EB Ridge, with SVD

Suppose $Y = Xb + e$

$X = \underset{n\times k}{U}\underset{k\times k}{D}\underset{k\times p}{V'}$ available $\quad (k<p)$

So $U'Y = DV'b + U'e$

$\tilde{Y} = \tilde{X}b + \tilde{e} \qquad (\tilde{X}\tilde{X} = D^2)$

$\tilde{Y} \sim N(0, s_b^2 D^2 + s^2 I)$

$\tilde{y}_j \sim N(0, s_b^2 d_j^2 + s^2)$

$$\log p(\tilde{y} \mid s_b, s) = \sum_j -\frac{1}{2}\log 2\pi(s_b^2 d_j^2 + s^2) - \frac{\tilde{y}_j^2}{2(s_b^2 d_j^2 + s^2)}$$

$$\frac{\partial}{\partial s_b^2} = \frac{1}{2}\sum_j \frac{-d_j^2}{s_b^2 d_j^2 + s^2} + \frac{\tilde{y}_j^2 d_j^2}{(s_b^2 d_j^2 + s^2)^2}$$

EM. $\tilde{y}_j \sim N(\theta_j, s^2)$

$\theta_j \sim N(0, s_b^2 d_j^2)$ ✓

eg Simple param as above: $\quad$ 1

$$\log(p(y, \theta \mid s^2, s_b^2)) = -\frac{\cancel{k}}{2}\log(2\pi s^2) - \frac{1}{2s^2}\sum_j (y_j - \theta_j)^2 + \sum_j\left(-\frac{1}{2}\log(2\pi s_b^2 d_j^2) - \frac{1}{2 s_b^2 d_j^2}\theta_j^2\right)$$

$$\frac{\partial}{\partial s_b^2} = -\frac{1}{2}\sum_j\left[\frac{1}{s_b^2} - \frac{\theta_j^2}{(s_b^2)^2 d_j^2}\right]$$

$$= 0 \quad \text{at} \quad s_b^2 = \frac{1}{k} \sum_j \frac{\theta_j^2}{d_j^2}$$

$$\frac{\partial}{\partial s^2} = -\frac{1}{2}\left[\frac{k}{s^2} - \frac{1}{(s^2)^2} \sum_j (y_j - \theta_j)^2 \right]$$

$$= 0 \quad \text{at} \quad s^2 = \frac{1}{k} \sum_j E\left((y_j - \bar{\theta}_j)^2 + (\bar{\theta}_j - \theta_j)^2\right)$$

(2) Scaled parameterization

$$y_j \sim N(s_b\theta_j, s^2) \implies \frac{y_j}{s_b} \sim N\left(\theta_j, \frac{s^2}{s_b^2}\right)$$

$$\theta_j \sim N(0, d_j^2)$$

$$\log p(y, \theta \mid s^2, s_b^2) = -\frac{k}{2}\log(2\pi s^2) - \frac{1}{2s^2}\sum_j (y_j - s_b\theta_j)^2$$

$$+ \cdots$$

$$\frac{\partial}{\partial s^2} = -\frac{1}{2}\left[\frac{k}{s^2} - \frac{1}{(s^2)^2}\sum_j (y_j - s_b\theta_j)^2\right]$$

$$\hat{s}^2 = \frac{1}{k}\sum_j E\left((y_j - s_b\theta_j)^2\right)$$

do s_b first as $\hat{s}^2$ depends on $\hat{s}_b^2$.

$$\frac{\partial}{\partial s_b} = +\frac{2}{2s^2}\sum_j (y_j - s_b\theta_j)\theta_j$$

$$\implies \hat{s}_b^2 = \left[\sum_j y_j\bar{\theta}_j \Big/ \sum_j E(\theta_j^2)\right]^2$$

(2b) if we put d_j in with y

$$\frac{\partial}{\partial s_b} = \frac{1}{s^2}\sum_j (y_j - d_j s_b \theta_j)\, d_j\theta_j$$

$$\hat{s}_b^2 = \left[\frac{\sum_j y_j d_j \bar{\theta}_j}{\sum_j d_j^2 E(\theta_j^2)}\right]^2$$

$$\left[\sum_j d_j^{2} E(\theta_j^{2}) \right)$$

(but this is the same update ...
so it makes no difference.

Basically because d_j is known the reparametrization $\theta_j \leftrightarrow \theta_j d_j$ makes no difference

(3) hybrid

$$y_j \sim N(s_b \theta_j, s^2)$$
$$\theta_j \sim N(0, d_j^2 \lambda^2)$$

$$\left.\begin{aligned} \hat{s}^2 &= \frac{1}{n} \sum_j E\left((y_j - s_b \theta_j)^2\right) \\ \hat{s}_b^2 &= \left[\sum_j y_j \bar{\theta}_j \Big/ \sum_j E(\theta_j^2) \right]^2 \end{aligned}\right\} \text{ as in (2)}$$

$$\hat{\lambda}^2 = \frac{1}{n} \sum_j \frac{E(\theta_j^2)}{d_j^2} \quad \text{as in (1)}$$