{aligned}$$

Hence $R_a f \in S_\varphi$. Therefore $R_a S_\varphi \subset S_\varphi$. Consequently S_φ is a finite dimensional closed right invariant ideal. □

29-2.7. **Theorem** Let $H = [h_1, \cdots, h_s]$ be an orthonormal basis of the right solution space S_φ of a projector φ in $A(G)$. Then we have $\varphi(xy^{-1}) = \sum_{i=1}^s h_i(x)\overline{h_i(y)}$ for all $x, y \in G$. Furthermore the dimension s of S_φ is equal to $\|\varphi\|^2$.

Proof. Fix $a \in G$. Define $f(x) = \varphi(xa^{-1}) - \sum_{i=1}^s h_i(x)\overline{h_i(a)}$ for every $x \in G$. Then the routine calculation

$$(\varphi \times f)(x) = \int \varphi(xy^{-1}) f(y) d_m y$$

$$= \int \varphi(xy^{-1})\varphi(ya^{-1})d_my - \sum_{i=1}^{s} \int \varphi(xy^{-1})h_i(y)\overline{h_i(a)}d_my$$
$$= \int \varphi(xa^{-1}y^{-1})\varphi(y)d_my - \sum_{i=1}^{s}(\varphi \times h_i)(x)\overline{h_i(a)}$$
$$= (\varphi \times \varphi)(xa^{-1}) - \sum_{i=1}^{s} h_i(x)\overline{h_i(a)}$$
$$= \varphi(xa^{-1}) - \sum_{i=1}^{s} h_i(x)\overline{h_i(a)} = f(x)$$

shows that $f \in S_\varphi$. Next observe that

$$< f, h_j >= \int \left[\varphi(xa^{-1}) - \sum_{i=1}^{s} h_i(x)\overline{h_i(a)}\ \right] \overline{h_j(x)}d_mx$$
$$= \int \varphi^*(xa^{-1})\overline{h_j(x)}d_mx - \sum_{i=1}^{s} \int h_i(x)\overline{h_i(a)}\ \overline{h_j(x)}d_mx$$
$$= \int \overline{\varphi(ax^{-1})}\ \overline{h_j(x)}d_mx - \sum_{i=1}^{s} \delta_{ij}\overline{h_i(a)}$$
$$= \overline{\varphi \times h_j(a)} - \overline{h_j(a)} = 0.$$

Since H is an orthonormal basis, we have $f = 0$, i.e. $\varphi(xa^{-1}) = \sum_{i=1}^{s} h_i(x)\overline{h_i(a)}$ for all $a, x \in G$. It follows from the proof of last theorem that

$$0 = \int \int |\pi(x, y)|^2 d_mxd_my = \|\varphi\|^2 - s, \text{ or } s = \|\varphi\|^2. \qquad \Box$$

29-2.8. **Lemma** Let $\{D^w : w \in J\}$ be a finite set of inequivalent irreducible unitary representations of G. For each w, let s_w be the order of D^w and A^w a square matrix of order s_w. If $\varphi = \sum_{w\in J} s_w < D^w, A^{w-} >$, then we have $\varphi^* = \sum_{w\in J} s_w < D^w, (A^{w-})^* >$ and $\varphi \times \varphi = \sum_{w\in J} s_w < D^w, (A^{w-})^2 >$.

Proof. For $D^w = [D^w_{ij}]$ and $A^w = [\alpha^w_{ij}]$, we have

$$\varphi^*(x) = \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha^{w-}_{ij}\ \overline{D^w_{ij}(x^{-1})}$$
$$= \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha^{w-}_{ij} D^w_{ji}(x) = \sum_{w\in J} s_w < D^w(x), (A^{w-})^* >$$

and

$$\varphi \times \varphi = \left(\sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha^w_{ij} D^w_{ij}\right) \times \left(\sum_{v\in J} s_v \sum_{p,q=1}^{s_v} \alpha^v_{pq} D^v_{pq}\right)$$
$$= \sum_{w,v\in J} \sum_{i,j=1}^{s_w} \sum_{p,q=1}^{s_v} s_w s_v \alpha^w_{ij}\alpha^v_{pq}\ \delta_{wv}\ \delta_{jp}\ \frac{1}{s_v}\ D_{iq}$$
$$= \sum_{w\in J} \sum_{i,j,q=1}^{s_w} s_w \alpha^w_{ij}\alpha^w_{jq}\ D_{iq} = \sum_{w\in J} s_w < D^w, (A^{w-})^2 >. \qquad \Box$$

29-2.9. **Theorem** An ap-function φ on G is a projector iff

$$\varphi = \sum_{w\in J} s_w < D^w, A^{w-} >$$

where $\{D^w : w \in J\}$ is a finite set of inequivalent irreducible unitary representations of G, each A^w is a projector matrix and each s_w is the order of D^w, A^w. In this case, we get $A^w =< \varphi, D^w >$, that is $[< \varphi, D^w_{ij} >]$.

Proof. ($\Rightarrow$) Let φ be a projector in $A(G)$. Then φ is a linear combination of entry functions of some unitary representation D of G. There is an invertible matrix P such that $P^{-1}D(x)P$ is the direct sum of some irreducible unitary representations among which the set of all inequivalent irreducible representations is denoted by $\{D^w : w \in J\}$. Then φ is also a linear combination of entry functions of $\{D^w : w \in J\}$. More precisely, for some $\alpha_{ij}^w \in \mathbb{C}$ we have $\varphi = \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w D_{ij}^w$ where s_w is the order of D_w. For $A^w = [\alpha_{ij}^w]$, we obtain the required equality. From $\varphi = \varphi^* = \varphi \times \varphi$, we get

$$\begin{aligned}\textstyle\sum_{w\in J} s_w < D^w(x), A^{w-} > &= \textstyle\sum_{w\in J} s_w < D^w(x), (A^{w-})^* > \\ &= \textstyle\sum_{w\in J} s_w < D^w(x), (A^{w-})^2 >. \qquad \text{(a)}\end{aligned}$$

Because the set $\{D_{ij}^w : w \in J,\ 1 \le i, j \le s_w\}$ is linearly independent, we obtain $(A^{w-}) = (A^{w-})^* = (A^{w-})^2$. Thus A^{w-} is a projector matrix and so is A^w.
($\Leftarrow$) Suppose that for all $x \in G$ we have $\varphi = \sum_{w\in J} s_w < D^w, A^{w-} >$ where $\{D^w : w \in J\}$ is a finite set of inequivalent irreducible unitary representations of G, each s_w is the order of D^w and each A^w is a projector matrix. Then we obtain the equation (a) above from which we get $\varphi = \varphi^* = \varphi \times \varphi$. Therefore φ is a projector in $A(G)$. Finally, the proof is completed by the computation:

$$\begin{aligned}< \varphi, D_{kn}^v > &= \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w < D_{ij}^w, D_{kn}^v > \\ &= \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w \frac{1}{s_v} \delta_{wv}\delta_{ik}\delta_{jn} = \alpha_{kn}^v.\end{aligned}$$

□

29-2.10. **Corollary** Let D, E be equivalent irreducible unitary representations of G of the same degree s and A, B be nonzero projector matrices of order s. If $\varphi = s < D, A^- > = s < E, B^- >$; then A, B are unitarily similar.

Proof. Let P be a unitary matrix such that $D = P^{-1}EP$. Then the complex conjugate $Q = P^-$ is also unitary and $D^- = Q^{-1}E^-Q$. Therefore we have

$$\begin{aligned}A = < \varphi, D > &= \textstyle\int \varphi(x)D(x)^- d_m x = \int \varphi(x)Q^{-1}E(x)^- Q d_m x \\ &= Q^{-1}[\textstyle\int \varphi(x)E(x)^- d_m x]Q = Q^{-1} < \varphi, E > Q = Q^{-1}BQ.\end{aligned}$$

□

29-3 Fourier Matrices

29-3.1. Let D be a bounded representation of a group G and f, g be almost periodic functions on a group G. Then the *Fourier matrix* of f corresponding to D is defined by $D(f) = < f, D > = \int f(x)D(x)^- d_m x$, the *Fourier inner product* of f, g by $D < f, g > = < D(f), D(g) > = tr\ D(g)^* D(f)$ and the *Fourier norm* of f by $\|D(f)\| = \sqrt{D < f, f >}$. The entries of $D(f)$ are called the *Fourier coefficients* of f.

29-3.2. **Theorem** (a) $(f \times D)(x) = D(f)^t D(x)$ and $(D \times f)(x) = D(x)D(f)^t$.
(b) $D(L_a f) = D(a)^t D(f)$ and $D(R_a f) = D(f)D(a)^t$.

Proof. We only prove the first equality of both parts as follow:

$$f \times D(x) = \int f(y)D(y^{-1}x)d_m y = \left[\int f(y)D(y^{-1})d_m y\right] D(x)$$
$$= \left[\int f(y)D(y)^{t-}d_m y\right] D(x) = \left[\int f(y)D(y)^{-}d_m y\right]^t D(x) = D(f)^t D(x)$$

and

$$D(L_a f) = \int L_a f(x)D(x)^- d_m x = \int f(ax)D(x)^- d_m x$$
$$= \int f(y)D(a^{-1}y)^- d_m y = D(a^{-1})^- \int f(y)D(y)^- d_m y = D(a)^t D(f). \qquad \square$$

29-3.3. **Theorem** Let D be a unitary representation of G of order s. Then the map $f \to D(f)$ is a star homomorphism of the convolution algebra $A(G)$ into the algebra $M(s)$ of square matrices of order s.

Proof. Clearly $f \to D(f)$ is a linear map. The proof is completed by the following calculation:

$$D(f^*) = \int f^*(x)D(x)^- d_m x = \int f(x^{-1})^- D(x^{-1})^{*-} d_m x$$
$$= \int f(x)^- D(x)^{*-} d_m x = \left[\int f(x)D(x)^- d_m x\right]^* = D(f)^*$$

and

$$D(f \times g) = \int (f \times g)(x)D(x)^- d_m x = \int \int f(xy^{-1})g(y)D(x)^- d_m x d_m y$$
$$= \int \int f(x)g(y)D(xy)^- d_m x d_m y$$
$$= \left[\int f(x)D(x)^- d_m x\right] \left[\int g(y)D(y)^- d_m y\right] = D(f)D(g). \qquad \square$$

29-3.4. **Theorem** If D, E are equivalent unitary representations of G, then the Fourier matrices $D(f), E(f)$ are unitarily similar and $D < f, g >= E < f, g >$ for all $f, g \in A(G)$.

Proof. Let P be a unitary matrix such that $D(x) = P^* E(x) P$ for all $x \in G$. Then $Q = P^-$ is unitary, $D(x)^- = Q^* E(x)^- Q$ and

$$D(f) = \int f(x)D(x)^- d_m x = \int f(x)Q^* E(x)^- Q d_m x$$
$$= Q^* \left[\int f(x)E(x)^- d_m x\right] Q = Q^* E(f) Q.$$

Hence $D(f), E(f)$ are unitarily similar. Finally for all $f, g \in A(G)$, we have

$$D < f, g >=< D(f), D(g) >= tr D(g)^* D(f)$$
$$= tr[Q^* E(g) Q]^* [Q^* E(g) Q] = tr[Q^* E(g)^* Q Q^* E(g) Q]$$
$$= tr E(g)^* E(g) =< E(f), E(g) >= E < f, g > . \qquad \square$$

29-3.5. **Theorem** Let $\{D^w : w \in J\}$ be a finite set of inequivalent irreducible unitary representations of G and let s_w be the degree of D^w for each w. Let f be an ap-function on G. Suppose that $g = \sum_{w\in J} s_w < D^w, A^w >$ where $\{A^w : w \in J\}$ is a set of constant matrices. Then we have

(a) $\|f-g\|^2 = \|f\|^2 - \sum_{w\in J} s_w\|D^w(f)\|^2 + \sum_{w\in J} s_w\|A^w - D^w(f)\|^2$,

(b) **Bessel's Inequality** $\sum_{w\in J} s_w\|D^w(f)\|^2 \le \|f\|^2$.

Proof. Let $A^w = [\alpha^w_{ij}]$, $D^w(x) = [D^w_{ij}(x)]$ and $D^w(f) = [\beta^w_{ij}]$. Then we have

$$\|f-g\|^2 =< f - \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha^w_{ij} D^w_{ij}\ ,\ f - \sum_{v\in J} s_v \sum_{k,n=1}^{s_v} \alpha^v_{kn} D^v_{kn} >$$

$$= \|f\|^2 - \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha^w_{ij} < D^w_{ij}, f > - \sum_{v\in J} s_v \sum_{k,n=1}^{s_v} \alpha^{v-}_{kn} < f, D^v_{kn} >$$

$$+ \sum_{w,v\in J} s_w s_v \sum_{i,j=1}^{s_w} \sum_{k,n=1}^{s_v} \alpha^w_{ij}\alpha^{v-}_{kn} < D^w_{ij}, D^v_{kn} >$$

$$= \|f\|^2 + \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \{-\alpha^w_{ij}\beta^{w-}_{ij} - \alpha^{w-}_{ij}\beta^w_{ij} + \alpha^{w-}_{ij}\alpha^{w-}_{ij}\}$$

$$= \|f\|^2 + \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \{(\alpha^{w-}_{ij} - \beta^{w-}_{ij})(\alpha^w_{ij} - \beta^w_{ij}) - \beta^{w-}_{ij}\beta^w_{ij}\}$$

$$= \|f\|^2 + \sum_{w\in J} s_w\|A^w - D^w(f)\|^2 - \sum_{w\in J} s_w\|D^w f\|^2.$$

Letting $A^w = D^w(f)$, we obtain $0 \le \|f-g\|^2 = \|f\|^2 - \sum_{w\in J} s_w\|D^w(f)\|^2$. □

29-3.6. **Theorem** Let D, E be bounded representations of G and TD, TE the vector subspaces of $A(G)$ spanned by the entry functions of D, E respectively. If D, E are equivalent, then $TD = TE$.

Proof. Let P be an invertible matrix such that $D(x) = P^{-1}E(x)P$ for all $x \in G$. Since every entry function of $D(x)$ is a linear combination of entry functions of $E(x)$, we have $TD \subset TE$. By symmetry, we have $TD = TE$. □

29-3.7. **Theorem** If D is an irreducible bounded representation of G of degree s, then TD is a minimal closed invariant ideal of $A(G)$. Furthermore the dimension of TD is s^2.

Proof. Since TD is a finite dimensional vector subspace of $A_\infty(G)$, it is closed in $A_\infty(G)$. Take any $a \in G$. Since $L_a D_{ij}(x) = D_{ij}(ax) = \sum_{k=1}^{s} D_{ik}(a)D_{kj}(x)$ and $R_a D_{ij}(x) = D_{ij}(xa) = \sum_{k=1}^{s} D_{ik}(x)D_{kj}(a)$, TD is translation invariant. Because $(D \times f)(x) = D(x)D(f)^t$ for any $f \in A(G)$, TD is a right ideal of the convolution algebra $A(G)$. Similarly, it is also a two-sided ideal. Since D is irreducible, the entry functions $\{D_{ij} : 1 \le i, j \le s\}$ are linearly independent. Hence the dimension of TD is s^2. Finally let M be an ideal of $A(G)$ such that

$\{0\} \neq M \subset TD$. There is a nonzero $g = \sum_{ij=1}^{s} \alpha_{ij} D_{ij}$ in M. Suppose $\alpha_{kn} \neq 0$. Then for all $1 \leq p, q \leq s$,

$$D_{pk} \times g \times D_{nq} = \sum_{ij=1}^{s} \alpha_{ij} D_{pk} \times D_{ij} \times D_{nq} = \frac{\alpha_{kn}}{s^2} D_{pq}$$

shows that $D_{pq} \in M$, i.e. $TD \subset M$. Therefore TD is minimal. □

29-3.8. **Lemma** If D, E are inequivalent irreducible bounded representations of G, then we have $T(D \oplus E) = TD \oplus TE$.

Proof. Clearly we have $T(D \oplus E) = TD + TE$. Because $D_{ij} \times E_{kn} = 0$, we have $TD \cap TE = \{0\}$. Therefore the sum is direct. □

29-3.9. **Theorem** For every bounded representation D of G, if TD is minimal, then D is irreducible.

Proof. If $D = E_1 \oplus E_2$, then $TD = TE_1 \oplus TE_2$ and $TE_1 \subset TD$ but $\{0\} \neq TE_1 \neq TD$. Therefore D is not minimal. □

29-3.10. **Corollary** For every bounded representation D of G, TD is a closed invariant ideal of $A(G)$.

Proof. Write $D = \oplus\{s_w D^w : w \in J\}$ where $\{D^w : w \in J\}$ is a finite family of inequivalent irreducible unitary representations of G. Then $TD = \oplus\{TD^w : w \in J\}$ is a closed invariant ideal. □

29-3.11. For the rest of this section, let Γ be the family of all equivalent classes of irreducible bounded representations of G. For each $w \in \Gamma$, choose any irreducible unitary representation $D^w \in w$. Let s_w denote the degree of D^w. For every subset $\triangle \subset \Gamma$, functions in the vector subspace $T\triangle$ of $A(G)$ spanned by $\cup\{Tw : w \in \triangle\}$ are called *trigonometric polynomials* on G or $\triangle$-trigonometric polynomials in order to be precise.

29-3.12. **Theorem** Every projector in $A(G)$ is a trigonometric polynomial. The translates of $\triangle$-trigonometric polynomials are $\triangle$-trigonometric polynomials. Furthermore, $T\triangle$ is an ideal of the convolution algebra $A(G)$.

29-3.13. **Lemma** For every $f \in A(G)$, the set $\{w \in \Gamma : \|D^w(f)\| \neq 0\}$ is countable.

Proof. For any integer $k > 0$, if the set $J(k) = \{w \in \Gamma : \|D^w(f)\| \geq \frac{1}{k}\}$ has more than $\|f\|^2 k^2 + 1$ elements, then we have

$$\|f\|^2 \geq \sum_{w \in J(k)} s_w \|D^w(f)\|^2 \geq \sum_{w \in J(k)} \frac{1}{k^2} \geq \|f\|^2 + \frac{1}{k^2}$$

which is a contradiction. Hence $J(k)$ is a finite set. Therefore the set $\{w \in \Gamma : \|D^w(f)\| \neq 0\} = \bigcup_{k=1}^{\infty} J(k)$ is countable. $\square$

29-3.14. **Parseval's Equation** For all ap-function f, g on G, we have

$$\|f\|^2 = \sum_{w\in\Gamma} s_w \|D^w(f)\|^2 \qquad \text{and} \qquad < f, g >= \sum_{w\in\Gamma} s_w D^w < f, g > .$$

Proof. It follows from Bessel's inequality that $\sum_{w\in\Gamma} s_w \|D^w(f)\|^2 \leq \|f\|^2$. On the other hand, for every $\varepsilon > 0$ there is a projector φ on G such that $\|f - \varphi \times f\| \leq \varepsilon$. Since $\varphi \times f$ is a trigonometric polynomial, write $\varphi \times f = \sum_{w\in J} s_w < D^w, A^{w-} >$ where J is a finite subset of Γ and A_w are constant matrices. From §29-3.5a, we obtain

$$\|f\|^2 - \sum_{w\in J} s_w \|D^w(f)\|^2 \leq \|f - \varphi \times f\|^2 \leq \varepsilon^2$$

or $\|f\|^2 \leq \varepsilon^2 + \sum_{w\in J} s_w \|D^w(f)\|^2$. Since $\varepsilon > 0$ is arbitrary, we have $\|f\|^2 \leq \sum_{w\in\Gamma} s_w \|D^w(f)\|^2$. This proves the first equality. The proof is completed by the following calculation:

$$4 < f, g >= \|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2$$

$$= \sum_{w\in\Gamma} s_w \{\|D^w(f+g)\|^2 - \|D^w(f-g)\|^2 + i\|D^w(f+ig)\|^2 - i\|D^w(f-ig)\|^2\}$$

$$= \sum_{w\in\Gamma} s_w \{4D^w < f, g >\}.$$ $\square$

29-3.15. Let M be a closed invariant ideal of G. A bounded representation D of G is called an *M-representation* of G if all entry functions of D belong to M.

29-3.16. **Lemma** Let D, E be irreducible bounded representations of G.
(a) If one of entry functions of D belongs to M, then D is an M-representation.
(b) If D, E are equivalent and if D is an M-representation, then so is E.

Proof. If $D_{ij} \in M$, then $D_{kn} = s^2 D_{ki} \times D_{ij} \times D_{jn} \in M$ where s is the degree of D. Part (b) is left as an exercise. $\square$

29-3.17. **Theorem** Let D be an irreducible bounded representation of G. If there is $f \in M$ such that the Fourier matrix $D(f)$ is nonzero, then D is an M-representation.

Proof. By §29-1.7a, we may assume that D is unitary of degree s. Given $D(f) = \int f(x)D(x)^- d_m x \neq 0$, we have $< f, D_{kn} >= \int f(x)D_{kn}(x)^- d_m x \neq 0$ for some $k, n \leq s$. Then for all $i, j \leq s$ and $a \in G$, we obtain

$$D_{im} \times f \times D_{nj}(a) = \int (D_{ik} \times f)(x) D_{nj}(x^{-1}a) d_m x$$

$$= \int\int D_{ik}(xy^{-1}) f(y) D_{nj}(x^{-1}a) d_m x d_m y$$

$$= \sum_{p,q=1}^{s} \int\int D_{ip}(x)D_{pk}(y^{-1})f(y)D_{nq}(x^{-1})D_{qj}(a)d_m x d_m y$$
$$= \sum_{p,q=1}^{s} \left\{\int D_{ip}(x)D_{nq}(x^{-1})d_m x\right\} \left\{\int D_{pk}(y^{-1})f(y)d_m y\right\} D_{qj}(a)$$
$$= \sum_{p,q=1}^{s} \left\{\int D_{ip}(x)D_{qn}(x)^{-}d_m x\right\} \left\{\int f(y)D_{kp}(y)^{-}d_m y\right\} D_{qj}(a)$$
$$= \sum_{p,q=1}^{s} < D_{ip}, D_{qn} >< f, D_{kp} > D_{qj}(a)$$
$$= \sum_{p,q=1}^{s} \tfrac{1}{s}\ \delta_{iq}\delta_{pn} < f, D_{kp} > D_{qj}(a) = \tfrac{1}{s} < f, D_{kn} > D_{ij}(a),$$

or $D_{ij} = \dfrac{s}{< f, D_{kn} >} D_{ik} \times f \times D_{nj} \in M$. Because D is irreducible, it is an M-representation. □

29-3.18. **Corollary** Every minimal closed invariant ideal M is of the form TD for some irreducible unitary representation D. Consequently all minimal closed invariant ideals of $A(G)$ are finite dimensional.

Proof. Choose any nonzero $f \in M$. There is an irreducible unitary representation D such that the Fourier matrix $D(f)$ is not zero. It follows that D must be an M-representation. Hence TD is a closed invariant ideal contained in M. Therefore $M = TD$. □

29-99. **References and Further Readings** : Boyer, Naimark, Huang, Sugiura and Hofman.

Chapter 30

Saturated Closed Invariant Ideals

30-1 Dual Objects

30-1.1. We are *comfortable* to work with sine-cosine functions which are real and imaginary parts of the exponential function $e^{i\theta x}$. We shall prove later in this chapter that every *continuous* irreducible unitary representation of the real line is of the form $e^{i\theta x}$. For vibrating strings with fixed end points, only periodic functions with fixed period are required. Continuity is normally imposed upon the study of topological groups. This is why $A(G)$ begins to disappear from the scene.

30-1.2. Let $C(G)$ be a closed invariant ideal of a group G. From now on, it will become the subject of our study. Members of $C(G)$ are called *comfortable almost periodic* functions, or *cap*-functions for convenience. By a representation, we actually mean a $C(G)$-representation defined in §29-3.15. By an uncomfortable representation, we indicate that it need not be a $C(G)$-representation. Write $C_p(G)$ to emphasize the norm in use.

30-1.3. The family Ω of all equivalent classes of irreducible bounded representations of G is called the *dual object* of G. The equivalent classes in Ω are called *dual classes*. A family $D\Omega = \{D^w : w \in \Omega\}$ is called a *unitary representative* of the dual object if every representation $D^w \in w$ is unitary. Clearly $D\Omega$ is a maximal system of inequivalent irreducible unitary representations of G. Conversely if a maximal system of inequivalent irreducible representations of G is given, then the equivalent classes containing representations of this system is the dual object of G.

30-1.4. Let w be a dual class. For all $D \in w$, define the *degree* s_w of w to be the degree of D; $Tw = TD$, $w < f,g >= D < f,g >$ and $\|w(f)\| = \|D(f)\|$. They are all well defined because they are independent of the choice of $D \in w$. Without loss of generality, we shall work with a particular unitary representative $D\Omega$ of the dual object.

30-1.5. **Theorem** For all cap-functions f, g; we have $\|f\|^2 = \sum_{w\in\Omega} s_w \|w(f)\|^2$ and $< f, g >= \sum_{w\in\Omega} s_w \|w < f, g > \|^2$.

Proof. Let Γ be the family of all equivalent classes of uncomfortable irreducible unitary representations and choose $D^w \in w$ for each $w \in \Gamma$. If $w(f) \neq 0$, then $D^w(f) \neq 0$ and hence D^w is a comfortable representation. Therefore for all $w \in \Gamma \setminus \Omega$, we have $w(f) = 0$. Consequently, we have

$$\|f\|^2 = \sum_{w\in\Gamma} s_w \|w(f)\|^2 = \sum_{w\in\Omega} s_w \|w(f)\|^2.$$

Part (b) is left as an exercise. □

30-1.6. **Corollary** The Fourier matrices uniquely determine the convolution algebra $C(G)$. More precisely, for all cap-functions f, g, h on G and $\alpha \in \mathbb{C}$; the following statements hold.

(a) $f = 0$ iff $D(f) = 0$, for all irreducible unitary representations D.
(b) $f = g$ iff $D(f) = D(g)$, for all D.
(c) $f = g^*$ iff $D(f) = D(g)^*$, for all D.
(d) $f = g + h$ iff $D(f) = D(g) + D(h)$, for all D.
(e) $f = \alpha g$ iff $D(f) = \alpha D(g)$, for all D.
(f) $f = g \times h$ iff $D(f) = D(g)D(h)$, for all D.

Proof. (a) $f = 0$ iff $\|f\|^2 = \sum_{w\in\Omega} s_w \|D^w(f)\|^2 = 0$ iff $D^w(f) = 0$. The rest follows because $f \to D(f)$ is a homomorphism from the convolution algebra $C(G)$ into the algebra of square matrices. □

30-1.7. The formal sum $\sum_{w\in\Omega} s_w < D^w, A^w >$ is called a *trigonometric series* where each A^w is a constant matrix of order s_w. The *Fourier series* of a cap-function f is defined by $\Omega(f) = \sum_{w\in\Omega} s_w < D^w, D^w(f)^- >$. There is some trigonometric series which is not a Fourier series of any cap-function. Fourier series of a cap-function need not be pointwise convergent. Example can be found from classical Fourier analysis on the circle group.

30-1.8. **Lemma** The Fourier series is independent of the choice of the dual representatives.

Proof. Let D, E be equivalent unitary representations and let P be a unitary matrix satisfying $D(x) = P^* E(x) P$ for all $x \in G$. Then we have

$$< D(x), D(f)^- >= tr\ D(f)^{-*} D(x) = tr\ \left[\int f(y) D(y)^- d_m y\right]^{-*} D(x)$$
$$= tr\ \left[\int f(y)^- D(y) d_m y\right]^* D(x) = tr\left[\int f(y)^- P^* E(y) P d_m y\right]^* P^* E(x) P$$
$$= tr\ \left[P^* \left\{\int f(y) E(y)^- d_m y\right\}^* P\right] P^* E(x) P$$

$= tr\ P[P^*E(f)^{-*}PP^*E(x)] =< E(x), E(f)^- >.$

Hence every term $< D^w(x), D^w(f)^- >$ is independent of the choice of $D^w \in w$. Therefore the Fourier series is independent of the choice of $\{D^w : w \in \Omega\}$. □

30-1.9. **Theorem** The Fourier series of f converges to f in $C_2(G)$. Therefore $\left\{\frac{1}{\sqrt{s_w}} D^w_{ij} : w \in \Omega\right\}$ is an orthonormal basis of the inner product space $C_2(G)$. For compact groups, this is frequently called *Peter-Weyl Theorem*.

Proof. For any $\varepsilon > 0$, there is a finite subset J of Ω such that

$$\left\|f - \sum_{w\in J} s_w < D^w, D^w(f)^- >\right\|^2 = \|f\|^2 - \sum_{w\in J} s_w \|D^w(f)\|^2 \leq \varepsilon$$

which is the statement of the theorem. □

30-1.10. **Theorem** Then set $T\Omega$ of all trigonometric polynomials is dense in $C_p(G)$ for all $1 \leq p \leq \infty$. This is also called *Weierstrass' Theorem* for $G = \mathbb{R}$.

Proof. Let f be a cap-function on G. For every $\varepsilon > 0$, there exist a projector φ and a cap-function g such that $\|f - \varphi \times g \times f\|_p \leq \|f - \varphi \times g \times f\|_\infty \leq \varepsilon$. Since $\varphi \times g \times f$ is a trigonometric polynomial, the proof is complete. □

30-1.11. **Corollary** If $f \in C(G)$, then the hermitian f^* is also in $C(G)$.

Proof. Since $D^*_{ij}(x) = D_{ij}(x^{-1})^- = D_{ji}(x)$ for every unitary representation D, the hermitian of trigonometric polynomials in $C(G)$ are in $C(G)$. The proof is completed by uniform limits. □

30-2 Characters

30-2.1. General comfortable almost periodic functions or cap-functions are described by their Fourier matrices. In this section, we specialize an important case when commutativity is available. Let $C(G)$ be the closed invariant ideal of comfortable almost periodic functions on a group G. The trace of a bounded representation D of G is called the *character* ρ_D of D. If D is irreducible, then ρ_D is said to be *irreducible*. The degree of D is also called the *degree* of ρ_D. A character of degree one is a unitary representation. A function f on G is said to be *central* if $f(xy) = f(yx)$ for all $x, y \in G$. Clearly it is equivalent to $f(y^{-1}xy) = f(x)$ for all $x, y \in G$. This section studies the relationship between central functions and characters.

30-2.2. **Lemma** (a) $\rho_{D\oplus E} = \rho_D + \rho_E$ for all bounded representations D, E. (b) If D, E are equivalent bounded representations, then $\rho_D = \rho_E$.

(c) Let ρ be the character of an uncomfortable irreducible representation D. If ρ is comfortable, then so is D.

Proof. (b) Let P be an invertible matrix such that $D = P^{-1}EP$. Then we have $\rho_D(x) = trD(x) = tr[P^{-1}E(x)P] = trPP^{-1}E(x) = trE(x) = \rho_E(x)$.
(c) Since $D_{11} = sD_{11} \times \rho \in C(G)$ where s is the degree of D, all entry functions of D belong to $C(G)$ because D is irreducible. □

30-2.3. For every dual class w in the dual object Ω of G, the function $\rho_w = \rho_D$ is independent of the choice of $D \in w$ and hence it is well defined. Since every bounded representation E is equivalent to a direct sum $\oplus_{w\in\Omega} m_w D^w$ where m_w is the multiplicity of D^w in E. The integers m_w and the functions $\rho_w = \rho_{D^w}$ are independent of the choice of $D\Omega$. We also called m_w the *multiplicity* of the dual class w in E. Except a finite number of dual classes w, we get $m_w = 0$.

30-2.4. **Theorem** Let w, v be dual classes of G and s_w the degree of w.
(a) $(s_w\rho_w) \times f = \delta_{wv} f = f \times (s_w\rho_w)$ for all trigonometric polynomials $f \in Tv$.
(b) $(s_w\rho_w) \times \rho_v = \delta_{wv}\rho_w$, $\rho_w(x^{-1}) = \rho_w^-$, $\rho_w^* = \rho_w$, $\rho_w(e) = s_w$ and also $< \rho_w, \rho_v >= \delta_{wv}$. In particular, $s_w\rho_w$ is a projector and the set $\{\rho_w : w \in \Omega\}$ is orthonormal in $C_2(G)$.

Proof. Choose any irreducible unitary representations $D \in w$ and $E \in v$. Let D_{ij}, E_{kn} be the entry functions of D, E respectively. Observe that

$$s_w\rho_w \times E_{kn} = s_w \sum_{i=1}^{s_w} D_{ii} \times E_{kn} = s_w \sum_{i=1}^{s_w} \frac{1}{s_w} \delta_{DE}\delta_{ik}E_{in} = \delta_{wv}E_{kn}.$$

Since Tv is spanned by $\{E_{kn} : k, n \le s_v\}$, both (a) and $s_w\rho_w \times \rho_v = \delta_{wv}\rho_w$ follow. Next, from $D_{ij}(x^{-1}) = D_{ji}(x)^-$ we obtain $\rho_w(x^{-1}) = \rho_w(x)^-$ and $\rho_w^* = \rho_w$. Clearly $\rho_w(e) = tr(I) = s_w$. Finally, we have

$$< \rho_w, \rho_v >= \int \rho_w(x)\rho_v(x)^- d_m x = \int \rho_w(x)\rho_v(x^{-1}) d_m x$$
$$= \rho_w \times \rho_v(e) = \frac{1}{s_w}\delta_{wv}\rho_w(e) = \delta_{wv}.$$ □

30-2.5. **Theorem** Let ρ_D, ρ_E be the characters of bounded representations D, E respectively.
(a) $\rho_D = \sum_{w\in\Omega} m_w\rho_w$ and $m_w =< \rho_D, \rho_w >$.
(b) The multiplicity of the identity representation in D is $\int \rho_D(x)d_m x$.
(c) D, E are equivalent iff $\rho_D = \rho_E$.
(d) $\|\rho_D\|^2 = \sum_{w\in\Omega} m_w^2$. Note that this is actually a finite sum.
(e) D is irreducible iff $\|\rho_D\| = 1$.

Proof. (a) Let P be an invertible matrix such that for all $x \in G$ we have $P^{-1}D(x)P = \sum_{w\in\Omega} m_w D^w(x)$. Then

$$\rho_D(x) = trD(x) = tr[P^{-1}D(x)P] = \sum_{w\in\Omega} m_w trD^w(x) = \sum_{w\in\Omega} m_w \rho_w(x)$$

and $$< \rho_D, \rho_w >= \sum_{v\in\Omega} m_v < \rho_v, \rho_w >= \sum_{v\in\Omega} m_v \delta_{vw} = m_w.$$

(b) The identity representation carries every element of G to the number 1. Its character is the constant function 1. Therefore its multiplicity in D is $< \rho_D, 1 >= \int \rho_D(x) d_m x$.

(c) Let $\rho_E = \sum_{w\in\Omega} n_w \rho_w$. If $\rho_D = \rho_E$, then $n_w =< \rho_E, \rho_w >=< \rho_D, \rho_w >= m_w$ for all $w \in \Omega$. Hence D, E are equivalent. The converse is obvious.

(d) $$\|\rho_D\|^2 =< \rho_D, \rho_D >=< \sum_{w\in\Omega} m_w \rho_w, \sum_{v\in\Omega} m_v \rho_v >$$
$$= \sum_{w,v\in\Omega} m_w m_v \delta_{wv} = \sum_{w\in\Omega} m_w^2.$$

(e) Suppose $\|\rho_D\| = 1$. Then $\sum_{w\in\Omega} m_w^2 = \|\rho_D\|^2 = 1$. There is some $v \in \Omega$ such that $m_v = 1$ and $m_w = 0$ for all $w \neq v$. Therefore D is equivalent to D_w. Consequently D is irreducible. Then converse is obvious. □

30-2.6. **Lemma** (a) The set $ZC(G)$ of all central cap-functions is a subalgebra of $C(G)$ under pointwise operations.

(b) If f is central, then so is $|f|$. If f, g are central and real, then both $f \vee g$ and $f \wedge g$ are central.

(c) Every character ρ_D is central.

Proof. (c) $\rho_D(xy) = trD(xy) = tr[D(x)D(y)] = tr[D(y)D(x)] = \rho_D(yx)$. □

30-2.7. **Theorem** A cap-function f is central iff $f \times g = g \times f$ for all $g \in C(G)$.

Proof. ($\Rightarrow$) Take any $g \in C(G)$. If f is central, then

$$(f \times g)(x) = \int f(xy)g(y^{-1})d_m y = \int g(y^{-1})f(yx)d_m y = (g \times f)(x).$$

($\Leftarrow$) Let D be any unitary representation of G. For every $a \in G$, we have

$$< L_a f, D_{ij} >= \int f(ax)D_{ij}(x)^- d_m x = \int f(ax)D_{ji}(x^{-1})d_m x$$
$$= (f \times D_{ji})(a) = (D_{ji} \times f)(a)$$
$$= \int D_{ji}(x^{-1})f(xa)d_m x = \int f(xa)D_{ij}(x)^- d_m x =< R_a f, D_{ij} > .$$

Hence $L_a f, R_a f$ have the same Fourier matrices. Therefore $L_a f = R_a f$, that is $f(ax) = f(xa)$ for all $a, x \in G$. □

30-2.8. **Theorem** Every central trigonometric polynomial f is a linear combination of characters.

Proof. Write $f = \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w D_{ij}^w$ where J is a finite subset of the dual object Ω of G and s_w the degree of w. Then for each $v \in J$, we have

$$f \times D_{kn}^v = \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w D_{ij}^w \times D_{kn}^v$$
$$= \sum_{w\in J} \sum_{i,j=1}^{s_w} \alpha_{ij}^w \delta_{wv}\delta_{jk} D_{in}^w = \sum_{i=1}^{s_v} \alpha_{ik}^v D_{in}^v$$

and

$$D_{kn}^v \times f = \sum_{w\in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w D_{kn}^v \times D_{ij}^v$$
$$= \sum_{w\in J} \sum_{i,j=1}^{s_w} \alpha_{ij}^w \delta_{wv}\delta_{ni} D_{kj}^w = \sum_{j=1}^{s_v} \alpha_{nj}^v D_{kj}^v.$$

Since f is central, we have $f \times D_{kn}^v = D_{kn}^v \times f$, i.e.

$$\sum_{i=1}^{s_v} \alpha_{ik}^v D_{in}^v = \sum_{j=1}^{s_v} \alpha_{nj}^v D_{kj}^v.$$

Since $\{D_{ij}^v\}$ are linearly independent, we have $\alpha_{ik} = 0$ if $i \neq k$ and $\alpha_{nj} = 0$ if $j \neq n$. This gives $\alpha_{kk}^v = \alpha_{nn}^v$ for all $k, n \leq s_v$. Therefore we obtain

$$f = \sum_{v\in J} s_v \sum_{i=1}^{s_v} \alpha_{ii}^v D_{ii}^v = \sum_{v\in J} s_v \alpha_{11}^v \rho_v. \qquad \square$$

30-2.9. **Lemma** If $f \in C(G)$, then the central function ξ_f associated with f given by $\xi_f(x) = \int f(y^{-1}xy)d_m y$ is a central function in $C(G)$. If f is a trigonometric polynomial, then so is ξ_f.

Proof. For every $a, x \in G$; we have

$$\xi_f(a^{-1}xa) = \int f(y^{-1}a^{-1}xay)d_m y$$
$$= \int f[(ay)^{-1}x(ay)]d_m y = \int f(y^{-1}xy)d_m y = \xi_f(x).$$

Therefore ξ_f is central. Observe that for any representation D,

$$\int D_{ij}(y^{-1}xy)d_m y = \sum_{kn}[\int D_{ik}(y^{-1})D_{nj}(y)d_m y]\ D_{kn}(x).$$

This proves the second statement. We shall show that $\xi_f \in C(G)$ as part of next proof. $\square$

30-2.10. **Theorem** The set of all linear combinations of irreducible characters is uniformly dense in $ZC_\infty(G)$.

Proof. Let f be a cap-function. For every $\varepsilon > 0$, choose a trigonometric polynomial g such that $\|f - g\|_\infty \leq \varepsilon$. Then for all $x, y \in G$ we obtain

$$|\xi_f(x) - \xi_g(x)| = \left| \int [f(y^{-1}xy) - g(y^{-1}xy)]\ d_m y \right| \leq \int \|f-g\|_\infty d_m y = \varepsilon.$$

Since ξ_g is a linear combination of characters in $C(G)$, the uniform limit ξ_f belongs to $C(G)$ as required by last lemma. Finally if f is central, then $\xi_f(x) = \int f(y^{-1}xy)d_m y = \int f(yy^{-1}x)d_m y = f(x)$ completes the proof. $\square$

30-2.11. **Lemma** Let D be an irreducible unitary representation of degree s, I the identity matrix of order s and f, g be central cap-functions on G. Then we have $D(f) = \frac{1}{s} < f, \rho_D > I$, $D(\rho_D) = \frac{1}{s}I$, $D < f, g >= \frac{1}{s} < f, \rho_D >< \rho_D, g >$ and $\|D(f)\| = \frac{1}{\sqrt{s}}| < f, \rho_D > |$.

Proof. For every $y \in G$, observe that

$$D(f) = \int f(x)D(x)^- d_m x = \int f(y^{-1}xy)D(y^{-1}xy)^- d_m x$$
$$= D(y^{-1})^- \left[\int f(x)D(x)^- d_m x\right] D(y)^- = D(y^{-1})^- D(f)D(y)^- .$$

Taking complex conjugate, we get $D(y)D(f)^- = D(f)^- D(y)$. It follows that $D(f)^-$ is a scalar matrix. Write $D(f) = \lambda I$ for some $\lambda \in \mathbb{C}$. Then

$$s\lambda = trD(f) = \sum_{i=1}^{s} \int f(x)D_{ii}(x)^- d_m x = \int f(x) \left[\sum_{i=1}^{s} D_{ii}(x)^-\right] d_m x$$
$$= \int f(x)\rho_D(x)^- d_m x =< f, \rho_D >,$$

i.e. $D(f) = \frac{1}{s} < f, \rho_D > I$. For $f = \rho_D$, we get $D(\rho_D) = \frac{1}{s} < \rho_D, \rho_D > I = \frac{1}{s}I$. Next for inner product,

$$D < f, g >=< D(f), D(g) >= tr\ D(g)^* D(f)$$
$$= tr\ \frac{1}{s^2} < g, \rho_D >^- < f, \rho_D > I = \frac{1}{s} < g, \rho_D >^- < f, \rho_D > .$$

The last formula follows by setting $g = f$. □

30-2.12. **Theorem** For all central cap-functions f, g on G, we have

$$\|f\|^2 = \sum_{w\in\Omega} | < f, \rho_w > |^2 \quad \text{and} \quad < f, g >= \sum_{w\in\Omega} < f, \rho_w >< \rho_w, g > .$$

Proof. It follows immediately from §30-1.5. □

30-2.13. **Theorem** Let f, g be cap-functions on G but they need not be central.

(a) The Fourier series of f is $\Omega(f) = \sum_{w\in\Omega} s_w \rho_w \times f$.

(b) $< f, g >= \sum_{w\in\Omega} s_w \rho_w \times g^* \times f(e)$.

(c) $\|f\|^2 = \sum_{w\in\Omega} s_w \rho_w \times f^* \times f(e)$.

(d) The trigonometric series $\sum_{w\in\Omega} s_w \rho_w \times f \times g$ converges to $f \times g$ uniformly and absolutely in $C_\infty(G)$.

Proof. (a) For any unitary representation D of G, we have

$$\rho_D \times f(a) = \int trD(ax^{-1})f(x)d_m x = trD(a) \int D(x^{-1})f(x)d_m x$$
$$= trD(a) \int f(x)D(x)^* d_m x = trD(a) \left[\int f(x)D(x)^- d_m x\right]^{-*}$$
$$= trD(a)D(f)^{-*} = trD(f)^{-*}D(a) =< D(a), D(f)^- > .$$

Hence $\sum_{w\in\Omega} s_w \rho_w \times f$ is the Fourier series of f.

(b) Observe that

$$D < f,g >=< D(f), D(g) >= tr D(g)^* D(f)$$
$$= tr \left[\int g(y)D(y)^- d_m y\right]^* \left[\int f(x)D(x)^- d_m x\right]$$
$$= tr \int \int g(y)^- D(y)^{-*} f(x)D(x)^- d_m x d_m y$$
$$= tr \int \int g(y)^- D(y^{-1})^- f(x)D(x)^- d_m x d_m y$$
$$= tr \int \int g(y)^- f(x)D(y^{-1}x)^- d_m x d_m y$$
$$= tr \int \int g(y^{-1})^* f(yx)D(x)^- d_m x d_m y, \qquad \text{replacing } x \text{ by } yx$$
$$= tr \int (g^* \times f)(x)D(x)^- d_m x = \int (g^* \times f)(x) tr D(x)^- d_m x$$
$$= \int (g^* \times f)(x)\rho(x)^- d_m x = \int (g^* \times f)(x)\rho(x^{-1}) d_m x = \rho \times g^* \times f(e).$$

Hence we have $w < f,g >= \rho_w \times g^* \times f(e)$. The result follows from §30-1.5.

(d) Let $f_J = \sum_{w\in J} s_w \rho_w \times f$ and $g_J = \sum_{w\in J} s_w \rho_w \times g$ for any finite subset J of Ω. Then $\|f_J\|^2 = \sum_{w\in J} s_w \rho_w \times f^* \times f(e) = \sum_{w\in J} s_w \|w(f)\|^2 \to \|f\|^2$ as $J \to \infty$. Now observe that

$$f_J \times g_J = \sum_{w,v\in J} s_w s_v \rho_w \times f \times \rho_v \times g$$
$$= \sum_{w,v\in J} s_w s_v \rho_w \times \rho_v \times f \times g = \sum_{w\in J} s_w \rho_w \times f \times g.$$

Therefore we obtain

$$\|f \times g - \sum_{w\in J} s_w \rho_w \times f \times g\|_\infty = \|f \times g - f_J \times g_J\|_\infty$$
$$\leq \|f - f_J\| \, \|g\| + \|f_J\| \, \|g - g_J\|$$
$$\leq \|f - f_J\| \, \|g\| + \|f\| \, \|g - g_J\| \to 0 \qquad \text{as } J \to \infty.$$

□

30-3 Saturated Dual Objects

30-3.1. Let $C(G)$ be the ideal of comfortable almost periodic functions or cap-functions on a group G. A closed invariant ideal of $C(G)$ is *saturated* if it is also closed under complex conjugation and pointwise multiplication. We shall prove that it is closely related to conjugate and tensor products of representations. Let N be a saturated closed invariant ideal of $C(G)$.

30-3.2. **Theorem** If $f \in N$ then $|f| \in N$. Furthermore if $f, g \in N$ are real, then both $f \vee g$ and $f \wedge g$ are in N.

Proof. For every $\varepsilon > 0$, choose a polynomial p without constant term by §3-8.9 such that $|p(t) - \sqrt{t}| \leq \varepsilon$ for all $0 \leq t \leq \|f^- f\|_\infty$. Since N is saturated, $f^- f \in N$ and hence $p(f^- f) \in N$. Therefore $|f|$ is a uniform limit of functions in N. Consequently, $|f| \in N$. The last statement follows from the standard formulas: $f \vee g = \frac{1}{2}(f + g + |f - g|)$ and $f \wedge g = \frac{1}{2}(f + g - |f - g|)$. □

30-3.3. **Lemma** Every saturated closed invariant ideal $N \neq \{0\}$ contains all constant functions.

Proof. Let $f \neq 0$ be a function in N. There is an irreducible unitary N-representation D such that $D(f) \neq 0$. Then D is an N-representation. Its character ρ belongs to N. Hence $1 = \rho^{-}\rho$ belongs to N. Since N is a vector space, it contains all constant functions. □

30-3.4. **Theorem** If N separates points of G, then for all distinct elements $a_1, a_2, \cdots, a_n \in G$ and arbitrary numbers α_i, there is $f \in N$ such that $f(a_i) = \alpha_i$ for each $i = 1, 2, \cdots, n$.

Proof. For each pair $i \neq j$, choose $g_{ij} \in N$ such that $g_{ij}(a_i) \neq g_{ij}(a_j)$. Then $h_{ij}(x) = \dfrac{g_{ij}(x) - g_{ij}(a_j)}{g_{ij}(a_i) - g_{ij}(a_j)}$ defines a function $h_{ij} \in N$ with $h_{ij}(a_i) = 1$ and $h_{ij}(a_j) = 0$. Clearly the function f given by $f(x) = \sum_{i=1}^{n} \alpha_i \prod_{j=1, j\neq i}^{n} h_{ij}(x)$ is a required cap-function in N. □

30-3.5. **Theorem** Let $f \in N$ and K a compact set containing $f(G)$. For every continuous function $\varphi : K \to \mathbb{C}$, the composite φf belongs to N.

Proof. Choose a sequence $\{g_n\}$ of polynomials in λ, λ^{-} such that $|g_n(\lambda, \lambda^{-}) - \varphi(\lambda)| \leq 1/n$ for all $\lambda \in K$. Since N is saturated, $f^{-} \in N$ and hence $g_n(f, f^{-}) \in N$. As the uniform limit, we have $\varphi f \in N$. □

30-3.6. **Corollary** If $f \in N$ with $|f| \geq r > 0$, then $1/f \in N$.

Proof. The compact set $K = \{\lambda \in \mathbb{K} : r \leq |t| \leq \|f\|_\infty\}$ contains $f(G)$ and the function $\varphi(\lambda) = 1/t$ is continuous on K. Therefore $\varphi f = 1/f \in N$. □

30-3.7. The sign-function is normally discontinuous and it is unlikely in $C(G)$. Hence it has to be paired with a function of suitable amplitude as in next theorem. Then we approximate the sign-function with elements in $C(G)$.

30-3.8. **Theorem** Let $h, f \in N$; $\operatorname{sgn}[f(x)] = \begin{cases} f(x)/|f(x)|, & \text{if } f(x) \neq 0\,; \\ 0, & \text{if } f(x) = 0 \end{cases}$ and $g(x) = h(x)\operatorname{sgn}[f(x)]$. If $|h| \leq |f|$, then $g \in N$.

Proof. Let $s_n(x) = \dfrac{h(x)f(x)}{\frac{1}{n^2} + |f(x)|}$ for all $x \in G$ and $n > 0$. Since N is saturated, all $s_n \in N$. Let $\varepsilon > 0$ be given. Take any $n \geq (1 + \|f\|_\infty)/\varepsilon$. If $0 < |f(x)| < \frac{1}{n}$, then $|g(x) - s_n(x)| = |h(x)|/[\ 1 + n^2|f(x)|\ ] \le \frac{1}{n} \le \varepsilon$. If $|f(x)| \ge \frac{1}{n}$, then $|g(x) - s_n(x)| = |h(x)|/[\ 1 + n^2|f(x)|\ ] \le \|f\|_\infty/(1+n) \le \varepsilon$. If $f(x) = 0$, then $|g(x) - s_n(x)| = 0 \le \varepsilon$. Therefore $s_n \to g$ uniformly. Consequently, $g \in N$. □

30-3.9. **Theorem** Let $h \in N$. Then for every $\varepsilon > 0$, there is $g \in N$ such that $\|g\|_\infty \le 1$, $g(x)h(x) \ge 0$ and $\|\ gh - |h|\ \|_\infty \le \varepsilon$.

Proof. For any $\varepsilon > 0$, choose a polynomial $p(t)$ with real coefficients and without constant term such that $|p(t) - \sqrt{t}| \le \varepsilon$ for all $0 \le t \le \|h^- h\|_\infty$. Write $p(t) = r(t)t$ where $r(t)$ is a polynomial with real coefficients. The real function $f = r(h^- h) \in N$ satisfies $\big|f(x)h^-(x)h(x) - |h(x)|\ \big| \le \varepsilon$ for all $x \in G$. Replacing f by f^+, we may assume that $f \ge 0$ as a result of $h^- h \ge 0$. Since the function $\lambda : \mathbb{C} \to \mathbb{C}$ defined by $\lambda(z) = \begin{cases} z, & \text{if } \ |z| \le 1 \\ z/|z|, & \text{if } \ |z| > 1 \end{cases}$ is continuous, the composite function given by $g(x) = \lambda[f(x)h^-(x)]$ for all $x \in G$ belongs to N. Clearly we have $\|g\|_\infty \le 1$ and $g(x)h(x) \ge 0$ for all $x \in G$. Furthermore if $|f(x)h^-(x)| \le 1$, then we have $\big|g(x)h(x) - |h(x)|\ \big| = \big|f(x)h^-(x)h(x) - |h(x)|\ \big| \le \varepsilon$. If $|f(x)h^-(x)| \ge 1$, then we also have

$$|g(x)h(x) - |h(x)|\ | = \left|\frac{f(x)h^-(x)h(x)}{|f(x)h^-(x)|} - |h(x)|\right| = \left|\frac{f(x)|h(x)|^2}{f(x)|h^-(x)|} - |h(x)|\right| = 0 \le \varepsilon.$$

This completes the proof. □

30-3.10. Let M be a closed invariant ideal of $C(G)$. As in §30-1.3, the family Ω_M of all equivalent classes of irreducible bounded representations of G is called the *M-dual object* of G. Because $C(G)$ is nothing more than a closed invariant ideal of $A(G)$, all results about $C(G)$ hold for M and Ω_M.

30-3.11. **Theorem** For every $\triangle \subset \Omega_M$, the following statements are equivalent.

(a) $\triangle = \Omega_M$.

(b) Every function in M is a uniform limit of $\triangle$-trigonometric polynomials.

(c) Every central function in M is a uniform limit of linear combinations of irreducible $\triangle$-characters.

Proof. Without loss of generality, we may assume $M = C(G)$ and $\Omega_M = \Omega$. We have proved $(a \Rightarrow b)$ by §30-1.10 and $(b \Rightarrow c)$ by §30-2.10. It remains to prove $(c \Rightarrow a)$. Suppose to the contrary that there is $v \in \Omega$ but $v \notin \triangle$. Then ρ_v is a central function in (G). For $\varepsilon = 1/2$, there is a finite subset J of $\triangle$ and some constants α_w such that $\|\rho_v - \sum_{w \in J} \alpha_w s_w \rho_w\|_\infty \le \varepsilon$. Then we have

$$s_v = \rho_v(e) \le \|\rho_v\|_\infty = \|s_v \rho_v \times (\rho_v - \textstyle\sum_{w \in J} \alpha_w s_w \rho_w)\|_\infty$$

$\leq \|s_v \rho_v\|_2 \left\|\rho_v - \sum_{w \in J} \alpha_w s_w \rho_w\right\|_2$
$\leq s_v \|\rho_v\|_2 \left\|\rho_v - \sum_{w \in J} \alpha_w s_w \rho_w\right\|_\infty \leq s_v \varepsilon \leq \frac{1}{2} s_v$

which is a contradiction. Therefore we have $\triangle = \Omega$. □

30-3.12. **Theorem** Let $\triangle$ be a nonempty subset of Ω. Then the uniform closure M of $T\triangle$ is a closed invariant ideal. Furthermore we have $\triangle = \Omega_M(G)$.

Proof. Clearly M is a closed vector subspace of $C_\infty(G)$. Suppose that $f \in M$ and $a \in G$. For every $\varepsilon > 0$, choose $g \in T\triangle$ such that $\|f - g\|_\infty \leq \varepsilon$, i.e. $\|R_a f - R_a g\|_\infty \leq \varepsilon$. Since $R_a g \in T\triangle$, we have $R_a f \in M$. Similarly, $L_a f \in M$. Hence M is a closed invariant ideal. Next, take any dual class $w \in \triangle$ and pick any irreducible unitary representation $D^w \in w$. Then all entry functions of D^w is in $Tw \subset M$. Hence D^w is an M-representation, i.e. $w \in \Omega_M(G)$. Therefore $\triangle \subset \Omega_M(G)$. It follows from last theorem that $\triangle = \Omega_M(G)$. □

30-3.13. **Theorem** For $M = \{0\}$, let $\Omega_M(G) = \emptyset$. Then the map $M \to \Omega_M(G)$ is a bijection from the family of all closed invariant ideals onto the family of all subsets of Ω.

Proof. The map $M \to \Omega_M(G)$ is surjective by last theorem. Suppose that M, N are closed invariant ideals with $\Omega_M(G) = \Omega_N(G)$. Let D be an irreducible M-representation of G. Then the dual class containing D is in $\Omega_M(G)$, and also in $\Omega_N(G)$. Therefore D is equivalent to some N-representation. Consequently, D is an N-representation. As a result, every M-trigonometric polynomial is also an N-trigonometric polynomial. We obtain $M \subset N$. By symmetry, we get $M = N$. □

30-3.14. **Theorem** Let M be a closed invariant ideal of $C(G)$.
(a) $M^\perp = \{f \in C_2(G) : < f, h >= 0,\ \forall\ h \in M\}$ is also a closed invariant ideal.
(b) For all $f, g \in C(G)$, we have $f \times g \in M \oplus M^\perp$.
(c) $M \oplus M^\perp$ is dense in $C_\infty(G)$.
(d) If M is self-conjugate, then so is $M^\perp$.
(e) Ω_M and $\Omega_{M^\perp}$ form a partition of the dual object Ω.

Proof. (a) For all $f, h \in M$, let $\mu_h(f) =< f, h >$. Then we have

$$|\mu_h(f)| = |< f, h >| \leq \|f\|_\infty \|h\|.$$

Hence $\mu_h : C_\infty(G) \to \mathbb{C}$ is a continuous linear form. Thus $M^\perp = \cap_{h \in M} \ker(\mu_h)$ is closed in $C_\infty(G)$. Observe that for every $f \in M^\perp$ and $h \in M$, we get $< L_a f, h >=< f, L_{a^{-1}} h >= 0$ and $< R_a f, h >=< f, R_{a^{-1}} h >= 0$. Hence $M^\perp$ is translation invariant. Therefore $M^\perp$ forms a closed invariant ideal.

(b) Observe that $f \times g = \sum_{w\in\Omega} s_w\rho_w \times f \times g$ converges in $C_\infty(G)$. Hence $h_1 = \sum_{w\in\Omega_M(G)} s_w\rho_w \times f \times g \in M$ and $h_2 = \sum_{w\in\Omega_{M^\perp}(G)} s_w\rho_w \times f \times g \in M^\perp$. Then $f\times g = h_1+h_2 \in M+M^\perp$. Next, if $f \in M\cap M^\perp$, then $< f, f >= \|f\|_2 = 0$, or $f = 0$. Hence the sum $M + M^\perp$ is direct.

(c) It follows from §28-4.11.

(d) Take any $f \in M^\perp$. Then for every $h \in M$, we have

$$< f^-, h >= \int f(x)^- g(x)^- d_m x = \left[\int f(x)g(x)d_m x\right]^- =< f, g^- >^- = 0.$$

This shows that $M^\perp$ is self-conjugate.

(e) Suppose $w \in \Omega_M(G)\cap\Omega_{M^\perp}(G)$. Then $\rho_w \in M\cap M^\perp$ which is a contraction. Next, suppose $w \notin \Omega_{M^\perp}(G)$. Then $\rho_w \notin M^\perp$. Since the M-trigonometric polynomials are uniformly dense in M, there is an irreducible M-representation D such that $< \rho_w, D_{ij} >\neq 0$ for some i, j. Then the Fourier matrix $D(\rho_w)$ is nonzero. Hence $< \rho_w, \rho_D >\neq 0$. It follows that $\rho_w = \rho_D$. Therefore $D \in w$. Consequently, $w \in \Omega_M(G)$. □

30-3.15. For the rest of this section, let $C(G)$ be a saturated closed invariant ideal, for example $A(G)$ itself. We want to characterize those ideals M of $C(G)$ that are saturated.

30-3.16. **Theorem** Let D, E be bounded representations of a group G.

(a) The map D^- is a bounded representation called the *conjugate* of D.

(b) $D^{--} = D$, $\rho_D^- = \rho_{D^-}$, $(D\oplus E)^- = D^- \oplus E^-$.

(c) D is irreducible iff D^- is irreducible.

(d) D is unitary iff D^- is unitary.

(e) D, E are equivalent iff D^-, E^- are equivalent.

(f) $f \in TD$ iff $f^- \in T(D^-)$.

(g) D, D^- are equivalent iff ρ_D is real.

Proof. (a) Because $C(G)$ is closed under conjugation, all entry functions of D^- belong to $C(G)$.

(c) D is irreducible iff $\|\rho_{D^-}\| = \|\rho_D\| = 1$ iff D^- is irreducible.

(g) D, D^- are equivalent iff $\rho_D = \rho_{D^-} = \rho_D^-$ iff ρ_D is real. □

30-3.17. Let Ω be the dual object of G. For every $w \in \Omega$, take any $D \in w$ and define w^- as the dual class containing D^-. Clearly w^- is independent of the choice of $D \in w$ and it is called the *dual class conjugate* to w. Clearly $w \to w^-$ is a bijection of Ω onto Ω. Also $w^{--} = w$, $\rho_{w^-} = \rho_w^-$, $T(w^-) = (Tw)^-$. Clearly

$w = w^-$ iff ρ_w is real. Finally for every closed invariant ideal M of $C(G)$, we have $\Omega_{M^-}(G) = \Omega_M(G)^-$.

30-3.18. The tensor product of two matrices was defined in §15-4.4. For conformable matrices, we list the following properties.

(a) $(A+B)\otimes C = A\otimes C + B\otimes C$ and $C\otimes(A+B) = C\otimes A + C\otimes B$.

(b) $\lambda(A\otimes B) = (\lambda A)\otimes B = A\otimes(\lambda B)$, $A\otimes(B\otimes C) = (A\otimes B)\otimes C$.

(c) $(A\otimes B)(C\otimes D) = AC\otimes BD$, $I_p\otimes I_q = I_{pq}$.

(d) $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$ if both A, B are invertible.

(e) $(A\otimes B)^t = A^t\otimes B^t$, $(A\otimes B)^- = A^-\otimes B^-$, $(A\otimes B)^* = A^*\otimes B^*$.

(f) $tr(A\otimes B) = (trA)(trB)$, $\|A\otimes B\| = \|A\|\,\|B\|$ for square norm.

(g) If both A, B are unitary, then so is $A\otimes B$.

30-3.19. **Theorem** Let D, E be bounded representations of G of degree s, t respectively. For every $x\in G$, let $(D\otimes E)(x) = D(x)\otimes E(x)$.

(a) $D\otimes E$ is a bounded representation of G of degree st. It is called the *tensor product representation* of D, E. Furthermore if D, E are unitary, then so is $D\otimes E$.

(b) $\rho_{D\otimes E} = \rho_D\rho_E$, $(D\otimes E)^- = D^-\otimes E^-$.

(c) $D\otimes E$ and $E\otimes D$ are equivalent.

(d) The multiplicity of the identity representation in $D\otimes D^-$ is equal to $\sum_{w\in\Omega} m_w^2$ where Ω is the dual object of G and s_w is the multiplicity of w in D.

Proof. (a) Because $C(G)$ is closed under multiplication, all entry functions $D_{ij}E_{kn}$ of $D\otimes E$ belong to $C(G)$.

(c) They have the same character.

(d) $\int \rho_{D\otimes D^-}(x)d_mx = \int \rho_D(x)\rho_D^-(x)d_mx$

$$= \int |\rho_D(x)|^2 d_mx = \|\rho_D\|^2 = \textstyle\sum_{w\in\Omega} m_w^2. \qquad \square$$

30-3.20. **Theorem** A closed invariant ideal M of $C(G)$ is saturated iff tensor products and conjugates of bounded M-representations are also M-representations.

Proof. If tensor products and conjugates of bounded M-representations are M-representations, then products and complex conjugate of M-trigonometric polynomials are M-trigonometric polynomials. Passing through limits, products and complex conjugate of functions in M belong to M. Hence M is saturated. The converse is obvious. □

30-3.21. Let $D\Omega = \{D^w : w \in \Omega\}$ be the dual system of distinct representatives of the dual object Ω. For all $u, v \in \Omega$, $D^u \otimes D^v$ is equivalent to a direct sum $\oplus_{w\in\Omega} m_w D^w$ where the multiplicity m_w of D^w in $D^u \otimes D^v$ is zero for all except only a finite number of $w \in \Omega$. Define $u \otimes v = \{w \in \Omega : m_w \neq 0\}$. Equivalently we have $u \otimes v = \{w \in \Omega :< \rho_{u\otimes v}, \rho_w >\neq 0\}$. Obviously $u \otimes v$ is independent of the choice of $\{D^w : w \in \Omega\}$. We identify elements in Ω with singletons. For any nonempty subset U, V of Ω, define $U \otimes V = \{u \otimes v : u \in U, v \in V\}$. If one of U, V is empty, define $U \otimes V = \emptyset$. Hence $\otimes$ is defined on the power set of Ω. A subset $\triangle$ of Ω is called a *saturated* of G if $\triangle \otimes \triangle \subset \triangle$ and $\triangle^- \subset \triangle$. Clearly the intersection of all saturated dual objects containing a subset $\triangle$ of Ω is a saturated dual object which is said to be *generated by* $\triangle$. It is denoted by $sa(\triangle)$. Clearly the intersection of all saturated closed invariant ideals containing a subset N of $C(G)$ is a saturated closed invariant ideal which is called the saturated closed invariant ideal *generated* by N. It is denoted by $sa(N)$.

30-3.22. **Exercise** Let U, V, W be subsets of the dual object Ω and let 1 denote the dual class containing the identity representation. Prove the following statements.

(a) $1 \otimes w = w$, for all $w \in \Omega$.
(b) $1 \in w \otimes w^-$ for every $w \in \Omega$.
(c) $U \otimes V = V \otimes U$.
(d) $U \otimes (U \otimes W) = (U \otimes U) \otimes W$.
(e) $(U \otimes V)^- = U^- \otimes V^-$.
(f) $sa(U \cup V) = sa(U) \otimes sa(V)$.

30-3.23. **Exercise** Prove that a closed invariant ideal M of $C(G)$ is saturated iff $\Omega_M(G)$ is saturated. Show that $\Omega_{sa(M)}(G) = sa[\Omega_M(G)]$ for every closed invariant ideal M.

30-4 Separating Points

30-4.1. In this section, we start with simple results of continuous almost periodic functions on topological groups which readers without the necessary background may restrict themselves to the additive groups of Banach spaces. Then we derive an explicit formula for mean-values on Banach spaces. Finally we work with compact groups which readers may skip without discontinuity.

30-4.2. **Theorem** Every continuous ap-function f on a topological group G

is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. Choose an ε-cover $\{A_1, A_2, \cdots, A_n\}$ of G for f. Pick $a_i \in A_i$ for each i. By continuity of f, there is a symmetric neighborhood V of e such that $|f(x) - f(a_i)| \le \varepsilon$ for all $x \in a_i V$ and all i. Now suppose that $x^{-1}y \in V$. There is some A_j containing x. Thus $|f(a_j y) - f(xy)| \le \varepsilon$ for all $y \in G$, or $|f(a_j x^{-1} z) - f(z)| \le \varepsilon$ for all $z \in G$. Observe that

$$|f(x) - f(y)| = |f(x) - f(a_j)| + |f(a_j) - f(a_j x^{-1} y)| + |f(a_j x^{-1} y) - f(y)| \le 3\varepsilon.$$

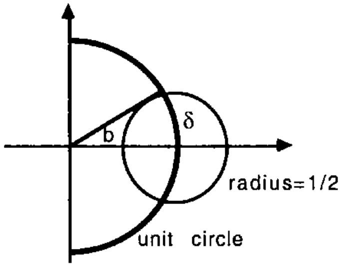
The same inequality also holds for $xy^{-1} \in V$. Therefore f is uniformly continuous with respect to both left and right uniformity of G. □

30-4.3. **Exercise** Prove that the family of all continuous ap-functions on a topological group forms a saturated closed invariant ideal.

30-4.4. **Theorem** The additive group of a Banach space E has enough continuous ap-function to separate points.

Proof. Let $a \neq b$ in E. Without loss of generality, assume that $a \neq 0$. There is a continuous linear form f on E such that $f(a) = \pi$ and $f(b) = 0$. For every $x \in E$, let $v(x) = \text{Re } f(x)$. Then $x \to e^{iv(x)}$ is a continuous character on E and hence it is a continuous ap-function. Furthermore $e^{iv(a)} = -1 \neq 1 = e^{iv(b)}$. □

30-4.5. **Lemma** Every continuous irreducible unitary representation ρ on the real line $\mathbb{R}$ is of the form $x \to e^{i\theta x}$ for some $\theta \in \mathbb{R}$.



Proof. Since $\mathbb{R}$ is commutative, the degree of ρ is one. By continuity at $x = 0$, there is $\delta > 0$ such that $|\rho(x) - 1| = |\rho(x) - \rho(0)| \le \frac{1}{2}$ for all $|x| \le \delta$. There is $b \in \mathbb{R}$ with $|b| < \frac{1}{2}\pi$ satisfying $\rho(\delta) = e^{ib}$. Now $\rho(\frac{1}{2}\delta)^2 = \rho(\frac{1}{2}\delta)\rho(\frac{1}{2}\delta) = \rho(\frac{1}{2}\delta + \frac{1}{2}\delta) = \rho(\delta) = e^{ib}$, i.e. $\rho(\frac{1}{2}\delta) = e^{i(b+2n\pi)/2}$. If $n = 2k + 1$ is odd, then

$$\begin{aligned}\left|\rho\left(\tfrac{1}{2}\delta\right) - 1\right| &= \left|e^{i(b+2n\pi)/2} - 1\right| = \left|e^{i(b/2+2k\pi+\pi)} - 1\right| \\ &= \left|1 - e^{ib/2} - 2\right| \ge 2 - \left|1 - e^{ib/2}\right| \ge 2 - \tfrac{1}{2} > 1\end{aligned}$$

which is a contradiction. Thus $n = 2k$ is even. Hence $\rho(\frac{1}{2}\delta) = e^{i(b+4k\pi)/2} = e^{ib/2}$. Similarly, we obtain $\rho(\delta/2^n) = e^{ib/2^n}$ for all integers $n > 1$. It follows that

$\rho[\delta(k/2^n)] = e^{ib(k/2^n)}$ for all integers k, n. Thus for all rationals r, we have $\rho(\delta r) = e^{ibr}$. By continuity, we have $\rho(\delta y) = e^{iby}$ for all real y. Therefore $\rho(x) = e^{i(b/\delta)x}$ for all $x \in \mathbb{R}$. For an alternative proof, see [Naimark, p152]. □

30-4.6. **Theorem** Every continuous irreducible unitary representation ρ on a Banach space E is of the form $x \to e^{iv(x)}$ for some *real* continuous linear form v on E. Note that v is a real part of a continuous linear form on E.

Proof. Fix $x \in E$. Define $\varphi_x(t) = \rho(tx)$ for all $t \in \mathbb{R}$. Then φ_x is a continuous representation on $\mathbb{R}$. There is $v(x) \in \mathbb{R}$ such that $\varphi_x(t) = \rho(tx) = e^{itv(x)}$ for all $t \in \mathbb{R}$. Given any $x, y \in E$, we have $\rho[t(x+y)] = \rho(tx+ty) = \rho(tx)\rho(ty)$, i.e. $e^{itv(x+y)} = e^{itv(x)}e^{itv(y)}$, or $e^{it[v(x+y)-v(x)-v(y)]} = 1$. Hence for every $t \in \mathbb{R}$, there is some integer n such that $t[v(x+y) - v(x) - v(y)] = 2n\pi$. Thus for all $t \in \mathbb{R}$, $t[v(x+y) - v(x) - v(y)]/\pi = 2n$ is an integer. This can happen only if $v(x+y) - v(x) - v(y) = 0$, or $v(x+y) = v(x) + v(y)$. Similarly, we have $v(\alpha x) = \alpha v(x)$ for all $\alpha \in \mathbb{R}$. Therefore v is a real linear form on E. Next, take any $0 < \varepsilon < 1$. There is a ball V with center $0 \in E$ such that for all $x \in V$, we have $|\rho(x) - 1| \leq \varepsilon$. Suppose to the contrary that $|v(x)| > \frac{1}{2}\pi$ for some $x \in V$. Choose $|t| \leq 1$ such that $tv(x) = \frac{1}{2}\pi$. Clearly $tx \in V$. Hence $|\rho(tx) - 1| = |e^{itv(x)} - 1| = \sqrt{2} > \varepsilon$. Therefore $|v(x)| \leq \frac{1}{2}\pi$ for all $x \in V$. Consequently, v is continuous on E. Note that replacing V by a balanced 0-neighborhood, the same proof works for separated locally convex spaces. □

30-4.7. **Corollary** Every continuous ap-function on E is a uniform limit of finite linear combinations of characters of the form $e^{iv(x)}$ for all $x \in E$ where v is a *real* continuous linear form on E.

30-4.8. **Example** The mean-value of a continuous ap-function f on $\mathbb{R}$ is given by $\int f d_m = \lim_{t\to\infty} \frac{1}{t}\int_a^{a+t} f(x)dx = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(x)dx$. The convergence is uniformly and is independent of a. Furthermore if h is continuous periodic with period $p > 0$, then we have $\int h d_m = \frac{1}{p}\int_0^p h(x)dx$.

Proof. For every $\varepsilon > 0$, choose a continuous trigonometric polynomial g with $|f(x) - g(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. Write $g(x) = a_0 + \sum_{j=1}^k a_j e^{i\theta_j x}$ where $\theta_j \neq 0$. Observe that

$$\left|\frac{1}{t}\int_a^{a+t} g(x)dx - a_0\right| = \left|\sum_{j=1}^k \frac{a_j e^{i\theta_j a}(e^{i\theta_j t} - 1)}{i\theta_j t}\right| \leq \frac{2}{t}\sum_{j=1}^k \left|\frac{a_j}{\theta_j}\right| \to 0$$

uniformly in a as $t \to \infty$. There is $r > 0$ such that for all $s, t \geq r$ and for all $a \in \mathbb{R}$ we have

$$\left|\frac{1}{s}\int_a^{a+s} g(x)dx - \frac{1}{t}\int_a^{a+t} g(x)dx\right| \le \varepsilon.$$

Hence

$$\left|\frac{1}{s}\int_a^{a+s} f(x)dx - \frac{1}{t}\int_a^{a+t} f(x)dx\right|$$

$$\le \frac{1}{s}\int_a^{a+s} |f(x)-g(x)|dx + \left|\frac{1}{s}\int_a^{a+s} g(x)dx - \frac{1}{t}\int_a^{a+t} g(x)dx\right|$$

$$+\frac{1}{t}\int_a^{a+t} |f(x)-g(x)|dx.$$

$$\le 3\varepsilon.$$

Therefore $\lim_{t\to\infty}\frac{1}{t}\int_a^{a+t} f(x)dx$ to some $\mu(f)\in\mathbb{C}$ converges uniformly in a. Remember that continuous ap-functions are bounded. From

$$\left|\frac{1}{t}\int_a^{a+t} f(x)dx - \frac{1}{t}\int_0^{t} f(x)dx\right| \le \frac{1}{t}\left|\int_t^{a+t} f(x)dx - \int_0^{a} f(x)dx\right| \le \frac{2a\|f\|_\infty}{t} \to 0,$$

we have $\mu(f)=\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x)dx$ which is independent of the choice of a. In particular,

$$\mu(h) = \lim_{t\to\infty}\frac{1}{t}\int_a^{a+t} h(x)dx = \lim_{n\to\infty}\frac{1}{np}\int_0^{np} h(x)dx = \frac{1}{p}\int_0^{p} h(x)dx.$$

Clearly μ is a normalized positive linear form on the invariant ideal of continuous ap-functions. Because

$$\lim_{t\to\infty}\frac{1}{t}\int_0^{t} f(x-a)dx = \lim_{t\to\infty}\frac{1}{t}\int_a^{a+t} f(y)dy = \mu(f),$$

μ is also translation invariant. Therefore we have $\int f d_m = \mu(f)$ by the uniqueness of mean-value. Finally, by inversion invariance,

$$\int f d_m = \int f(-x)d_m x = \lim_{t\to\infty}\frac{1}{t}\int_0^{t} f(-x)dx = \lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0} f(y)dy$$

from which we get

$$\int f d_m = \frac{1}{2}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^{t} f(x)dx + \lim_{t\to\infty}\frac{1}{t}\int_{-t}^{0} f(x)dx\right\} = \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} f(x)dx. \ \square$$

30-4.9. **Corollary** For every nonzero real $v\in E'$, we have $\int e^{iv(x)}d_m x = 0$.

Proof. Let $a\in E$ satisfy $v(a)=1$ and let $H=\ker(v)$. Then we may identify $E=\mathbb{R}a\oplus H=\mathbb{R}\times H$. For $x=sa+y$ where $s\in\mathbb{R}$ and $y\in H$, we have

$$\int e^{iv(x)}d_m(x) = \int\int e^{is}d_m s d_m y$$

$$= \lim_{t\to\infty}\frac{1}{2t}\int_{-t}^{t} e^{is}ds = \lim_{t\to\infty}\frac{1}{2t\,i}(e^{it}-e^{-it}) = 0. \qquad \square$$

30-4.10. **Theorem** Let $C(G)$ be a closed invariant ideal of ap-functions on a group G and let $a, b \in G$.
(a) $f(a) = f(b)$ for all $f \in C(G)$ iff $D(a) = D(b)$ for every irreducible unitary representation D.
(b) $f(a) = f(b)$ for all central functions $f \in C(G)$ iff $\rho(a) = \rho(b)$ for all irreducible characters $\rho \in C(G)$.

Proof. It follows because the trigonometric polynomials are dense in $C(G)$.□

30-4.11. **Theorem** The set $H = \{x \in G : f(x) = f(e), \quad \forall f \in A(G)\}$ is a normal subgroup of G where e is the identity element of G. If K is a normal subgroup such that G/K has enough ap-functions to separate points, then we have $H \subset K$. Without continuity of quotient map, we work with $C(G) = A(G)$.

Proof. Every unitary representation D of G is a homomorphism from G into some unitary group. Hence $\ker(D)$ is a normal subgroup of G. It follows that $H = \cap_D \ker(D)$ is also a normal subgroup of G. Next, suppose $a \in G \setminus K$. Let φ be the quotient map from G onto G/K. Then $\varphi(a) \neq \varphi(e)$. There is some ap-function g on G/K such that $g\varphi(a) \neq g\varphi(e)$. It follows that $f = g\varphi$ is an ap-function on G with $f(a) \neq f(e)$. Hence $a \notin H$. Therefore $H \subset K$. □

30-4.12. **Theorem** Every continuous function f on a (separated) compact group G is almost periodic.

Proof. Since G is compact, f is uniformly continuous. There is a neighborhood of e such that $|f(x) - f(y)| \leq \varepsilon$ for all $x^{-1}y \in V$. There is an open neighborhood W of e such that $W^{-1}W \subset V$. By compactness of G, we have $G \subset JW$ for some finite subset J of G. Now for all $x \in G$, $a \in J$ and $u, v \in W$; we have $(xau)^{-1}(xav) = u^{-1}v \in W^{-1}W \subset V$. Hence $|f(xau) - f(xav)| \leq \varepsilon$. Consequently the family $\{aW : a \in J\}$ is a right ε-cover of G for f. Therefore f is almost periodic on G. □

30-4.13. **Corollary** The set of all continuous functions on a compact group forms a saturated closed invariant ideal.

30-4.14. **Corollary** The family of all continuous irreducible unitary matrix representations of a compact group separates points.

30-4.15. Note that there are groups without finite dimensional unitary representations, e.g. [Hewitt-63; p348].

30-99. **References and Further Readings** : Hewitt-63, GilDeLamadrid, Dunkl, Rno, Cukerman, Das, Talman and Vilenkin.

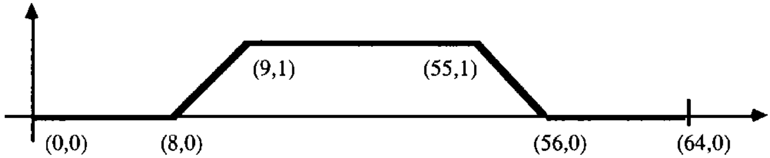
Chapter 31

Mean Spaces

31-1 Representations of Product Groups

31-1.1. Nontrivial translation invariant measures on σ-algebras generated by open subsets of infinite dimensional Banach spaces do not exist. Continuous functions with compact support on infinite dimensional Banach spaces must be zero. Although mean-values behave like integrals, yet Monotone Convergence Theorem fails as shown by a counter example below. Harmonic analysis on unitary groups is largely restricted to finite dimensional cases. In short, we have to look for *new* objects in order to handle infinite dimensional Banach spaces and possibly some subgroups of infinite dimensional unitary groups of Hilbert spaces. In this section, we develop unitary representations on product groups as preparation for the new objects.

31-1.2. A function f on $\mathbb{R}$ is *piecewise differentiable* if $f'(x)$ exists except only a finite number of points in every compact interval and both left-right limits $f'(x-), f'(x+)$ exist everywhere.



31-1.3. **<u>Failure of Monotonic Convergence Theorem</u>** There are continuous periodic piecewise differentiable functions f_n on $\mathbb{R}$ such that for all $1 \leq p < \infty$ and all $n \geq 1$ we have $0 \leq f_{n+1}^p \leq f_n^p \to 0$ pointwise and $\int f_n^p d_m \geq 1/6$.

<u>Proof</u>. Define f_1 on $[0, 8^2]$ by joining the points $(0,0)$, $(8,0)$, $(9,1)$, $(55,1)$, $(56,0)$, $(64,0)$ and then extend it periodically over $\mathbb{R}$. By §30-4.8, we have

$$\int f_1^p d_m = \frac{1}{64} \int_0^{64} f_1^p(x)dx \geq \frac{1}{64} \int_9^{55} f_1^p(x)dx \geq \frac{55-9}{64} > \frac{1}{2}.$$

Inductively, define f_{n+1} as the periodic function of period $8^{2^{n+1}}$ given by

$$f_{n+1}(x) = \begin{cases} 0, & \text{if } \ 0 \le x \le 8^{2^n}; \\ f_n(x), & \text{if } \ 8^{2^n} \le x \le 8^{2^{n+1}} - 8^{2^n}; \\ 0, & \text{if } \ 8^{2^{n+1}} - 8^{2^n} \le x \le 8^{2^{n+1}}. \end{cases}$$

Basically, we take 8^{2^n} lots of consecutive intervals of length 8^{2^n} and change the value of f_n to zero on the first and the last intervals. Clearly all f_n are continuous periodic and piecewise differentiable. Obviously $0 \le f_{n+1}^p \le f_n^p \to 0$ pointwise. We claim $\int f_n^p d_m \ge \prod_{j=1}^n (1-2^{-j})$. In fact, the mean-value of f_{n+1}^p is the average on $[0, 8^{2^{n+1}}]$, that is

$$\int f_{n+1}^p d_m = \frac{8^{2^n}-2}{8^{2^n}} \int f_n^p d_m \ge (1 - 2/8^{2^n}) \prod_{j=1}^n (1-2^{-j}) \ge \prod_{j=1}^{n+1} (1-2^{-j}).$$

Observe that for $0 < t \le \frac{1}{2}$, we have $0 < t^2 < \frac{1}{2}$ and $1 + t < e^t$; that is $1 - t = (1-t^2)/(1+t) \ge 1/(2e^t)$. Hence

$$\int f_n^p d_m \ge \prod_{j=1}^n (1-2^{-j}) \ge \frac{1}{2e^{2^{-1}+2^{-2}+\cdots+2^{-n}}} \ge \frac{1}{2e} \ge \frac{1}{6} > 0.$$

Therefore we do not have $\int f_n^p d_m \to 0$. □

31-1.4. **Exercise** For each $n \ge 2$, let g_n be the real function on $\mathbb{R}$ joining the points $(0,1)$, $(n+1,1)$, $(n+2,0)$, $(\alpha_n - n - 3, 0)$, $(\alpha_n - n - 2, 1)$ and $(\alpha_n, 1)$ and extend it periodically over $\mathbb{R}$ where $\alpha_n = 4n(n+1)(n+2)$. Verify that $\int g_n d_m = \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})$. Let $f_n = \max\{g_1, g_2, \cdots, g_n\}$. Show that f_n are continuous periodic functions satisfying $f_n \uparrow 1$, $0 \le f_n \le g_1 + g_2 + \cdots + g_n$ and $\int f_n d_m \le \frac{1}{2} < 1 = \int 1 d_m$. Prove that $\{f_n\}$ is Cauchy in the Banach space $C_1(\mathbb{R})$ of all continuous ap-functions.

31-1.5. Let G, H be groups and let $A(G), A(H), A(G \times H)$ be the sets of *all* almost periodic functions on $G, H, G \times H$ respectively. After we prepare the ground work, then we deal with closed invariant ideals $C(G), C(H)$ and define $C(G \times H)$ later.

31-1.6. **Theorem** Let f be an ap-function on their product group $G \times H$. Then for every $\varepsilon > 0$ there are ap-functions g_i, h_i on G, H respectively such that $|f(x,y) - \sum_{i=1}^n g_i(x) h_i(y)| \le \varepsilon$ for all $(x,y) \in G \times H$.

Proof. Choose a trigonometric polynomial φ on $G \times H$ such that $\|f - \varphi\|_\infty \le \varepsilon$. Write $\varphi = \sum_{w \in J} s_w \sum_{i,j=1}^{s_w} \alpha_{ij}^w D_{ij}^w$ where J is a finite index set, each D^w is a bounded representation of $G \times H$ of degree s_w and all α_{ij}^w are constants. For simplicity, let e denote the identity element of both G, H. From $D^w(x,y) = D^w[(x,e)(e,y)] = D^w(x,e) D^w(e,y)$, we have

$$D_{ij}^w(x,y) = \sum_{k=1}^{s_w} D_{ik}^w(x,e)D_{kj}^w(e,y).$$

Clearly, $D^w(x,e), D^w(e,y)$ in x,y are bounded representations of G,H respectively. Therefore φ is of the form $\sum_{i=1}^n \beta_i g_i(x)h_i(y)$ where g_i, h_i are entry functions of some bounded representations of G,H respectively and all β_i are constants. □

31-1.7. Let $C(G), C(H)$ be closed invariant ideals of ap-functions on G,H respectively. For every $g \in C(G), h \in C(H)$; let $(g \otimes h)(x,y) = g(x)h(y)$ for all $(x,y) \in G \times H$. It follows from §28-3.4 that $g \otimes h$ is an ap-function on $G \times H$. Let $C(G \times H)$ be the uniform closure of all linear combinations of functions on $G \times H$ of the form $g \otimes h$ for $g \in C(G), h \in C(H)$. For convenience, $C(G \times H)$ is called the *tensor product* of $C(G), C(H)$ and is denoted by $C(G) \otimes C(H)$. As usual, functions in $C(G), C(H), C(G \times H)$ are called comfortable almost periodic functions or cap-functions. It is a reminder of continuous objects on topological groups.

31-1.8. **Theorem** If f is real cap-functions on $G \times H$, then for every $\varepsilon > 0$ there are real cap-functions g_i, h_i on G,H respectively such that for all $(x,y) \in G \times H$ we have $|f(x,y) - \sum_{i=1}^n g_i(x)h_i(y)| \le \varepsilon$.

Proof. Choose complex cap-functions g_i, h_i such that $\|f - \sum_{i=1}^n g_i \otimes h_i\|_\infty \le \varepsilon$. Then $\text{Re}[g_i(x)h_i(y)] = [\text{Re } g_i(x)]\ [\text{Re } h_i(y)] - [\text{Im } g_i(x)]\ [\text{Im } h_i(y)]$ is a finite sum of products of real cap-functions on G,H respectively. Hence

$$|f(x,y) - \textstyle\sum_{i=1}^n \text{Re}[g_i(x)h_i(y)]| = |\text{Re}\,\{f(x,y) - \sum_{i=1}^n g_i(x)h_i(y)\}|$$
$$\le |f(x,y) - \textstyle\sum_{i=1}^n g_i(x)h_i(y)| \le \varepsilon.$$

The result follows by renaming. □

31-1.9. **Problem** If $f \ge 0$, can we choose $g_i, h_i \ge 0$? It seems to be affirmative if we invoke Bohr-compactification but we want a proof acceptable within our framework.

31-1.10. **Theorem** Let f be a cap-function on $G \times H$. If f is central, then for every $\varepsilon > 0$ there are irreducible characters ρ_i, ξ_i on G,H respectively and constants α_i such that $|f(x,y) - \sum_{i=1}^n \alpha_i \rho_i(x)\xi_i(y)| \le \varepsilon$ for all $(x,y) \in G \times H$.

Proof. Choose cap-functions g_i, h_i on G,H respectively such that $\|f - \sum_{i=1}^n g_i \otimes h_i\|_\infty \le \varepsilon$. As in §30-2.10, we have $\|f - \sum_{i=1}^n g_i' \otimes h_i'\|_\infty \le \varepsilon$ where $g_i'(x) = \int g_i(u^{-1}xu)d_m u$ and $h_i'(y) = \int h_i(v^{-1}yv)d_m v$ are central cap-functions which can be uniformly approximated by linear combinations of irreducible characters on G,H respectively, the result follows. □

31-1.11. **Theorem** Let D, E be (comfortable) bounded representations of G, H respectively.
(a) The map $D \times E : (x, y) \to D(x) \otimes E(y)$ is a (comfortable) bounded representation of the product group $G \times H$. It is called the *product representation* of D, E.
(b) If both D, E are unitary, then so is $D \times E$.
(c) $\rho_{D\times E}(x, y) = \rho_D(x)\rho_E(y)$, $\|\rho_{D\times E}\| = \|\rho_D\| \, \|\rho_E\|$ and $(D \times E)^- = D^- \times E^-$.
(d) $D \times E$ is irreducible iff both D, E are irreducible.

Proof. (a) For all $(a, b), (x, y) \in G \times H$, we have

$$(D \times E)[(a, b)(x, y)] = (D \times E)(ax, by) = D(ax) \otimes E(by)$$
$$= [D(a)D(x)] \otimes [E(b)E(y)] = [D(a) \otimes E(b)][D(x) \otimes E(y)]$$
$$= [(D \times E)(a, b)] \, [(D \times E)(x, y)].$$

As products of cap-functions on G, H; all entry functions of $D \times E$ are cap-functions on $G \times H$. Hence $D \times E$ is a comfortable representation.
(b) It follows immediately from tensor product of matrices.
(c) They all follow from simple calculation:

$$\rho_{D\times E}(x, y) = tr(D \times E)(x, y) = tr D(x) \otimes E(y)$$
$$= [tr D(x)] \, [tr E(y)] = \rho_D(x)\rho_E(y),$$
$$\|\rho_{D\times E}\|^2 = \int_{G\times H} |\rho_{D\times E}(x, y)|^2 d_m(x, y)$$
$$= \int_G \int_H |\rho_D(x)|^2 |\rho_E(y)|^2 d_m y d_m x$$
$$= \int_G |\rho_D(x)|^2 d_m x \ \int_H |\rho_E(y)|^2 d_m y = \|\rho_D\|^2 \|\rho_E\|^2$$

and the last one is trivial.
(d) Remember that $\|\rho_D\|^2$ is an integer. Hence $D \times E$ is irreducible iff $\|\rho_{D\times E}\| = 1$ iff $\|\rho_D\| \, \|\rho_E\| = 1$ iff $\|\rho_D\| = 1$ and $\|\rho_E\| = 1$ iff D, E are irreducible. □

31-1.12. **Theorem** $(D \oplus D') \times E$ and $D \times E \oplus D' \times E$ are equivalent representations of $G \times H$. Also $D \times (E \oplus E')$ and $D \times E \oplus D \times E'$ are equivalent.

Proof. The first statement follows from $\rho_{(D\oplus D')\times E} = \rho_{D\otimes E + D'\otimes E}$ as a result of the simple calculation:

$$\rho_{(D\oplus D')\times E}(x, y) = \rho_{D\oplus D'}(x) \otimes E(y) = [\rho_D(x) + \rho_{D'}(x)]\rho_E(y)$$
$$= D(x)\rho_E(y) + \rho_{D'}(x)\rho_E(y) = \rho_{D\otimes E}(x, y) + \rho_{D'\otimes E}(x, y) = \rho_{D\otimes E + D'\otimes E}(x, y).$$

The second statement follows in a similar way. □

31-1.13. **Theorem** $(D\otimes D')\times(E\otimes E')$ and $(D\otimes E)\times(D'\otimes E')$ are equivalent.

Proof. $\rho_{(D\otimes D')\times(E\otimes E')}(x,y) = \rho_{D\otimes D'}(x)\rho_{E\otimes E'}(y) = \rho_D(x)\rho_{D'}(x)\rho_E(y)\rho_{E'}(y)$ $= \rho_D(x)\rho_E(y)\rho_{D'}(x)\rho_{E'}(y) = \rho_{D\otimes E}(x,y)\rho_{D'\otimes E'}(x,y) = \rho_{(D\otimes E)\oplus(D'\otimes E')}(x,y)$. □

31-1.14. **Exercise** Prove that if D, D' are equivalent and E, E' are equivalent, then $D \times E$, $D' \times E'$ are also equivalent.

31-1.15. **Exercise** Prove that every irreducible bounded representation of $G \times H$ is equivalent to some representation of the form $D \times E$ where D, E are irreducible unitary representations of G, H respectively.

31-1.16. **Exercise** Prove that every irreducible character on $G \times H$ is for the form $(x, y) \to \varphi(x)\xi(y)$ for $(x, y) \in G \times H$ where φ, ξ are irreducible characters on G, H respectively.

31-1.17. **Exercise** For every $u \in \Omega(G)$ and $v \in \Omega(H)$, choose any $D_u \in u$ and $E_v \in v$. Let $u \times v$ be the dual class of $G \times H$ containing $D_u \times E_v$. Prove that $u \times v$ is independent of the choice of D_u, E_v. Show that the map $(u, v) \to u \times v$ is a bijection from $\Omega(G) \times \Omega(H)$ onto $\Omega(G \times H)$. Hence we may identify $\Omega(G) \times \Omega(H)$ with $\Omega(G \times H)$.

31-2 Means on Groups

31-2.1. Let $C(G)$ be a closed invariant ideal of comfortable almost periodic functions or cap-functions on a group G. The dual space $M(G)$ of $C_\infty(G)$ is called the *mean space* of G. Members of $M(G)$ are called a *means* on G. The norm of a mean μ is given by $\|\mu\| = \sup\{|\mu(f)| : f \in C(G),\ \|f\|_\infty \le 1\}$ for every $\mu \in M(G)$. Write $M_\beta(G)$ to emphasize this norm.

31-2.2. Write $\mu(f) = \int f d_m\mu = \int f(x)d_m\mu(x)$, $\forall\ \mu \in M(G)$ and $f \in C(G)$. Of course, x is merely a dummy variable. Obviously for all $\mu, \nu \in M(G)$ and all $\alpha \in \mathbb{C}$, we have $\left|\int f d_m\mu\right| \le \|f\|_\infty\|\mu\|$, $\int f d_m(\mu + \nu) = \int f d_m\mu + \int f d_m\nu$ and $\int f d_m(\alpha\mu) = \alpha \int f d_m\mu$.

31-2.3. **Theorem** Let G, H be groups and f a cap-function on the product group $G \times H$. Let μ, ν be means on G, H respectively.

(a) $y \to \int f(x, y)d_m\mu(x)$ is a cap-function in $y \in H$.

(b) $\int\int f(x, y)d_m\mu(x)d_m\nu(y) = \int\int f(x, y)d_m\nu(y)d_m\mu(x)$.

Proof. For every $\varepsilon > 0$, choose $g_i \in C(G), h_i \in C(H)$ such that

$$|f(x,y)-\sum_{i=1}^n g_i(x)h_i(y)|\le\varepsilon,\qquad \forall\, x\in G, y\in H.$$

Applying μ, we have

$$\left|\int f(x,y)d_m\mu(x)-\sum_{i=1}^n\left\{\int g_i(x)d_m\mu(x)\right\}h_i(y)\right|\le\varepsilon\|\mu\|,\qquad\forall\, y\in H.$$

Thus $y\to\int f(x,y)d_m\mu(x)$ is a uniform limit of cap-functions on H and hence it is almost periodic on H.

(b) Applying ν to the last inequality, we get

$$\left|\int\int f(x,y)d_m\mu(x)d_m\nu(y)-\sum_{i=1}^n\int g_i(x)d_m\mu(x)\ \int h_i(y)d_m\nu(y)\right|\le\varepsilon\|\mu\|\ \|\nu\|.$$

Similarly by symmetry, we get

$$\left|\int\int f(x,y)d_m\nu(y)d_m\mu(x)-\sum_{i=1}^n\int g_i(x)d_m\mu(x)\ \int h_i(y)d_m\nu(y)\right|\le\varepsilon\|\mu\|\ \|\nu\|.$$

Combining these two inequalities, we obtain

$$\left|\int\int f(x,y)d_m\mu(x)d_m\nu(y)-\int\int f(x,y)d_m\nu(y)d_m\mu(x)\right|\le 2\varepsilon\|\mu\|\ \|\nu\|.$$

Since $\varepsilon>0$ is arbitrary, the result follows. □

31-2.4. **Theorem** Let μ,ν be means on a group G. For all $f\in C(G)$, let $\int f(x)d_m(\mu\times\nu)(x)=\int\int f(xy)d_m\mu(x)d_m\nu(y)$ and $\int fd_m\mu^*=\left[\int f^*d_m\mu\right]^-$.

(a) $\mu\times\nu$ is a mean on G. It is called the *convolution* of μ,ν.

(b) μ^* is a mean on G. It is called the *hermitian* of μ.

(c) $\|\mu\times\nu\|\le\|\mu\|\ \|\nu\|$ and $\|\mu^*\|=\|\mu\|$.

(d) If f is a central function, then $\int fd_m(\mu\times\nu)=\int fd_m(\nu\times\mu)$.

Proof. Clearly both $\mu\times\nu$ and μ^* are linear forms on $C(G)$. Take any $f\in C(G)$. Since

$$\begin{aligned}\left|\int fd_m(\mu\times\nu)\right|&=\left|\int\int f(xy)d_m\mu(x)d_m\nu(y)\right|\le\|\nu\|\sup_{y\in H}\left|\int f(xy)d_m\mu(x)\right|\\&\le\|\nu\|\sup_{y\in H}\left\{\sup_{x\in G}\|f\|_\infty\|\mu\|\right\}=\|\mu\|\ \|\nu\|\ \|f\|_\infty,\end{aligned}$$

the linear form $\mu\times\nu$ is continuous on $C_\infty(G)$ and hence it is a mean on G. Furthermore we have $\|\mu\times\nu\|\le\|\mu\|\ \|\nu\|$. Next, since

$$\left|\int fd_m\mu^*\right|=\left|\left\{\int f^*d_m\mu\right\}^-\right|\le\|f^*\|\ \|\mu\|=\|f\|_\infty\|\mu\|,$$

μ^* is a continuous linear form on $C_\infty(G)$ and hence it is a mean on G. Furthermore we have $\|\mu^*\|\le\|\mu\|$. It is routine to verify that $\mu^{**}=\mu$. Thus replacing μ by μ^*, we have $\|\mu\|=\|\mu^{**}\|\le\|\mu^*\|$. Therefore we obtain $\|\mu^*\|=\|\mu\|$. Finally, for central function f, we have

$$\begin{aligned}&\int f(x)d_m(\mu\times\nu)(x)=\int f(xy)d_m\mu(x)d_m\nu(y)\\&=\int f(yx)d_m\nu(y)d_m\mu(x)=\int f(x)d_m(\nu\times\mu)(x).\end{aligned}$$

□

31-2.5. **Theorem** The mean space $M(G)$ is an associative algebra with involution under convolution. If G is abelian, then $M(G)$ is commutative. When we want to emphasize the convolution, it is called the *mean algebra* of G.

Proof. Let $\mu, \nu, \omega \in M(G)$ and $f \in C(G)$. From

$$\int f(x)d_m[\mu \times (\nu \times \omega)](x) = \int \left\{\int f(xy)d_m(\nu \times \omega)(y)\right\} d_m\mu(x)$$
$$= \int \left\{\int \int f(xyz)d_m\nu(y)d_m\omega(z)\right\} d_m\mu(x)$$
$$= \int \int \int f(xyz)d_m\mu(x)d_m\nu(y)d_m\omega(z) = \int f(x)d_m[(\mu \times \nu) \times \omega](x),$$

we obtain $\mu \times (\nu \times \omega) = (\mu \times \nu) \times \omega$. Next from

$$\int f(x)d_m[(\mu + \nu) \times \omega](x) = \int \int f(xy)d_m(\mu + \nu)(x)d_m\omega(y)$$
$$= \int \left\{\int f(xy)d_m\mu(x) + \int f(xy)d_m\nu(x)\right\} d_m\omega(y)$$
$$= \int \int f(xy)d_m\mu(x)d_m\omega(y) + \int \int f(xy)d_m\nu(x)d_m\omega(y)$$
$$= \int f(x)d_m(\mu \times \omega)(x) + \int f(x)d_m(\nu \times \omega)(x)$$
$$= \int f(x)d_m(\mu \times \omega + \nu \times \omega)(x),$$

we get $(\mu+\nu)\times\omega = \mu\times\omega+\nu\times\omega$. Similarly we can prove $\mu\times(\nu+\omega) = \mu\times\omega+\nu\times\omega$, $\alpha(\mu \times \nu) = (\alpha\mu) \times \nu = \mu \times (\alpha\nu)$ for $\alpha \in \mathbb{C}$, $(\mu + \nu)^* = \mu^* + \nu^*$, $(\mu \times \nu)^* = \nu^* \times \mu^*$. Therefore $M(G)$ is an associative algebra with involution. If G is abelian, then every cap-function is central and hence $\int f d_m(\mu \times \nu) = \int f d_m(\nu \times \mu)$, or $\mu \times \nu = \nu \times \mu$. □

31-2.6. **Theorem** For each $a \in G$ and $f \in C(G)$, let $\delta_a(f) = f(a)$.

(a) δ_a is a mean on G. It is called the *point-mean* at $a \in G$. To conform with out notation, we also write $\int f(x)d_m\delta_a(x) = f(a)$.

(b) $\|\delta_a\| = 1$, $\mu \times \delta_a = \mu R_a$, $\delta_a \times \mu = \mu L_a$, $\delta_e \times \mu = \mu \times \delta_e = \mu$, $\delta_{ab} = \delta_a \times \delta_b$ and $\delta_a^* = \delta_{a^{-1}}$.

(c) $M(G)$ is a unital algebra with δ_e as the multiplicative identity.

(d) The map $a \to \delta_a$ is a homomorphism from the group G into the semigroup $M(G)$ under convolution.

Proof. Clearly δ_a is a linear form on $C(G)$. Since $|\delta_a(f)| = |f(a)| \leq \|f\|_\infty$, δ_a is on $C_\infty(G)$. Furthermore we have $\|\delta_a\| \leq 1$. By considering the constant function $f = 1$, we obtain $\|\delta_a\| = 1$. Next, since

$$(\mu \times \delta_a)(f) = \int f(x)d_m(\mu \times \delta_a)(x) = \int \int f(xy)d_m\mu(x)d_m\delta_a(y)$$
$$= \int f(xa)d_m\mu(x) = \int R_a f(x)d_m\mu(x) = (\mu R_a)(f),$$

we have $\mu \times \delta_a = \mu R_a$. Similarly, we obtain $\delta_a \times \mu = \mu L_a$. In particular when a is the identity element e of G, we have $\delta_e \times \mu = \mu \times \delta_e = \mu$. Next, since

$$\int f(x) d_m \delta_{ab}(x) = f(ab) = \int \int f(xy) d_m \delta_a(x) d_m \delta_b(y) = \int f(x) d_m(\delta_a \times \delta_b)(x),$$

we get $\delta_{ab} = \delta_a \times \delta_b$. Finally, $\delta_a^* = \delta_{a^{-1}}$ because of

$$\int f(x) d_m \delta_a^*(x) = \left[\int f^*(x) d_m \delta_a(x)\right]^- = [f^*(a)]^- = f(a^{-1}) = \int f(x) d_m \delta_{a^{-1}}. \square$$

31-2.7. **Corollary** Let $C(G)$ separate points of G. Then $a \to \delta_a$ is injective. Consequently, G is abelian iff $M(G)$ is commutative.

Proof. If $\delta_a = \delta_b$, then $f(a) = \delta_a(f) = \delta_b(f) = f(b)$ for all $f \in C(G)$, i.e. $a = b$. Thus $a \to \delta_a$ is injective. If $M(G)$ is commutative, then from $\delta_{ab} = \delta_a \times \delta_b = \delta_b \times \delta_a = \delta_{ba}$, we get $ab = ba$ for all $a, b \in G$ and hence G is abelian. □

31-3 Order Structure on Mean Spaces

31-3.1. In this section, we show that Monotone Convergence Theorem and Fatou's Lemma hold if we use the order structure of linear forms. Readers should review Chapter 16 if necessary. Let $C(G)$ be a *saturated* closed invariant ideal of comfortable almost periodic functions or cap-functions on a group G.

31-3.2. **Lemma** $C(G)$ is a breakable complex vector lattice. See §16-3.11.

Proof. Since $C(G)$ is saturated, it is a complex vector lattice. Let $f, g, h \in C(G)$ such that $|h| \le f + g$ and $f, g \ge 0$. Since the real vector lattice $C^r(G)$ is breakable, write $|h| = f_1 + g_1$ for some $f_1, g_1 \in C^+(G)$. From $0 \le f_1 \le |h|$, we have $f_2 = f_1 \text{sgn}(h) \in C(G)$ by §30-3.8. Similarly, we obtain $g_2 = g_1 \text{sgn}(h) \in C(G)$. Now $f_2 + g_2 = f_1 \text{sgn}(h) + f_2 \text{sgn}(h) = (f_1 + f_2)\text{sgn}(h) = |h|\text{sgn}(h) = h$ completes the proof. □

31-3.3. **Theorem** The mean space $M(G)$ is the order dual space $C^b(G)$ of all order bounded linear forms on $C(G)$.

Proof. Let $\mu \in M(G)$. If $|g| \le f \in C(G)$, then $|\mu(g)| \le \|\mu\| \, \|g\|_\infty \le \|\mu\| \, \|f\|_\infty$. Hence $\sup\{|\mu(g)| : |g| \le f\} \le \|\mu\| \, \|f\|_\infty$. Therefore μ is order bounded. Conversely, let μ be a positive linear form on $C(G)$. Then for every $f \in C(G)$, we get $|\mu(f)| \le \mu(|f|) \le \mu(\|f\|_\infty) \le \mu(1)\|f\|_\infty$. Hence μ is continuous on $C_\infty(G)$, or $\mu \in M(G)$. □

31-3.4. As a result of the general theory of complex vector lattices, for every mean μ on G, its conjugate μ^- and valuation $|\mu|$ are well defined by

(a) $\mu^-(f) = [\mu(f^-)]^-$ for all $f \in C(G)$,

(b) $|\mu|(f) = \sup\{|\mu(g)| : |g| \leq f\}$ for all $f \in C^+(G)$.
Standard properties can be found in Chapter 16.

31-3.5. **Lemma** If $\mu \in M^+(G)$, then $\|\mu\| = \mu(1)$.

Proof. Since $|\mu(f)| \leq \mu(|f|) \leq \|f\|_\infty \mu(1)$, we have $\|\mu\| \leq \mu(1)$. The result follows from $\mu(1) \leq \|\mu\| \, \|1\|_\infty = \|\mu\|$. □

31-3.6. **Theorem** If μ, ν are positive means, then so is $\mu \times \nu$.

Proof. It follows immediately from the trivial computation

$$\int f(x) d_m(\mu \times \nu)(x) = \int \int f(xy) d_m\mu(x) d_m\nu(y) \geq 0, \qquad \text{for all } f \in C^+(G). \ \square$$

31-3.7. Clearly δ_a is a positive mean for every $a \in G$. We also have the following formulas:
(a) $\mu^{--} = \mu$, $(\mu + \nu)^- = \mu^- + \nu^-$, $(\alpha\mu)^- = \alpha^-\mu^-$, $\|\mu^-\| = \|\mu\|$.
(b) $|\alpha\mu| = |\alpha| \, |\mu|$, $|\mu + \nu| \leq |\mu| + |\nu|$, $\big| \, |\mu| - |\nu| \, \big| \leq |\mu - \nu|$, $|\mu^*| = |\mu^-| = |\mu|$, $|\mu \times \nu| \leq |\mu| \times |\nu|$.

Proof. To prove the last inequality, for any $f \in C(G)$, we have

$$\left|\int f(x) d_m(\mu \times \nu)(x)\right| = \left|\int \int f(xy) d_m\mu(x) d_m\nu(y)\right|$$
$$\leq \int \int |f(xy)| d_m|\mu|(x) d_m|\nu|(y) = \int |f|(x) d_m(|\mu| \times |\nu|)(x).$$

It follows that $|\mu \times \nu| \leq |\mu| \times |\nu|$. □

31-3.8. **Theorem** Let $\mu, \nu \in M(G)$.
(a) $\|\mu\| = \int 1 d_m|\mu|(x) = \big\| \, |\mu| \, \big\|$.
(b) If $\mu \leq \nu$ in $M^r(G)$, then for all $f \in C^+(G)$, $\int f(x) d_m\mu(x) \leq \int f(x) d_m\nu(x)$.
(c) If $\mu, \nu \geq 0$, then $\|\mu + \nu\| = \|\mu\| + \|\nu\|$.
(d) The map $\mu \to |\mu|$ is continuous on $M_\beta(G)$.
(e) For all real means μ_n, ν_n, μ, ν on G, if $\mu_n \to \mu$ and $\nu_n \to \nu$ in $M_\beta(G)$, then $\mu_n \vee \nu_n \to \mu \vee \nu$ and $\mu_n \wedge \nu_n \to \mu \wedge \nu$. Furthermore if $\mu_n \geq 0$, then $\mu \geq 0$.

Proof. (a) Observe that $\int 1 d_m|\mu|(x) = \sup\{|\mu(g)| : |g| \leq 1, \ g \in C(G)\} = \|\mu\|$. Replacing μ by $|\mu|$, we obtain $\big\| \, |\mu| \, \big\| = \int 1 d_m \big| \, |u| \, \big|(x) = \int 1 d_m|\mu|(x) = \|\mu\|$.
(c) $\|\mu + \nu\| = \int 1 d_m(\mu + \nu)(x) = \int 1 d_m\mu(x) + \int 1 d_m\nu(x) = \|\mu\| + \|\nu\|$.
(d) From $\big\| \, |\mu_n| - |\mu| \, \big\| = \int 1 d_m \big| \, |\mu_n| - |\mu| \, \big|(x) \leq \int 1 d_m|\mu_n - \mu|(x) = \|\mu_n - \mu\| \to 0$, the map $\mu \to |\mu|$ is continuous.
(e) It follows immediately from $\mu \vee \nu = \frac{1}{2}(\mu + \nu + |\mu - \nu|)$ and $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. Finally, take any $f \in C^+(G)$, we have $0 \leq \mu_n(f) \to \mu(f)$. Therefore $\mu(f) \geq 0$. Consequently, $\mu \geq 0$. □

31-3.9. For every mean μ on G, the *mean-value* of μ is defined and denoted by $\mu(1) = \int \mu d_m = \int \mu(x) d_m x$ although $\mu(x)$ has no meaning at this moment. The linear form $\mu \to \int \mu d_m$ is called the *mean-value form.* The following properties follow immediately from simple calculation. The mean-value form is norm-continuous on $M_\beta(G)$. Furthermore for all $\mu \in M(G)$ we have

(a) $\int \delta_a \times \mu d_m = \int \mu d_m = \int \mu \times \delta_a d_m$ for all $a \in G$, translation invariant;

(b) $\int \mu^* d_m = \int \mu^- d_m = [\int \mu d_m]^-$;

(c) $\|\mu\| = \int |\mu| d_m$.

31-3.10. **Monotonic Convergence Theorem for Means** If $\mu_n \le \mu_{n+1}$ is a monotonic increasing sequence of real means satisfying $\sup \int \mu_n d_m < \infty$, then $\mu = \sup \mu_n$ exists in $M(G)$. Furthermore we obtain $\int f d_m \mu_n \to \int f d_m \mu$ for all $f \in C(G)$. In particular, we also have $\int \mu_n d_m \to \int \mu d_m$.

Proof. Let $\alpha = \sup \int \mu_n d_m$. For all $k \ge 1$, observe that

$$\begin{aligned}\|\mu_{n+k} - \mu_n\| &= \int |\mu_{n+k} - \mu_n| d_m = \int (\mu_{n+k} - \mu_n) d_m \\ &= \int \mu_{n+k} d_m - \int \mu_n d_m \le \alpha - \int \mu_n d_m \to 0\end{aligned}$$

as $n \to \infty$. Therefore $\{\mu_n\}$ is a Cauchy sequence in $M_\beta(G)$. Let $\mu = \lim \mu_n \in M_\beta(G)$. Then $\mu_n \to \mu$ weakly, that is $\int f d_m \mu_n \to \int f d_m \mu$. Consequently $\mu = \sup \mu_n$. Letting $f = 1$, we obtain $\int \mu_n d_m \to \int \mu d_m$. □

31-3.11. **Fatou's Lemma for Means** Let $\mu_n \in M^+(G)$. If $\liminf \int \mu_n d_m < \infty$, then $\liminf \mu_n = \sup_{n \ge 1} \inf_{k \ge n} \mu_k$ exists in $M(G)$. Furthermore we have

$$\int \liminf \mu_n d_m \le \liminf \int \mu_n d_m.$$

Proof. For each n, define $\nu_n = \inf_{k \ge n} \mu_k$. Since $0 \le \nu_n \le \mu_k$ for all $k \ge n$, we have $0 \le \int \nu_n d_m \le \int \mu_k d_m$. Hence $0 \le \int \nu_n d_m \le \inf_{k \ge n} \int \mu_k d_m$. Since ν_n is monotonic increasing, we have

$$\sup_{n \ge 1} \int \nu_n d_m \le \lim_{n \to \infty} \inf_{k \ge n} \int \mu_k d_m = \liminf \int \mu_n d_m < \infty.$$

Hence $\sup \nu_n$ exists in $M(G)$ and satisfies

$$\int \liminf \mu_n d_m = \int \sup_{n \ge 1} \nu_n d_m = \sup_{n \ge 1} \int \nu_n d_m \le \liminf \int \mu_n d_m. \quad \square$$

31-4 Identification of Functions as Means

31-4.1. Let $C(G)$ be a *saturated* closed invariant ideal of comfortable almost periodic functions or cap-functions on a group G and $M(G)$ the mean space of G. In this section, we embed the normed space $C_1(G)$ into $M_\beta(G)$. As an analogue of L_1-spaces, we also develop $\ell_1(G)$ on arbitrary group G including the additive groups of infinite dimensional Banach spaces. Note that $\ell_1(G)$ has nothing to do with $\ell_p(X)$ in §21-3.7.

31-4.2. **Theorem** For every $h \in C(G)$, let $d_m h : C(G) \to \mathbb{K}$ be defined by $\int f(x) d_m h(x) = \int f(x)h(x) d_m x$ for all $f \in C(G)$. Then $d_m h$ is a mean on G called the *mean from h*. Furthermore, $(d_m h)^- = d_m(h^-)$, $(d_m h)^* = d_m(h^*)$ and $\|d_m h\| = \|h\|_1$. The mean-value forms on $C(G), M(G)$ agree with each other.

Proof. Clearly $d_m h$ is a linear form on $C(G)$. From

$$\left|\int f(x) d_m h(x)\right| = \left|\int f(x)h(x) d_m x\right| \leq \|f\|_\infty \|h\|_1$$

$d_m h$ is continuous on $C_\infty(G)$. Therefore $d_m h$ is a mean on G with $\|d_m h\| \leq \|h\|_1$. To prove the equality, for every $\varepsilon > 0$ choose $g \in C(G)$ by §30-3.9 such that $|g(x)| \leq 1$, $g(x)h(x) \geq 0$ and $\big| \, g(x)h(x) - |h(x)| \, \big| \leq \varepsilon$ for all $x \in G$. Thus

$$\begin{aligned} \|d_m h\| &\geq \left|\int g(x) d_m h(x)\right| = \left|\int g(x)h(x) d_m x\right| \\ &= \int g(x)h(x) d_m x \geq \int \big\{|h(x)| - \varepsilon\big\} d_m x = \|h\|_1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\|d_m h\| \geq \|h\|_1$. To prove $(d_m h)^* = d_m(h^*)$, consider that for all $f \in C(G)$ we have

$$\begin{aligned} &\int f(x)(d_m h)^*(x) = \left[\int f^*(x) d_m h(x)\right]^- = \left[\int f^-(x^{-1})h(x) d_m x\right]^- \\ &= \int f(x)h^-(x^{-1}) d_m x = \int f(x)h^*(x) d_m x = \int f(x)(d_m h^*)(x). \end{aligned}$$

Similarly we obtain $(d_m h)^- = d_m(h^-)$. □

31-4.3. The map $d_m : C_1(G) \to M_\beta(G)$ is a norm preserving star-algebra isomorphism which is called the *natural embedding*. The closure of its image in $M_\beta(G)$ is denoted by $\ell_1(G)$. Elements of $\ell_1(G)$ are called ℓ_1-means. As a closed subspace of the Banach space $M_\beta(G)$, $\ell_1(G)$ is itself a Banach space. We identify $C(G) \subset \ell_1(G) \subset M(G)$. Clearly $\mu \to \mu^-$ and $\mu \to \mu^*$ are norm-preserving automorphism on $\ell_1(G)$.

31-4.4. **Theorem** The natural embedding of $C(G)$ into $M(G)$ preserves convolution. The space $\ell_1(G)$ is a two-sided ideal of $M(G)$. Furthermore for all $\mu \in M(G)$ and $f \in C(G)$, we have $(\mu \times h)(x) = \int h(y^{-1}x) d_m \mu(y)$ and $(h \times \mu)(x) = \int h(xy^{-1}) d_m \mu(y)$.

Proof. By §31-2.3a, $\mu \times h$ is a cap-function on G. For all $f, h \in C(G)$ and $\mu \in M(G)$; we have

$$\int f(x) d_m(\mu \times h)(x) = \int\int f(xy) d_m \mu(x) d_m h(y) = \int\int f(xy)h(y) d_m \mu(x) d_m y$$

$= \int \int f(y)h(x^{-1}y)d_m\mu(x)d_my,$ replacing y by $x^{-1}y$

$= \int f(y)\left[\int h(x^{-1}y)d_m\mu(x)\right]d_my.$

Hence $(\mu \times h)(x) = \int h(x^{-1}y)d_m\mu(x)$, or $\mu \times C(G) \subset C(G)$. Since the map $\nu \to \mu \times \nu : M_\beta(G) \to M_\beta(G)$ is continuous, we have $\mu \times \ell_1(G) \subset \ell_1(G)$. Similarly, we obtain $\ell_1(G) \times \mu \subset \ell_1(G)$. □

31-4.5. **Theorem** For every $h \in C(G)$, we have $d_m|h| = |d_mh|$. The natural embedding $d_m : C(G) \to M(G)$ preserves all lattice operations.

Proof. Let $h \in C(G)$ and $\mu = d_mh$. Take any $f \in C^+(G)$. For every $g \in C(G)$ with $|g| \leq f$, we have

$$\left|\int g(x)h(x)d_mx\right| \leq \int |g(x)h(x)|d_mx \leq \int f(x)|h(x)|d_mx = \int f(x)d_m|h|(x).$$

Taking the supremum, we obtain $\int f(x)d_m|\mu|(x) \leq \int f(x)d_m|h|(x)$, or $|\mu| \leq d_m|h|$. On the other hand, for every $\varepsilon > 0$ choose $k \in C(G)$ such that for all $x \in G$, we have $|k(x)| \leq 1$, $k(x)h(x) \geq 0$ and $\big|\, k(x)h(x) - |h(x)| \,\big| \leq \varepsilon$. Thus

$$\left|\int \left\{f(x)k(x)h(x) - f(x)|h(x)|\right\}d_mx\right| \leq \varepsilon \int f(x)d_mx.$$

The function g defined by $g(x) = f(x)k(x)$ satisfies $g \in C(G)$, $|g| \leq f$ and

$$\int f(x)|h(x)|d_mx - \varepsilon \int f(x)d_mx \leq \int f(x)k(x)h(x)d_mx$$
$$= \int g(x)h(x)d_mx \leq \int f(x)d_m|\mu|(x).$$

Since $\varepsilon > 0$ is arbitrary, we have $d_m|h| \leq |\mu|$. Consequently, we have proved that $|\mu| = d_m|h|$. As a result, the natural embedding preserves all lattice operations. □

31-4.6. **Theorem** If $\mu \in \ell_1(g)$, then $|\mu| \in \ell_1(G)$. As a result, $\ell_1(G)$ is closed under lattice operations.

Proof. Since $\ell_1(G)$ is the closure of $C(G)$ in $M_\beta(G)$, there are $h_n \in C(G)$ such that $d_mh_n \to \mu$ in $M_\beta(G)$. Hence $d_m|h_n| = |d_mh_n| \to |\mu|$ in $M_\beta(G)$. Therefore $|\mu| \in \ell_1(G)$. □

31-4.7. **Monotone Convergence Theorem for ℓ_1-Means** If $\mu_n \leq \mu_{n+1}$ in $\ell_1(G)$ and if $\sup \int \mu_n d_m < \infty$, then we have $\mu = \sup \mu_n \in \ell_1(G)$.

Proof. The sequence $\{\mu_n\}$ is Cauchy in $M_\beta(G)$ and hence it is also Cauchy in $\ell_1(G)$. Since $\ell_1(G)$ is a closed subspace of $M_\beta(G)$, the limit $\mu = \lim \mu_n = \sup \mu_n$ also belongs to $\ell_1(G)$. □

31-4.8. **Exercise** Prove that if $\mu_n \geq 0$ and if $\liminf \int \mu_n d_m < \infty$, then we have $\liminf \mu_n \in \ell_1(G)$.

31-4.9. **Example** Let $C(\mathbb{R})$ be the space of continuous ap-functions on $\mathbb{R}$. By §31-1.3, there are $h_n \in C(\mathbb{R})$ such that $0 \leq h_n \downarrow 0$ and $\int h_n d_m \geq \frac{1}{6}$. Then $\inf h_n = 0$ in $C(\mathbb{R})$ under pointwise order but $\inf d_m h_n \neq 0$ in $M(G)$.

Proof. For every $f \in C(G)$, define $\mu(f) = \lim \int f(x)h_n(x)d_m x$. If $f \geq 0$, then the limit exists because $\int f(x)h_n(x)d_m x$ is monotonic decreasing in n and bounded below by zero. Since every cap-function is a linear combination of positive ones, $\mu(f) = \lim \int f(x)h_n(x)d_m x$ exists for all $f \in C(G)$. Clearly μ is a linear form on $C(\mathbb{R})$ and $\mu = \inf \mu_n$. From $\left|\int f(x)h_n(x)d_m x\right| \leq \|fh_n\|_\infty \leq \|f\|_\infty \|h_n\|_\infty \leq \|f\|_\infty$, μ is continuous on $C(G)$, i.e. $\mu \in M(\mathbb{R})$. However $\mu \neq 0$ because $\mu(1) = \lim \int h_n(x)d_m x \geq \frac{1}{6}$. □

31-4.98. **Project** Let $1 < p \leq \infty$ and $1 \leq q < \infty$ be conjugate indices given by $\frac{1}{p} + \frac{1}{q} = 1$. Members of the dual space $\ell_p(G)$ of $C_q(G)$ are called *ℓ_p-means* of G. The norm of a p-mean μ is given by $\|\mu\|_p = \sup\{ |\mu(f)| : \|f\|_q \leq 1\}$. Since $|\mu(f)| \leq \|f\|_q \|\mu\| \leq \|f\|_\infty \|\mu\|$, we have $\ell_p(G) \subset M(G)$. We also have $C(G) \subset \ell_p(G)$ through the identification $h \to d_m h : C(G) \to M(G)$ defined by $\int f d_m h = \int f h d_m$ for all $f \in C(G)$. What can we say about $\ell_p(G)$? Consult the web-page of this book for further information.

31-4.99. **Project** Let $C(G), C(H)$ be closed invariant ideals of ap-functions on groups G, H respectively and $C(G\times H) = C(G)\otimes C(H)$ their tensor product. For every $\mu \in M(G)$, $\nu \in M(H)$ and $f \in C(G \times H)$,

$$\int f(x,y)d_m(\mu \otimes \nu)(x) = \int \int f(x,y)d_m\mu(x)d_m\nu(y)$$

is well-defined by §31-4.3. It is easy to show that $\mu \otimes \nu$ is a continuous linear form on $C_\infty(G \times H)$ and hence it is a mean on $G \times H$. What can we say about the relationship among $M(G), M(H), M(G\times H)$; among $\ell_p(G), \ell_p(H), \ell_p(G\times H)$ and among Fourier matrices of next section?

31-5 Fourier Matrices of Means

31-5.1. Let $C(G)$ be a closed invariant ideal of comfortable almost periodic functions or cap-functions on a group G, $M(G)$ the mean space of G and $mat(s,t)$ the set of all $s \times t$-matrices. Let $A = [A_{ij}] : G \to mat(s,t)$ be *comfortable almost periodic* or a *cap-map*, i.e. all entry functions $A_{ij} : G \to \mathbb{C}$ are cap-functions. For every $\mu \in M(G)$, define $\int A(x)d_m\mu(x) = \left[\int A_{ij}(x)d_m\mu(x)\right]$. The *Fourier matrix* of μ corresponding to a bounded representation D of G is defined by $D(\mu) = \int D^-(x)d_m\mu(x)$. The entries of $D(f)$ are called the *Fourier coefficients* of μ.

31-5.2. **Theorem** Let D be a bounded representation of G.
(a) $(\mu \times D)(x) = D(\mu)^t D(x)$ and $(D \times \mu)(x) = D(x)D(\mu)^t$, $\forall\ \mu \in M(G)$, $x \in G$.
(b) $D(\delta_a) = D^-(a)$ for all $a \in G$.
(c) $D(d_m h) = D(h)$ for all $h \in C(G)$.

Proof. (a) $(\mu \times D)(x) = \int D(y^{-1}x)d_m\mu(y) = \int D(y^{-1})D(x)d_m\mu(y)$
$= \left[\int D^{-t}(y)d_m\mu(y)\right] D(x) = \left[\int D^-(y)d_m\mu(y)\right]^t D(x) = D(\mu)^t D(x)$.
(b) $D(\delta_a) = \int D^-(x)d_m\delta_a(x) = D^-(a)$.
(c) $D(d_m h) = \int D^-(x)d_m h(x) = \int D^-(x)h(x)d_m x = D(h)$. □

31-5.3. **Theorem** Let D be a unitary representation of G of order s. Then the map $\mu \to D(\mu)$ is a star-homomorphism of the mean algebra $M(G)$ into the matrix algebra.

Proof. Clearly $\mu \to D(\mu)$ is a linear map. For all $\mu, \nu \in M(G)$, we have

$$D(\mu^*) = \int D(x)^- d_m\mu^*(x) = \left[\int D(x)^{-*}d_m\mu(x)\right]^- = \left[\int D(x^{-1})^{--}d_m\mu(x)\right]^-$$
$$= \left[\int D(x)^{-t}d_m\mu(x)\right]^- = \left[\int D(x)^- d_m\mu(x)\right]^{-t} = D(\mu)^*$$

and

$$D(\mu \times \nu) = \int D(x)^- d_m(\mu \times \nu)(x) = \int\int D(xy)^- d_m\mu(x)d_m\nu(y)$$
$$= \int\int D(x)^- D(y)^- d_m\mu(x)d_m\nu(y) = D(\mu)D(\nu).$$ □

31-5.4. **Theorem** If D, E are equivalent unitary representations, then the Fourier matrices $D(\mu), D(\nu)$ are unitarily similar.

31-5.5. **Theorem** The Fourier matrices uniquely determine the mean algebra $M(G)$. More precisely, for all means μ, ν, λ on G, the following statements hold.
(a) $\mu = 0$ iff $D(\mu) = 0$, for all irreducible unitary representations D.
(b) $\mu = \nu$ iff $D(\mu) = D(\nu)$ for all D.
(c) $\mu = \nu^*$ iff $D(\mu) = D(\nu)^*$ for all D.
(d) $\lambda = \mu + \nu$ iff $D(\lambda) = D(\mu) + D(\nu)$ for all D.
(e) $\mu = \alpha\nu$ iff $D(\mu) = \alpha D(\nu)$ for all D.
(f) $\lambda = \mu \times \nu$ iff $D(\lambda) = D(\mu)D(\nu)$ for all D.

Proof. We shall prove (a$\Leftarrow$) only. Suppose that $D(\mu) = 0$ for all irreducible unitary representation D. Since every trigonometric polynomial f is a linear combination of entry functions of irreducible unitary representations, we have $\int f(x)d_m\mu(x) = 0$. Since the set of all trigonometric polynomials are dense in $C_\infty(G)$ and μ is continuous on $C_\infty(G)$, we have $\int f(x)d_m\mu(x) = 0$ for all $f \in C(G)$, that is $\mu = 0$. The rest of the theorem is obvious. □

31-5.6. A mean μ on G is said to be *central* if $\mu \times \nu = \nu \times \mu$ for all $\nu \in M(G)$. The set of all central means is denoted by $ZM(G)$.

31-5.7. **Theorem** The following statements are equivalent for any $\mu \in M(G)$.
(a) μ is central.
(b) $\mu \times f = f \times \mu$ for all $f \in C(G)$.
(c) For every irreducible unitary representation D of G of degree s, the Fourier matrix $D(\mu)$ is a constant multiple of an identity matrix.
(d) $\mu \times \delta_a = \delta_a \times \mu$ for all $a \in G$.

Proof. $(a \Rightarrow b)$ It is obvious from the natural embedding $C(G) \subset M(G)$.
$(b \Rightarrow c)$ Let $D_m k$ denote the entry functions of D. Then we have

$$D(\mu)\left[\tfrac{1}{s}\delta_{im}\delta_{jk}\right] = D(\mu)D(D_{mk}) = D(\mu \times D_{mk})$$
$$= D(D_{mk} \times \mu) = D(D_{mk})D(\mu) = \left[\tfrac{1}{s}\delta_{im}\delta_{jk}\right] D(\mu).$$

Since for every $x \in G$, $D(x)$ is a linear combination of matrices $\left[\frac{1}{s}\delta_{im}\delta_{jk}\right]$, we have $D(\mu)D(x) = D(x)D(\mu)$. Because D is irreducible, $D(\mu)$ is a constant multiple of an identity matrix.
$(c \Rightarrow a)$ Take any $\nu \in M(G)$. Since $D(\mu)$ is a constant multiple of identity, we have $D(\mu)D(\nu) = D(\nu)D(\mu)$, i.e. $D(\mu \times \nu) = D(\nu \times \mu)$ for every irreducible unitary representation D. Hence $\mu \times \nu = \nu \times \mu$. Therefore μ is central.
$(a \Rightarrow d)$ It is obvious for δ_a is a mean.
$(d \Rightarrow c)$ Let D be any irreducible unitary representation of G. Then we have

$$D(\mu)D^-(a) = D(\mu)D(\delta_a) = D(\mu \times \delta_a) = D(\delta_a \times \mu) = D^-(a)D(\mu).$$

Since $D^-(a)$ is an irreducible representation of G, $D(\mu)$ is a constant multiple of an identity matrix.

31-99. **References and Further Readings** : Ma-75, Trimeche, Houdre, Vakhania and Jones.

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