UNIT – I

FOURIER SERIES

PROBLEM 1:

The turning moment T lb feet on the crankshaft of steam engine is given for a series of values of the crank angle θ degrees

θ	0	30	60	90	120	150	180
T	0	5224	8097	7850	5499	2626	0

Expand T in a series of sines.

Solution:

Let $T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots\ldots\ldots,$
Since the first and last values of T are repeated neglect last one

θ	**T**	**T sinθ**	**T sin2θ**	**T sin3θ**
0	0	0	0	0
30	5224	2612	4524.11	5224
60	8097	7012.2	7012.2	0
90	7850	7850	0	-7850
120	5499	4762.27	- 4762.27	0
150	2624	1313	- 2274.18	2626
Total		23549.47	4499.86	0

$b_1 = 2$ [mean value of T sinθ]

$$= 2\left[\frac{23549.47}{6}\right] = 7849.8$$

$b_2 = 2$ [mean value of T sin2θ]

$$= 2\left[\frac{4499.86}{6}\right] = 1499.95$$

$b_3 = 2$ [mean value of T sin2θ]

$= 0$

$$\therefore f(x) = 7849.8 \sin\theta + 1499.95 \sin 2\theta$$

PROBLEM 2:

Analyze harmonically the given below and express y in Fourier series upto the third harmonic.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1

Solution:

Since the last value of y is a repetition of the first, only the six values will be used. The length of the interval is 2π.

Let $y = \frac{a0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \ldots$ **(1)**

x	**y**	cosx	sinx	cos2x	sin2x	cos3x	sin3x
0	1.0	1	0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	- 0.5	0.866	-1	0
$\frac{2\pi}{3}$	1.9	- 0.5	0.866	- 0.5	- 0.866	1	0
π	1.7	-1	0	1	0	-1	0
$\frac{4\pi}{3}$	1.5	- 0.5	- 0.866	- 0.5	0.866	1	0
$\frac{5\pi}{3}$	1.2	0.5	- 0.866	- 0.5	- 0.866	-1	0

$a_0 = 2$ [mean value of y]

$= \frac{2}{6}(8.7) = \mathbf{2.9}$

$a_1 = 2$ [mean value of y cosx]

$= \frac{2}{6}[1\,(1.0) + 0.5\,(1.4) - 0.5\,(1.9) - 1\,(1.7) - 0.5\,(1.5) + 0.5\,(1.2)]$

$= \mathbf{-\,0.37}$

$b_1 = 2$ [mean value of y sinx]

$= \frac{2}{6}[0.866\,(1.4 + 1.9 - 1.5 - 1.2)]$

$= 0.17$

$a_2 = 2$ [mean value of y cos2x]

$= \frac{2}{6}[1(1.0 + 1.7) – 0.5 (1.4 + 1.9 + 1.5 + 1.2)]$

= **- 0.1**

b_2 = 2 [mean value of y sin2x]

$= \frac{2}{6}[0.866(1.4 – 1.9 + 1.5 – 1.2)]$

= **- 0.06**

a_3 = 2 [mean value of y cos3x]

$= \frac{2}{6}[1.0 – 1.4 + 1.9 – 1.7 + 1.5 – 1.2]$

= **0.03**

b_3 = 2 [mean value of y sin3x]

= **0**

Hence y = 1.45 + (-0.37 cosx + 0.17 sinx) – (0.1 cos2x + 0.06 sin2x) + 0.03 cos3x.

PROBLEM 3:

Find the Fourier series expansion for the function f(x) = x sinx in 0 < x < 2π and deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ....\infty = \frac{\pi - 2}{4}$

SOLUTION:

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)..................(1)$$

Given f(x) = x sinx

$$\mathbf{a_0} = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}(-2\pi) = \mathbf{-2}$$

$\mathbf{a_n}$ $= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx dx$

$= \frac{1}{2\pi}\int_0^{2\pi} [x\sin(1+n)x + \sin(1-n)x]dx$

$[\because 2SinACosB = Sin(A+B) + Sin(A-B)]$

$= \frac{1}{2\pi}\left[x(\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}) - (-\frac{\sin(1+n)x}{(1+n)^2} - \frac{\sin(1-n)x}{(1-n)^2})\right]_0^{2\pi}$

$= = \frac{1}{2\pi}[-2\pi(\frac{\cos 2(1+n)\pi}{1+n} + \frac{\cos 2(1-n)\pi}{1-n})]($

$= -[\frac{1}{1+n} + \frac{1}{1-n}]$

$= -[\frac{1-n+1+n}{1\text{-}n^2}]$

$= a_n = \frac{2}{n^2-1}$ where $n \neq 1$

a_1 $= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x dx$

$= \frac{1}{2\pi}\left[x(\frac{-\cos 2x}{2}) - (1)(-\frac{\sin 2x}{4})\right]_0^{2\pi}$

$= \frac{1}{2\pi}[2\pi\,(-\frac{\cos 4\pi}{2}) = -\frac{1}{2}$

$\frac{1}{2\pi}[2\pi(-\frac{\cos 4\pi}{2})$

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b_n $= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(1-n)x - \cos(1+n)x]dx$$

$$[\because 2SinASinB = Cos(A-B) - Cos(A+B)]$$

$$= \frac{1}{2\pi}\left[x(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}) - (-\frac{\cos(1-n)x}{(1-n)^2} - \frac{\cos(1+n)x}{(1+n)^2})\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(1-n)\pi}{(1-n)^2} - \frac{\cos 2(1+n)\pi}{(1+n)^2} - \frac{\cos 0}{(1-n)^2} + \frac{\cos 0}{(1+n)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} - \frac{1}{(1-n)^2} + \frac{1}{(1+n)^2}\right]$$

= 0 where $n \neq 1$

$$\therefore b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x(\frac{1-\cos 2x}{2})dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx$$

$$\frac{1}{2\pi}\left[x(x - \frac{\sin 2x}{2}) - (1)(\frac{x^2}{2} - \frac{\cos 2x}{4})\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[2\pi^2 - \frac{1}{4} + \frac{1}{4}]$$

$$= \pi$$

Substitute in (1), we get

$$\textbf{f (x) =} -1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx+\pi\sin x+.......(2)+ \textbf{………………..(2)}$$

Deduction Part:

Put $x=\frac{\pi}{2}$[x is a point of continuity] [x= $\frac{\pi}{2}$ is a point of continuity]

$$f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2}=\frac{\pi}{2}=\frac{\pi}{2} \qquad (\because \sin\frac{\pi}{2} = 1)$$

$$(2)\Rightarrow (2)\Rightarrow \frac{\pi}{2}=-1-0+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos\frac{n\pi}{2}+\pi \;= \text{-1-0+}$$

$$\frac{\pi}{2}-\pi+1=2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2} \text{-}\pi+1 =$$

$$--\frac{\pi}{2}+1=2\sum_{n=2}^{\infty}\frac{1}{(n-1)(n+1)}\cos\frac{n\pi}{2}+1 =$$

$$-\frac{\pi}{4}+\frac{1}{2}=\sum_{n=2}^{\infty}\frac{1}{(n-1)(n+1)}\cos\frac{n\pi}{2}=$$

$$=\sum_{n=2}^{\infty}\frac{1}{(n-1)(n+1)}\cos\frac{n\pi}{2}=\frac{-\pi+2}{4}$$

$$\frac{1}{1.3}(-1)+\frac{1}{2.4}(0)+\frac{1}{3.5}(1)+........=\frac{-\pi+2}{4}$$

$$(-1)[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-........]=-[\frac{\pi-2}{4}]$$

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-........=[\frac{\pi-2}{4}]$$

Problem:4

Find the Fourier series for $f(x) = |\sin x|$ in $-\pi < x < \pi$

Solution:

Given $f(x) = |\sin x|$

This is an even function. $\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos} nx \text{..................................(1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\text{Sin}x| dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= -\frac{2}{\pi}(\cos\pi - \cos 0)$$

$$= -\frac{2}{\pi}[-1-1][-1-1]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\text{Sin}x| \cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]dx$$

$$= \frac{1}{\pi}[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}]_0^{\pi}$$

$$= \frac{1}{\pi}[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}]$$

$$= \frac{1}{\pi}[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{1+n} - \frac{1}{n-1}]$$

$$= \frac{1}{\pi}[\frac{(-1)^{n}}{n+1} - \frac{(-1)^{n}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}]$$

$$= \frac{1}{\pi}[\frac{(-1)^{n}+1}{n+1} - \frac{(-1)^{n}+1}{n-1}]$$

$$= \frac{(-1)^{n}+1}{\pi}[\frac{1}{n+1} - \frac{1}{n-1}]$$

$$= \frac{(-1)^{n}+1}{\pi}[\frac{n-1-n-1}{n^2-1}]$$

$$= \frac{(-1)^{n}+1}{\pi}[\frac{-2}{n^2-1}]$$

$$= \frac{-2[(-1)^{n}+1]}{(n^2-1)\pi} \text{ if } n \neq 1.$$

$a_1 \quad = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |\mathrm{Sin}x|\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\cos x\,dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{\pi}[-\frac{\cos 2x}{2}]_0^{\pi}$$

$$= \frac{1}{\pi}[-\frac{1}{2} + \frac{1}{2}] = 0$$

Substitute in equation (1), we get

$$f(x) = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty}\frac{-2[(-1)^{n}+1]}{(n^2-1)\pi}\cos nx$$

Hence f(x) = $\frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-2[(-1)^n + 1]}{(n^2 - 1)\pi} \cos nx$

Problem:5

Expand f(x) = $|\cos x|$ in a Fourier series in the interval (-π,π)

Solution:

Given f(x) = $|\cos x|$.

This is an even function, $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \text{Cos} nx \text{.....................................(1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| dx$$

$$= \frac{2}{\pi}[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -(\cos x) dx]$$

$$= \frac{2}{\pi}[\int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{\pi} \cos x dx]$$

$$= \frac{2}{\pi}[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}]$$

$$= \frac{2}{\pi}[(1-0)-(0-1)]$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi}[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^{\pi} -(\cos x)\cos nx dx]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx dx - \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos x \cos nx dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x]dx - \frac{1}{\pi}\int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x]dx \text{ -}$$

$$= \frac{1}{\pi}[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}]_0^{\pi/2} - \frac{1}{\pi}[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}] + \frac{1}{\pi}[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}]$$

$$= \frac{2}{\pi}[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}]$$

$$= \frac{2}{\pi}[\frac{\sin(\pi/2 + n\pi/2)}{n+1} - \frac{\sin(\pi/2 - \pi/2)}{n-1}]$$

$$= \frac{2}{\pi}[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}[\frac{1}{n+1} - \frac{1}{n-1}]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}[\frac{n-1-n-1}{n^2-1}]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}[\frac{-2}{n^2-1}]$$

$$= \frac{-4}{\pi}\cos\frac{n\pi}{2}[\frac{1}{n^2-1}]$$

$$= \frac{-4}{\pi}[\frac{1}{n^2-1}] \text{ if n is even}$$

$= 0$ if n is odd $[n \neq 1]$

Hence

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

Complex Form of a Fourier series:

Problem 6:

Find the complex form of the Fourier series of the function f(x) = e^x when $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$.

Solution:

We know that $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ ------------------------------(1)

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e(^{(1-in)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{1-in}\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{2\pi(1-in)}\left[e^{(1-in)x}\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{2\pi(1-in)}[e^{(1-in)\Pi} - e^{-(1-in)\Pi}]$$

$$= \frac{1}{2\pi(1-in)}[e^{\Pi}e^{-in\Pi} - e^{-\Pi}e^{in\Pi}]$$

$$e^{in\Pi} = \cos n\Pi + i\sin n\Pi = (-1)^n + i0 = (-1)^n$$
$$e^{in\Pi} = \cos n\Pi - i\sin n\Pi = (-1)^n - i0 = (-1)^n$$

$$= \frac{1}{2\pi(1-in)}[e^{\Pi}(-1)^n - e^{-\Pi}(-1)^n]$$

$$= \frac{(-1)^n}{2\pi(1-in)}[\frac{e^{\Pi} - e^{-\Pi}}{2}]$$

$$C_n = \frac{(-1)^n (1+in)}{\pi(1^2+n^2)} \sinh \prod$$

$$(1) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1+in)}{\pi(1^2+n^2)} \sinh \prod \ e^{inx}$$

i.e., $$e^x = \frac{\sinh \prod}{\prod} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{1^2+n^2} e^{inx}$$

Problem7:
Find the complex form of the Fourier series of f(x) = Cosax in (-π, π) where 'a' is neither zero nor an integer.

Solution:

Here 2c = 2π or c = π.

We know that $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$ -----------------------------(1)

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Cos}ax e^{-inx} dx$$

$$= \frac{1}{2\pi} [\frac{e^{-inx}}{a^2-n^2} (-in \cos ax + a \sin ax)]_{-\prod}^{\prod}$$

$$= \frac{1}{2\pi(a^2-n^2)} [e^{-in\prod} (-in \cos a\prod + a \sin a\prod) - e^{in\prod} (-in \cos a\prod - a \sin a\prod)$$

$$= \frac{1}{2\pi(a^2-n^2)} [in \cos a\prod (e^{in\prod} - e^{-in\prod}) + a \sin a\prod (e^{in\prod} - e^{-in\prod})]$$

$$= \frac{1}{2\pi(a^2-n^2)} [in \cos a\prod (2i \sin n\prod) + a \sin a\prod (2 \cos n\prod)]$$

$$C_n = \frac{1}{2\pi(a^2-n^2)} (-1)^n 2a \sin a\prod$$

Hence (1) becomes

$$\text{Cosax} = \frac{a\sin a\prod}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(a^2-n^2)}e^{inx}$$

Problem8:

Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$

Solution:

We know that $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\prod x}$ -----------------------------(1)

$$C_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\prod x}dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\prod x}dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\prod)x}dx$$

$$= \frac{1}{2}[\frac{e^{-(1+in\prod)x}}{-(1+in\prod)}]_{-\prod}^{\prod}$$

$$= \frac{e^{1+in\prod} - e^{-(1+in\prod)}}{2(1+in\prod)}$$

$$= \frac{e(\cos n\prod + i\sin n\prod) - e^{-1}(\cos n\prod - i\sin n\prod)}{2(1+in\prod)}$$

$$= \frac{e(-1)^n - e^{-1}(-1)^n}{2(1+in\prod)} =$$

$$= \frac{(e-e^{-1})(-1)^n}{2} \quad \frac{(1-in\prod)}{1+n^2\prod^2}$$

$$= \frac{(-1)^n(1-in\prod)}{1+n^2\prod^2}\sinh 1$$

Hence (1) becomes e^{-x} $= \sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-in\prod)}{1+n^2\prod^2}\sinh 1 e^{in\prod x}$

UNIT- 2

FOURIER TRANSFORMS

1. Find the Fourier transform of $e^{-a|x|}$, if $a > 0$. Deduce that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$

Sol:

The Fourier transform of the function $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$.

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} e^{-a|x|} \cos sxdx + i \int_{-\infty}^{\infty} e^{-a|x|} \sin sxdx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^{\infty} e^{-a|x|} \cos sxdx + 0 \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\int_0^{\infty} e^{-ax} \cos sxdx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{(s^2+a^2)}$$

By Parseval's identity $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\therefore \quad \int_{-\infty}^{\infty} \left| \sqrt{\frac{2}{\pi}} \frac{a}{(s^2+a^2)} \right|^2 ds = \int_{-\infty}^{\infty} \left| e^{-a|x|} \right|^2 dx$$

$$\frac{2a^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(s^2+a^2)^2} ds = 2\int_0^{\infty} e^{-2ax} dx = 2\left(\frac{e^{-2ax}}{-2a} \right)_0^{\infty}$$

$$= \frac{1}{-a}(0-1)$$

$$\therefore \quad 2\int_0^{\infty} \frac{1}{(s^2+a^2)^2} ds = \frac{\pi}{2a^3}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

2.Show that the Fourier transform of $f(x)=\begin{cases} a^2-x^2;\ |x|<a \\ 0;\ |x|>a>0 \end{cases}$ **is** $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]$**. Hence deduce that** $\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt = \frac{\pi}{4}$**.**

Sol:

The Fourier transform of the function $f(x)$ is $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

Given $f(x) = a^2 - x^2$ in $-a < x < a$ and 0 otherwise.

$$\therefore F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-a}^{a}(a^2-x^2)\cos sxdx + i\int_{-a}^{a}(a^2-x^2)\sin sxdx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2\int_{0}^{a}(a^2-x^2)\cos sxdx + 0\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{(a^2-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right\}_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[0 - \frac{2a\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right] - [0-0+0]\right\}$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}e^{-isx}ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}(\cos sx - i\sin sx)ds$$

$$= \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\cos sxds - i\int_{-\infty}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\sin sxds\right\}$$

$$f(x) = \frac{2}{\pi}\left\{2\int_{0}^{\infty}\left\{\frac{\sin sa - sa\cos sa}{s^3}\right\}\cos sxds - 0\right\}$$

Put $x = 0$ & $a = 1$, we get

$$1 = \frac{4}{\pi}\int_{0}^{\infty}\left\{\frac{\sin s - s\cos s}{s^3}\right\}ds$$

i.e $$\int_{0}^{\infty}\left\{\frac{\sin s - s\cos s}{s^3}\right\}ds = \frac{\pi}{4}$$

Replace s by t, we get $\int_{0}^{\infty}\left\{\frac{\sin t - t\cos t}{t^3}\right\}dt = \frac{\pi}{4}$

3.Find the Fourier transform of $f(x) = \begin{cases} 1-|x| \; ; \; if \; |x| < 1 \\ 0 \; ; \; if \; |x| \geq 1 \end{cases}$. ***Hence deduce that*** $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$ ***and*** $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$.

<u>Sol:</u>

The Fourier transform of the function $f(x)$ is $F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

Given $f(x) = 1-|x|$ in $-1 < x < 1$ and 0 otherwise.

$$\therefore F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1}(1-|x|)\cos sxdx + i\int_{-1}^{1}(1-|x|)\sin sxdx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2\int_{0}^{1}(1-x)\cos sxdx + 0\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{(1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right\}_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[0-\frac{\cos s}{s^2}\right]-\left[0-\frac{1}{s^2}\right]\right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left\{\frac{1-\cos s}{s^2}\right\}e^{-isx}ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}(\cos sx - i\sin sx)ds$$

$$= \frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\cos sxds - i\int_{-\infty}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\sin sxds\right\}$$

$$f(x) = \frac{1}{\pi}\left\{2\int_{0}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}\cos sxds - 0\right\}$$

Put $x = 0$, we get

$$1 = \frac{2}{\pi}\int_{0}^{\infty}\left\{\frac{1-\cos s}{s^2}\right\}ds$$

$$i.e \quad \int_{0}^{\infty}\left\{\frac{2\sin^2\left(s/2\right)}{s^2}\right\}ds = \frac{\pi}{2}.1$$

Let $\frac{s}{2} = t$, then $ds = 2dt$ $\quad s = 0 \Rightarrow t = 0$

$s = \infty \Rightarrow t = \infty$

$$\int_0^\infty \left\{\frac{2\sin^2 t}{(2t)^2}\right\} 2dt = \frac{\pi}{2}$$

$$\int_0^\infty \left\{\frac{\sin^2 t}{t^2}\right\} dt = \frac{\pi}{2}$$

By Parseval's identity $\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$

$$\therefore \quad \int_{-\infty}^{\infty} \left| \sqrt{\frac{2}{\pi}} \left\{\frac{1-\cos s}{s^2}\right\} \right|^2 ds = \int_{-1}^{1} |(1-|x|)|^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left| \left\{\frac{1-\cos s}{s^2}\right\} \right|^2 ds = 2\int_0^1 (1-|x|)^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = 2\int_0^1 (1-x)^2 dx$$

$$\frac{2}{\pi} 2\int_0^{\infty} \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = 2\int_0^1 (1+x^2-2x) dx$$

$$\frac{2}{\pi}\int_0^{\infty} \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = \left\{x+\frac{x^3}{3}-\frac{2x^2}{2}\right\}_0^1$$

$$\frac{2}{\pi}\int_0^{\infty} \left\{\frac{1-\cos s}{s^2}\right\}^2 ds = \left\{\left[1+\frac{1}{3}-1\right]-[0+0-0]\right\}$$

$$2\int_0^{\infty} \left\{\frac{2\sin^2 (s/2)}{s^2}\right\}^2 ds = \frac{\pi}{3}$$

Let $\frac{s}{2} = t$, then $ds = 2dt$ $\quad s = 0 \Rightarrow t = 0$, $\quad s = \infty \Rightarrow t = \infty$

$$2\int_0^{\infty} \left\{\frac{2\sin^2 t}{(2t)^2}\right\}^2 2dt = \frac{\pi}{3}$$

$$\int_0^{\infty} \left\{\frac{\sin^2 t}{t^2}\right\}^2 dt = \frac{\pi}{3}$$

$$\int_0^{\infty} \left\{\frac{\sin t}{t}\right\}^4 dt = \frac{\pi}{3}$$

4.Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{-\frac{x^2}{2}}$ is self reciprocal.

<u>Sol:</u>

The Fourier transform of the function $f(x)$ is $F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$.

$$\therefore F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2+\frac{s^2}{4a^2}\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

Put $ax-\frac{is}{2a}=y$, then $dx=\frac{dy}{a}$

$x=-\infty \Rightarrow y=-\infty$

$x=\infty \Rightarrow y=\infty$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{a}$$

$$= \frac{1}{a\sqrt{2\pi}}e^{-\frac{s^2}{4a^2}}\sqrt{\pi}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}}e^{-\frac{s^2}{4a^2}}$$

Put $a=\frac{1}{\sqrt{2}}$, then

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2x^2}\right] = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2}}e^{-\frac{s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$$

$\therefore\ e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier transform.

5. ***Find Fourier sine and cosine transform of*** x^{n-1} **and hence prove that** $F_C\left[\frac{1}{\sqrt{x}}\right]=\frac{1}{\sqrt{s}}$

Sol:

We know that $\int_0^\infty e^{-ax}x^{n-1}dx=\frac{\Gamma_{(n)}}{a^n}$

Put $a=is$, $\therefore \int_0^\infty e^{-isx}x^{n-1}dx \quad =\frac{\Gamma_{(n)}}{(is)^n}$

$$\int_0^\infty(\cos sx-i\sin sx)x^{n-1}dx \quad =\frac{\Gamma_{(n)}}{i^n s^n}$$

$$\int_0^\infty x^{n-1}\cos sxdx-\int_0^\infty x^{n-1}i\sin sxdx \ =\frac{\Gamma_{(n)}}{i^n s^n}$$

$$\int_0^\infty x^{n-1}\cos sxdx-i\int_0^\infty x^{n-1}\sin sxdx \ =\frac{\Gamma_{(n)}}{\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)^n s^n}$$

$$\int_0^\infty x^{n-1}\cos sxdx-i\int_0^\infty x^{n-1}\sin sxdx \ =\frac{\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)^{-n}\Gamma_{(n)}}{s^n}$$

$$\int_0^\infty x^{n-1}\cos sxdx-i\int_0^\infty x^{n-1}\sin sxdx \ =\frac{\left(\cos\left(\frac{n\pi}{2}\right)-i\sin\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$\int_0^\infty x^{n-1}\cos sxdx-i\int_0^\infty x^{n-1}\sin sxdx \ =\frac{\left(\cos\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}-\frac{i\left(\sin\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

Equating Real and imaginary parts, we get

$$\int_0^\infty x^{n-1}\cos sxdx \quad =\frac{\left(\cos\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$\int_0^\infty x^{n-1}\sin sxdx \ =\frac{\left(\sin\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx dx \quad = \sqrt{\frac{2}{\pi}}\frac{\left(\cos\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx dx \quad = \sqrt{\frac{2}{\pi}}\frac{\left(\sin\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$\Rightarrow \qquad F_C\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\left(\cos\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

$$F_S\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\left(\sin\left(\frac{n\pi}{2}\right)\right)\Gamma_{(n)}}{s^n}$$

Put $n = \frac{1}{2}$ in the above results, we get

$$F_C\left[x^{\frac{1}{2}-1}\right] = \sqrt{\frac{2}{\pi}}\frac{\left(\cos\left(\frac{\frac{1}{2}\pi}{2}\right)\right)\Gamma_{\left(\frac{1}{2}\right)}}{s^{\frac{1}{2}}}$$

$$F_C\left[x^{-\frac{1}{2}}\right] = \sqrt{\frac{2}{\pi}}\frac{\left(\cos\left(\frac{\pi}{4}\right)\right)\sqrt{\pi}}{\sqrt{s}}$$

$$F_C\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\frac{\frac{1}{\sqrt{2}}\sqrt{\pi}}{\sqrt{s}}$$

$$F_C\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

Therefore $\frac{1}{\sqrt{x}}$ is self reciprocal with respect to Fourier cosine transform.

5. Find Fourier sine transform and Fourier cosine transform of $e^{-ax}, a > 0$***. Hence evaluate*** $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx$ ***and*** $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$**.**

<u>Sol:</u>

The Fourier sine transform of the function $f(x)$ is $F_S\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$.

$$F_S\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

The Fourier cosine transform of the function $f(x)$ is $F_C\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx dx$.

$$\therefore F_C\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

By parseval's identity $\int_0^{\infty}\left|F_S\left[f(x)\right]\right|^2 ds = \int_0^{\infty}\left|f(x)\right|^2 dx$

$$\int_0^{\infty}\left|\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right|^2 ds = \int_0^{\infty}\left|e^{-ax}\right|^2 dx$$

$$\frac{2}{\pi}\int_0^{\infty}\left|\frac{s}{s^2+a^2}\right|^2 ds = \int_0^{\infty}\left|e^{-2ax}\right| dx$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{s^2}{\left(s^2+a^2\right)^2} ds = \int_0^{\infty} e^{-2ax} dx$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{s^2}{\left(s^2+a^2\right)^2} ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{s^2}{\left(s^2+a^2\right)^2} ds = \left[\frac{e^{-2a\infty}-e^{-2a0}}{-2a}\right]$$

$$\frac{2}{\pi}\int_0^{\infty}\frac{s^2}{\left(s^2+a^2\right)^2} ds = \left[\frac{-1}{-2a}\right]$$

$$\int_0^{\infty}\frac{s^2}{\left(s^2+a^2\right)^2} ds = \frac{\pi}{4a}$$

Replace s by x, we get

$$\int_0^{\infty}\frac{x^2}{\left(x^2+a^2\right)^2} dx = \frac{\pi}{4a}$$

Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

By parseval's identity $\int_0^{\infty} F_S\left[f(x)\right]F_C\left[g(x)\right] ds = \int_0^{\infty} f(x)g(x) dx$

$$\therefore \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{b}{s^2+a^2} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$i.e. \frac{2}{\pi} \int_0^{\infty} \frac{ab}{(s^2+a^2)(s^2+a^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$$

$$\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+a^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+a^2)} = \left[\frac{e^{-(a+b)\infty} - e^{-(a+b)0}}{-(a+b)} \right]$$

$$\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+a^2)} = \left[\frac{-1}{-(a+b)} \right]$$

$$\int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+a^2)} = \frac{\pi}{2ab(a+b)}$$

Replace s by x, we get

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+a^2)} = \frac{\pi}{2ab(a+b)}$$

6. Verify Parseval's theorem of Fourier transform for the function $f(x) = \begin{cases} 0 \ ; \ x < 0 \\ e^{-x} \ ; \ x > 0 \end{cases}$.

<u>Sol:</u>

By Parseval's theorem $\int_{-\infty}^{\infty} |f(x)|^2 ds = \int_{-\infty}^{\infty} |F(s)|^2 ds$ …………(1)

Given $f(x) = \begin{cases} 0 \ ; \ x < 0 \\ e^{-x} \ ; \ x > 0 \end{cases}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(1-is)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-(1-is)x}}{-(1-is)} \right)_0^{\infty}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-(1-is)\infty}}{-(1-is)} - \frac{e^{-(1-is)0}}{-(1-is)}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{0}{-(1-is)} - \frac{1}{-(1-is)}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{(1-is)}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1+is}{(1-is)(1+is)}\right)$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left(\frac{1+is}{s^2+1}\right)$$

L. H. S. of (1)

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{0}^{\infty} |e^{-x}|^2\, dx$$

$$= \int_{0}^{\infty} e^{-2x} dx$$

$$= \left(\frac{e^{-2x}}{-2}\right)_0^{\infty}$$

$$= \left(\frac{e^{-2\infty} - e^{-0}}{-2}\right)$$

$$= \frac{1}{2}$$

R. H. S. of (1)

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} \left|\frac{1}{\sqrt{2\pi}}\left(\frac{1+is}{s^2+1}\right)\right|^2 ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1+is}{s^2+1}\right|^2 ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|1+is|^2}{(s^2+1)^2} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1+s^2)}{(s^2+1)^2} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{ds}{s^2+1}$$

$$= \frac{1}{2\pi}\left[\tan^{-1} s\right]_{-\infty}^{\infty}$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \frac{1}{2\pi}\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{2\pi}\left[\tan^{-1}\infty + \tan^{-1}\infty\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{1}{2\pi}[\pi] = \frac{1}{2}$$

L. H. S of (1) = R. H. S of (1)

Hence Parseval's theorem verified.

7. Solve for $f(x)$ from the integral equation $\int_0^{\infty} f(x)\cos\alpha x dx = e^{-\alpha}$.

Sol:

$$\int_0^{\infty} f(x)\cos\alpha x dx = e^{-\alpha}$$

$$\therefore \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-\alpha}\cos\alpha x\ dx = \sqrt{\frac{2}{\pi}}e^{-\alpha}$$

Inverse Fourier cosine transform is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}e^{-\alpha}\cos\alpha x\ d\alpha$$

$$= \frac{2}{\pi}\left[\frac{e^{-\alpha}}{1+x^2}(1.\cos\alpha x + x\sin\alpha x)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{1+x^2}(-1)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+x^2}\right]$$

$$= \frac{2}{\pi(1+x^2)}$$

UNIT-3

PARTIAL DIFFERENTIAL EQUATIONS

Problems on Lagrange's equations

1. Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Solution:

$$p\sqrt{x} + q\sqrt{y} = \sqrt{z} \text{ ------ } (1)$$

This is of the form Pp + Qq = R

Here $P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$

Subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Grouping the first two members

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

Integrating

$$\int \frac{dx}{x^{\frac{1}{2}}} = \int \frac{dx}{y^{\frac{1}{2}}}$$

$$\int x^{-\frac{1}{2}} dx = \int y^{-\frac{1}{2}} dy$$

$$2\sqrt{x} = 2\sqrt{y} + a$$

$$2(\sqrt{x} - \sqrt{y}) = a$$

$$\sqrt{x} - \sqrt{y} = \frac{a}{2}$$

$$\sqrt{x} - \sqrt{y} = c_1 \left(c_1 = \frac{a}{2} \right)$$

$$\therefore u = \sqrt{x} - \sqrt{y}$$

|||ly Grouping the another two members

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating

$$2\sqrt{y} = 2\sqrt{z} + b$$

$$2(\sqrt{y} - \sqrt{z}) = b$$

$$\sqrt{y} - \sqrt{z} = \frac{b}{2}$$

$$\sqrt{y} - \sqrt{z} = c_2 \left(c_2 = \frac{b}{2}\right)$$

$$\therefore v = \sqrt{y} - \sqrt{z}$$

The general solution of (1) is ϕ(u,v) = 0

$$\phi\left(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}\right) = 0$$

2. Obtain the general solution of pzx + qzy = xy.

Solution**:**

Given pzx + qzy = xy

This is of the form Pp + Qq = R

Here P = zx, Q= zy, R = xy

Subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{zx} = \frac{dy}{zy} = \frac{dz}{xy}$$

Grouping the first two members

$$\frac{dx}{zx} = \frac{dy}{zy}$$

Integrating

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$/og\, x = /og\, y + /og\, a$$

$$/og \frac{x}{y} = /og\ a$$

$$\frac{x}{y} = a ----(1)$$

$$\therefore u = \frac{x}{y}$$

Grouping the another two members

$$\frac{dy}{zy} = \frac{dz}{xy}$$

$$\frac{dy}{z} = \frac{dz}{x}$$

$$\frac{dy}{z} = \frac{dz}{ay} \quad (\ from\ (1)\ x = ay)$$

$$ay\, dy = z\, dz$$

Integrating

$$\frac{ay^2}{2} = \frac{z^2}{2} + c$$

$$ay^2 - z^2 = 2c$$

$$ay^2 - z^2 = b \quad (\ b = 2c\ )$$

$$\frac{x}{y}.y^2 - z^2 = b$$

$$xy - z^2 = b$$

$$\therefore v = xy - z^2$$

The general solution is ϕ (u,v) = 0

$$\phi\left(\frac{x}{y}\ ,\ xy - z^2\right) = 0$$

3. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$

Solution:

Given $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$

This is of the form Pp + Qq = R

Here $P = x(y^2 - z^2)$, $Q = y(z^2 - x^2)$, $R = z(x^2 - y^2)$

Subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \quad ------(1)$$

Choose the set of multipliers x,y,z

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \; x\,dx + y\,dy + z\,dz = 0$$

Integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = a \quad (let\ 2c = a)$$

$$\therefore u = x^2 + y^2 + z^2$$

|||ly consider another set of multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

Each member of (1)

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

Integrating we get

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$xyz = b$$

$$v = xyz$$

The general solution is $\phi(u,v) = 0$

$$\phi\left(x^2+y^2+z^2, xyz\right) = 0$$

4. Solve $(x^2-yz)\, p + (y^2-zx)\, q = z^2 - xy$

Solution:

Given $(x^2-yz)\, p + (y^2-zx)\, q = z^2 - xy$

This is of the form Pp + Qq = R

Here $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

Subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx)} = \frac{dz}{z^2-xy)} \quad -----(1)$$

Each of (1)

$$= \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)} = \frac{dx-dz}{x^2-z^2+y(x-z)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} = \frac{d(x-z)}{(x-z)(x+y+z)}$$

i.e., $$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z} = \frac{d(x-z)}{x-z}$$

Grouping the members

$$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

Integrating we get

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log\frac{x-y}{y-z} = \log c_1$$

$$\frac{x-y}{y-z} = c_1$$

$$\therefore\ u = \frac{x-y}{y-z}$$

Choose multipliers x , y , z

Each of $(1) = \dfrac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz}$ $\quad ------\rightarrow (2)$

Choose another set of multipliers 1,1,1

Each of $(1) = \dfrac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2-yz-zx-xy}$ $\quad ------\rightarrow (3)$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

i.e., $$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)} = \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)} = dx + dy + dz$$

$$(x+y+z)\,d(x+y+z) = x\,dx + y\,dy + z\,dz$$

Integrating we get

$$\frac{(x+y+z)^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + b$$

$$\frac{(x+y+z)^2 - (x^2 + y^2 + z^2)}{2} = b$$

$$xy + yz + zx = 2b$$

$$xy + yz + zx = c_2$$

$$v = xy + yz + zx$$

The general solution is $\phi\left(\frac{x-y}{y-z}, xy + yz + zx\right) = 0.$

5. Solve (mz – ny) p + (nx – *l*z) q = *l*y – mx

Solution:

Given (mz – ny) p + (nx – *l*z) q = *l*y – mx

This is of the form Pp + Qq = R

Here P = mz – ny, Q = nx – lz, R = ly – mx

The Subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad -----(1)$$

Choose a set of multipliers x, y ,z

Each of (1) $= \dfrac{x\,dx + y\,dy + z\,dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$

$= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$x\,dx + y\,dy + z\,dz = 0$

Integrating we get

$$x^2 + y^2 + z^2 = a$$

$$v = x^2 + y^2 + z^2$$

Choose a set of multipliers l, m ,n

$$\text{Each of } (1) = \frac{l\,dx + m\,dy + n\,dz}{l\,(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$l\,dx + m\,dy + n\,dz = 0$

Integrating we get

$l\,x + m\,y + n\,z = b$

$v = l\,x + m\,y + n\,z$

The general solution is $\phi(u,v) = 0$

$$\phi\left(x^2 + y^2 + z^2, lx + my + nz \right) = 0$$

Equations reducible to standard types:

Type V:

Equations of the form

$$f(x^m p, y^n q) = 0 \text{ or } f(x^m p, y^n q, z) = 0 \text{ ------------(1)}$$

where m and n are constants

This type of equations can be reduced to Type I or Type III by the following transformations.

Case (i):

If $m \neq 1$, $n \neq 1$

Put $x^{1-m} = X$, $y^{1-n} = Y$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = (1-m)\,x^{-m}\,P, \text{ where } P = \frac{\partial z}{\partial X}$$

i.e., $x^m p = (1 - m)P$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = (1-n)\,y^{-n}\,Q, \text{ where } Q = \frac{\partial z}{\partial Y}$$

i.e., $y^n q = (1 - n)Q$

The equation $f\left(x^m p,\ y^n q\right) = 0$ reduces to $f\left((1-m)P, (1-n)Q\right) = 0$ which is a Type I eqn.

The equation $f\left(x^m p,\ y^n q, z\right) = 0$ reduces to $f\left((1-m)P, (1-n)Q, z\right) = 0$ which is aType III eqn.

Case (ii):

If m = 1, n = 1 then (1)$\Rightarrow$ $f(xp, yq) = 0$ or $f(xp, yq, z) = 0$

Put log x = X, log y = Y.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x} = \frac{1}{x} P, \text{ where } P = \frac{\partial z}{\partial X}$$

i.e., $x\,p = P$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{1}{y} = \frac{1}{y} Q, \text{ where } Q = \frac{\partial z}{\partial Y}$$

i.e., $y\ q = Q$

The equation $f(px, qy) = 0$ reduces to $f(P, Q) = 0$ which Type I eqn.

The equation $f(px, qy, z) = 0$ reduces to $f(P, Q, z) = 0$ which Type III eqn.

Type VI :

Equations of the form $f(z^k p, z^n q) = 0$ or $f(z^k p, y^n q, x, y) = 0$ ------------(1)

Where k is constant,

This can be reduced to Type I or Type IV equations by the following substitutions.

Case (i):

If $k \neq -1$

Put $Z = z^{k+1}$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = (k+1) z^k p$$

$$\therefore z^k p = \frac{P}{k+1}$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = (k+1) z^k q$$

$$\therefore z^k q = \frac{Q}{k+1}$$

The equation $f\left(z^k p,\ z^k q\right) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}\right) = 0$ which is a Type I eqn.

The equation $f\left(z^k p,\ z^k q, x, y\right) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}, x, y\right) = 0$ which is a Type IV eqn..

Case (ii):

If $k = -1$

$$f\left(\frac{p}{z}, \frac{q}{z}\right) = 0 \text{ or } f\left(\frac{p}{z}, \frac{q}{z}, x, y\right) = 0$$

put $z = \log z$

$$P = \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{1}{z} p.$$

$$\therefore \frac{p}{z} = P$$

(1) $\Rightarrow$ $$Q = \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{1}{z} q.$$

$$\therefore \frac{q}{z} = Q$$

The eqn $f\left(\frac{p}{z}, \frac{q}{z}\right) = 0$ reduces to Type I eqn.

The eqn $f\left(\frac{p}{z}, \frac{q}{z}, x, y\right) = 0$ reduces to Type IV eqn.

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = (k+1) z^k p$$

$$\therefore z^k p = \frac{P}{k+1}$$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = (k+1) z^k q$$

$$\therefore z^k q = \frac{Q}{k+1}$$

The equation $f\left(z^k p, z^k q\right) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}\right) = 0$ which is a Type I eqn.

The equation $f\left(z^k p, z^k q, x, y\right) = 0$ reduces to $f\left(\frac{P}{k+1}, \frac{Q}{k+1}, x, y\right) = 0$ which is a Type IV eqn..

Example :

1. Solve $p^2 + x^2y^2q^2 = x^2z^2$

Solution :

Given $p^2 + x^2y^2q^2 = x^2z^2$

$$x^{-2}p^2 + y^2q^2 = z^2$$

$$x^{-1}p)^2 + (yq)^2 = z^2 \quad \text{---------------} (1)$$

This is of the form $f(x^m p, y^n q, z) = 0$ **(Type V)**

Here $m = -1$, $n = 1$.

Put $X = x^{1-m}$

$X = x^2$ $\qquad\qquad Y = \log y$

$\frac{\partial X}{\partial x} = 2x$ $\qquad\qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$

$p = \frac{\partial z}{\partial x}$ $\qquad\qquad q = \frac{\partial z}{\partial y}$

$= \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$ $\qquad\qquad = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$

$p = P \cdot 2x$ $\qquad\qquad q = Q \cdot \frac{1}{y}$

$x^{-1}p = 2P$ $\qquad\qquad yq = Q$

(1) Reduces to $(2P)^2 + Q^2 = z^2$ ------------(2)

This is of the form $f(p, q, z) = 0$ which is TypeIII eqn.

Let $z = f(X + aY)$ be a trial Solution.

$z = f(u)$

$u = X + aY$

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = a$$

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial X} \qquad Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial Y}$$

$$P = \frac{\partial z}{\partial u} \qquad Q = a\frac{\partial z}{\partial u}$$

(2) reduces to

$$\left(2\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$(4 + a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{(4 + a^2)}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{4 + a^2}}.$$

$$\frac{dz}{z} = \frac{du}{\sqrt{4 + a^2}}$$

Integrating

$$\int \frac{dz}{z} = \int \frac{du}{\sqrt{4+a^2}}$$

$$\log z = \frac{1}{\sqrt{4+a^2}} u + c$$

$$\log z = \frac{X + aY}{\sqrt{4+a^2}} + c$$

$$\log z = \frac{x^2 + a \log y}{\sqrt{4+a^2}} + c \qquad (\because X = x^2,\ Y = \log y)$$

which is the complete integral

2 . Solve $z^2 \left(p^2 + q^2 \right) = x^2 + y^2$

Solution :

Given $z^2 \left(p^2 + q^2 \right) = x^2 + y^2$

$$(zp)^2 + (zq)^2 = x^2 + y^2 \text{ -------------- } (1)$$

This is of the form $f\left(z^k p,\ z^k q,\ x,\ y \right) = 0 \quad (\text{Type VI})$

Here $k = 1$

Put $Z = z^{1+1}$

$Z = z^2$

$$P = \frac{\partial Z}{\partial x} \qquad\qquad Q = \frac{\partial z}{\partial y}$$

$$= \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \qquad\qquad = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$P = 2z \cdot p \qquad\qquad Q = 2z \cdot q$$

$$zp = \frac{P}{2} \qquad\qquad zq = \frac{Q}{2}$$

Eqn (1) reduces to $P^2 + Q^2 = 4\left(x^2 + y^2\right)$

$$\text{i.e.,} P^2 - 4x^2 = 4y^2 - Q^2$$

This is of Standard Type IV $f_1(x, p) = f_2(y, q)$

Let $P^2 - 4x^2 = 4y^2 - Q^2 = 4c$ (say)

$$P^2 - 4x^2 = 4c \qquad\qquad 4y^2 - Q^2 = 4c$$

$$P^2 = 4c + 4x^2 \qquad\qquad Q^2 = 4y^2 - 4c$$

$$P = \sqrt{4(c + x^2)} \qquad\qquad Q = \sqrt{4(y^2 - c)}$$

$$P = 2\sqrt{(c + x^2)} \qquad\qquad Q = 2\sqrt{(y^2 - c)}$$

We know

$$dZ = \frac{\partial Z}{\partial x} \cdot dx + \frac{\partial Z}{\partial y} \cdot dy$$

$$dZ = P\,dx + Q\,dy \qquad\qquad \left(\because P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial x}\right)$$

$$dZ = 2\sqrt{(c + x^2)}\,dx + 2\sqrt{(y^2 - c)}\,dy$$

$$\int dZ = 2\int \sqrt{(c + x^2)}\, dx + 2\int \sqrt{(y^2 - c)}\, dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{(c + x^2)} + \frac{c}{2}\sinh^{-1}\frac{x}{\sqrt{C}} + \frac{y}{2}\sqrt{(y^2 - c)} - \frac{c}{2}\cosh^{-1}\frac{x}{\sqrt{C}}\right] + a$$

$$\therefore z^2 = x\sqrt{(c + x^2)} + c\sinh^{-1}\frac{x}{\sqrt{C}} + y\sqrt{(y^2 - c)} - c\cosh^{-1}\frac{x}{\sqrt{C}} + a \quad (\because Z = z^2)$$

Examples:

*1.Form the **p d e** by eliminating the arbitrary*

function f

from the relation $z = f(x^2 + y^2)$

Solution:

Given $z = f(x^2 + y^2)$*--------------(1)*

Differentiating (1) partially w. r.t. x and y,

$$p = \frac{\partial z}{\partial x} = f'\ (x^2 + y^2).2x, \quad -----(2)$$

$$q = \frac{\partial z}{\partial y} = f'\ (x^2 + y^2).2y \quad ----(3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{x}{y}$$

$$yp - xq = 0$$

$\therefore\ yp - xq = 0$ is the required partial differential equation.

***2**.Form the **p d e** by eliminating the arbitrary functions f and g from the relation z = f (x +ct) + g (x -ct) .*

Solution :

Given z = f (x +ct) + g (x -ct) -------- (1)

Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = f'(x + ct) + g'(x - ct),$$

Again differentiating partially with respect to x

$$\frac{\partial^2 z}{\partial x^2} = f''(x + ct) + g''(x - ct) \quad ----------(2)$$

Differentiating (1) partially with respect to t

$$\frac{\partial z}{\partial t} = cf'(x+it) - cg'(x-it),$$

Again differentiating partially with respect to t

$$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+it) + c^2 g''(x-it)$$

$$= c^2 \frac{\partial^2 z}{\partial x^2} \text{ ----------- } (3)$$

$$(2)+(3) \Rightarrow \frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} = 0$$

UNIT-IV

Applications of Partial Differential Equations

1. ***A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position. Find the displacement at any time $'t'$.***

Sol:

One dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

The conditions are

(i) $y(0,t)=0$ (ii) $y(l,t)=0$ (iii) $\frac{\partial y}{\partial t}(x,0)=0$

(iv) $y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right)$

The correct solution which satisfying the given boundary conditions is

$y(x,t)=(c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat)$ -------------(1)

Apply the boundary condition (i) in (1)

$$y(0,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$

weget $c_1 = 0$, then equation (1) becomes

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat) \text{ -------------(2)}$$

Apply the boundary condition (ii) in (2)

$$y(l,t) = c_2 \sin pl(c_3 \cos pat + c_4 \sin pat) = 0$$

weget $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$

Then equation (2) becomes

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right) \text{-------------- (3)}$$

Differentiate Partially (3) w.r.to t

$$\frac{\partial y}{\partial t}(x,t) = c_2 \sin \frac{n\pi x}{l}\left(-c_3 \frac{n\pi a}{l} \sin \frac{n\pi at}{l} + c_4 \frac{n\pi a}{l} \cos \frac{n\pi at}{l} \right) \text{-------- (4)}$$

Apply the boundary condition (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = c_2 \sin \frac{n\pi x}{l} c_4 \frac{n\pi a}{l} = 0$$

weget $c_4 = 0$, then equation (3) becomes

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} c_3 \cos \frac{n\pi at}{l}$$

$$= c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \qquad \text{where } c_n = c_2 c_3$$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \text{ ----------------(5)}$$

Apply the boundary condition (iv) in (5)

$$y(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.1 = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$
$$=\frac{y_0}{4}\left(3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right)$$

$$\Rightarrow c_1\sin\frac{\pi x}{l}+c_2\sin\frac{2\pi x}{l}+c_3\sin\frac{3\pi x}{l}+c_4\sin\frac{4\pi x}{l}+..........$$
$$=\frac{y_0}{4}\left(3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right)$$

$$c_1=\frac{3y_0}{4}, c_2=0, c_3=-\frac{y_0}{4}, c_4=c_5=.........=0$$

$\therefore$ The equation (5) becomes

$$y(x,t)=\frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l}-\frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

2. ***A tightly stretched string of length l has its ends fastened at $x=0$ and $x=l$. The midpoint of the string is then taken to a height h and then released from rest in that position. Obtain an expression for the displacement of the string at any subsequent time.***

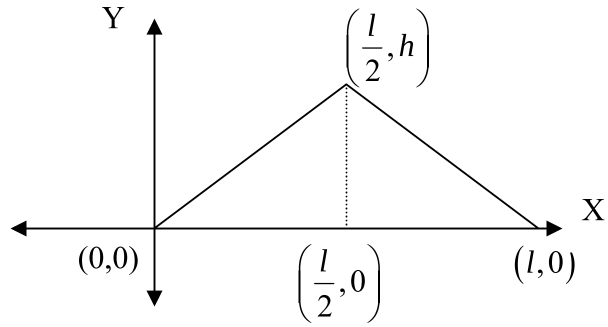
<u>Sol:</u>

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

The conditions are

(i) $y(0,t)=0$ (ii) $y(l,t)=0$ (iii) $\frac{\partial y}{\partial t}(x,0)=0$

(iv) $y(x,0)=f(x)$



Consider the interval $(0, \frac{l}{2})$, the end points are $(0,0)$, $(\frac{l}{2}, h)$

Using two point formula for the straight line

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-0}{0-h}=\frac{x-0}{0-\frac{l}{2}} \Rightarrow y=\frac{2hx}{l}$$

Consider the interval $(\frac{l}{2}, l)$, the end points are $(\frac{l}{2}, h)$, $(l,0)$

Again by two point formula for the straight line

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-h}{h-0}=\frac{x-\frac{l}{2}}{\frac{l}{2}-l} \Rightarrow y=\frac{2h}{l}(l-x)$$

$$\therefore\ f(x)=\begin{cases}\frac{2hx}{l}, & 0\le x\le \frac{l}{2}\\ \frac{2h}{l}(l-x), & \frac{l}{2}\le x\le l\end{cases}$$

The correct solution which satisfying the given boundary conditions is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat) \text{ -------- } (1)$$

Apply the boundary condition (i) in (1)

$$y(0,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$

weget $c_1=0$, then equation (1) becomes

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat) \text{ ------- } (2)$$

Apply the boundary condition (ii) in (2)

$$y(l,t)=c_2\sin pl(c_3\cos pat+c_4\sin pat)=0$$

weget $\sin pl=0 \Rightarrow pl=n\pi \Rightarrow p=\frac{n\pi}{l}$, then equation (2) becomes

$$y(x,t)=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l}+c_4\sin\frac{n\pi at}{l}\right) \text{ ------- } (3)$$

Differentiate Partially (3) w. r. to t

$$\frac{\partial y}{\partial t}(x,t) = c_2 \sin\frac{n\pi x}{l}\left(-c_3 \frac{n\pi a}{l}\sin\frac{n\pi at}{l} + c_4 \frac{n\pi\pi a}{l}\cos\frac{n\pi at}{l}\right) \text{-------} (4)$$

Apply the boundary condition (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = c_2 \sin\frac{n\pi x}{l} c_4 \frac{n\pi a}{l} = 0$$

We get $c_4 = 0$ then equation (3) becomes

$$y(x,t) = c_2 \sin\frac{n\pi x}{l} c_3 \cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad \text{where } c_n = c_2 c_3$$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad (5)$$

Apply the boundary condition (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.1 = \begin{cases} \frac{2hx}{l}, & 0 \le x \le \frac{l}{2} \\ \frac{2h}{l}(l-x), & \frac{l}{2} \le x \le l \end{cases} = f(x)$$

To find the value of c_n, expand $f(x)$ in a half range Fourier sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (7)$$

By comparing (6) and (7), we get $b_n = c_n$

To find c_n it is enough to find b_n

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4h}{l^2}\left[\left(x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right)_0^{1/2}\right.$$

$$\left.+\left((l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right)_{1/2}^{l}\right]$$

$$= \frac{4h}{l^2}\left[\frac{l^2}{2n\pi}\left(-\cos\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)+\frac{l^2}{2n\pi}\left(\cos\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{8h}{l^2}\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= c_n$$

$\therefore$ The equation (5) becomes

$$y(x,t)=\sum_{n=1}^{\infty}\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}\,\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

3 . The points of trisection of a string are pulled aside through a distance 'b' on opposite sides of the position of equilibrium and the string is released from rest. Find an expression for the displacement.

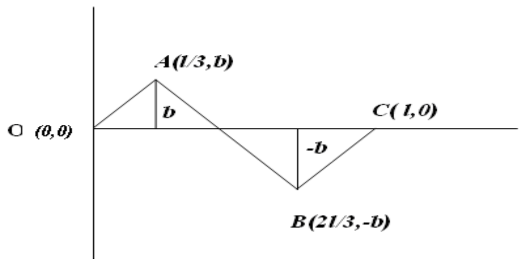
Sol:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

The conditions are

(i) $y(0,t)=0$ (ii) $y(l,t)=0$ (iii) $\frac{\partial y}{\partial t}(x,0)=0$

(iv) $y(x,0)=f(x)$



Equation of OA:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-0}{0-b}=\frac{x-0}{0-\ell/3}\Rightarrow y=\frac{3bx}{\ell}\quad \text{in}\quad (0,\ell/3)$$

Equation of AB:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-b}{b-(-b)}=\frac{x-\ell/3}{-\ell/3}\Rightarrow y=\frac{3b}{\ell}(\ell-2x)\quad \text{in}\quad (\ell/3,2\ell/3)$$

Equation of BC:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-0}{0-b}=\frac{x-0}{0-\ell/3}\Rightarrow y=\frac{3b}{\ell}(x-\ell)\quad \text{in}\quad (2\ell/3\ell,)$$

$$\therefore\ f(x)=\begin{cases}\dfrac{3bx}{\ell} & \text{in}\quad (0,\ell/3)\\ \dfrac{3b}{\ell}(\ell-2x) & \text{in}\quad (\ell/3,2\ell/3)\\ \dfrac{3b}{\ell}(x-\ell) & \text{in}\quad (2\ell/3,\ell)\end{cases}$$

The correct solution which satisfying the given boundary conditions is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat) \text{ -------- } (1)$$

Apply the boundary condition (i) in (1)

$$y(0,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$

weget $c_1=0$, then equation (1) becomes

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat) \text{ ------- } (2)$$

Apply the boundary condition (ii) in (2)

$$y(l,t)=c_2\sin pl(c_3\cos pat+c_4\sin pat)=0$$

weget $\sin pl=0\ \Rightarrow pl=n\pi\ \Rightarrow p=\dfrac{n\pi}{l}$, then equation (2) becomes

$$y(x,t)=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l}+c_4\sin\frac{n\pi at}{l}\right) \text{ ------- } (3)$$

Differentiate Partially (3) w. r. to t

$$\frac{\partial y}{\partial t}(x,t) = c_2 \sin\frac{n\pi x}{l}\left(-c_3\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + c_4\frac{n\pi\pi a}{l}\cos\frac{n\pi at}{l}\right) \text{------- (4)}$$

Apply the boundary condition (iii) in (4)

$$\frac{\partial y}{\partial t}(x,0) = c_2 \sin\frac{n\pi x}{l} c_4 \frac{n\pi a}{l} = 0$$

We get $c_4 = 0$ then equation (3) becomes

$$y(x,t) = c_2 \sin\frac{n\pi x}{l} c_3 \cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \qquad \text{where } c_n = c_2 c_3$$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{-------------- (5)}$$

Apply the boundary condition (iv) in (5)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.1 = \text{f(x)} = \begin{cases} \dfrac{3\text{bx}}{\ell} & \text{in } (0,\ell/3) \\ \dfrac{3\text{b}}{\ell}(\ell - 2\text{x}) & \text{in } (\ell/3, 2\ell/3) \\ \dfrac{3\text{b}}{\ell}(\text{x} - \ell) & \text{in } (2\ell/3, \ell) \end{cases}$$

To find the value of c_n, expand $f(x)$ in a half range Fourier sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (7)$$

By comparing (6) and (7), we get $b_n = c_n$

To find c_n it is enough to find b_n

$$\text{b}_\text{n} = \frac{2}{\ell}\int_0^{\ell} \text{f(x)}\sin\frac{\text{n}\pi\text{x}}{\ell}\,\text{dx}$$

$$\text{b}_\text{n} = \frac{2}{\ell}\left[\int_0^{\ell/3} \frac{3\text{bx}}{\ell}\sin\frac{\text{n}\pi\text{x}}{\ell}\,\text{dx} + \int_{\ell/3}^{2\ell/3} \frac{3\text{b}}{\ell}(\ell - 2\text{x})\sin\frac{\text{n}\pi\text{x}}{\ell}\,\text{dx} + \int_{2\ell/3}^{\ell} \frac{3\text{b}}{\ell}(\text{x} - \ell)\sin\frac{\text{n}\pi\text{x}}{\ell}\,\text{dx}\right]$$

$$b_n = \frac{18b}{n^2\pi^2}\sin\frac{n\pi}{3}\left(1+(-1)^n\right)$$

$$\therefore b_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{36b}{(n\pi)^2}\sin\dfrac{n\pi}{3} & \text{if } n \text{ is even} \end{cases}$$

Substitute b_n values in equation (5) we get the required solution

$$y(x,t) = \sum_{n=2,4}^{\infty} \frac{36b}{(n\pi)^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi at}{\ell}$$

4. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity $kx(l-x)$. Find the displacement of the string at any time.

Sol:

One dimensional wave equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$

The conditions are

(i) $y(0,t) = 0$ (ii) $y(l,t) = 0$ (iii) $y(x,0) = 0$

(iv) $\dfrac{\partial y}{\partial t}(x,0) = kx(l-x)$

The correct solution which satisfying the given boundary conditions is

$$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat) \text{ --------(1)}$$

apply the boundary condition (i) in (1)

$$y(0,t) = c_1(c_3\cos pat + c_4\sin pat) = 0$$

weget $c_1 = 0$ then equation (1) becomes

$$y(x,t) = c_2\sin px(c_3\cos pat + c_4\sin pat) \text{ -----------------(2)}$$

Apply the boundary condition (ii) in (2)

$$y(l,t) = c_2\sin pl(c_3\cos pat + c_4\sin pat) = 0$$

weget $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = \dfrac{n\pi}{l}$,

then equation (2) becomes

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \text{--------------(3)}$$

Apply the boundary condition (iii) in (3)

$$y(x,0) = c_2 \sin\frac{n\pi x}{l} c_3 = 0$$

here $c_3 = 0$

then equation (3) becomes

$$y(x,t) = c_2 \sin\frac{n\pi x}{l} c_4 \sin\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \qquad \text{where} \quad c_n = c_2 c_4$$

By the superposition principle, the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \text{ ----------------(4)}$$

Differentiate Partially (4) w.r.to t

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \frac{n\pi a}{l} \text{ ------------------------(5)}$$

Apply the boundary condition (iv) in (5)

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.\frac{n\pi a}{l} = kx(l-x) = f(x) \text{ -------------(6)}$$

To find the value of c_n, expand $f(x)$ in a half range Fourier sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ --------------------------(7)}$$

By comparing (6) and (7), we get $b_n = c_n \frac{n\pi a}{l} \Rightarrow c_n = \frac{l}{n\pi a} b_n$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[0+0-2.\frac{l^3}{n^3\pi^3}\cos n\pi + 0 - 0 + 2.\frac{l^3}{n^3\pi^3}.1\right]$$

$$= \frac{2k}{l}\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & if\ n\ is\ even \\ \dfrac{8kl^2}{n^3\pi^3} & if\ n\ is\ odd \end{cases}$$

$$\therefore \quad c_n = \frac{l}{n\pi a} b_n = \frac{l}{n\pi a}\frac{8l^2k}{n^3\pi^3} = \frac{8l^3k}{n^4\pi^4}$$

$\therefore$ The equation (4) becomes $\quad y(x,t) = \sum_{n=1,3,5}^{\infty} \frac{8kl^3}{n^4\pi^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

4. A rod of length l has its ends A and B kept at $0^\circ c$ and $120^\circ c$ respectively until steady state conditions prevail. If the temperature at B is reduced to $0^\circ c$ and so while that of A is maintained, find the temperature distribution of the rod.

   Sol:

   One dimensional heat equation $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$ -----(1)

Steady state equation is $\frac{\partial^2 u}{\partial x^2} = 0$ and the steady state solution is $\quad u(x) = ax + b$ --(2)

The boundary conditions are $(i)\ u(0) = 0 \quad (ii)\ u(l) = 120$

Apply (i) in (2), $\quad u(0) = b = 0$

The equation (2) becomes $u(x) = ax$ -------- (3)

Apply (ii) in (3)

$$u(l) = al = 120$$

$$\Rightarrow a = \frac{120}{l}$$

The equation (3) becomes $u(x) = \frac{120x}{l}$

Now consider the unsteady state condition.

The conditions are

$(iii)\ u(0,t) = 0 \qquad (iv)\ u(l,t) = 0 \qquad (v)\ u(x,0) = \frac{120x}{l}$

The correct solution which satisfying the given boundary conditions is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-a^2 p^2 t} \text{ ----------- } (4)$$

Apply the boundary condition (iii) in (4)

$$u(0,t) = c_1\, c_3 e^{-a^2 p^2 t} = 0$$

weget $c_1 = 0$

then equation (4) becomes $u(x,t) = c_2 \sin px\, c_3 e^{-a^2 p^2 t}$ ---------- (5)

Apply the boundary condition (iv) in (5)

$$u(l,t) = c_2 \sin pl\, c_3 e^{-a^2 p^2 t} = 0$$

weget $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l},$

The equation (5) becomes

$$u(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$$

$$= c_n \sin \frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t} \qquad \text{where } c_2 c_3 = c_n$$

By the super position principle, the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t} \qquad \text{------------------} \qquad (6)$$

Apply the boundary condition (v) in (6)

$$u(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.1=\frac{120x}{l}= f(x) \text{------------------} \qquad (7)$$

To find c_n, expand $f(x)$ in half range sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{------------------ (8)}$$

From the equations (7) & (8) we get $b_n = c_n$

Now

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\int_0^l \frac{120x}{l}\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{240}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{240}{l}\left[\frac{-l^2}{n\pi}\cos n\pi\right]$$

$$=\frac{-240l}{n\pi}(-1)^n$$

$$=\frac{240l}{n\pi}(-1)^{n+1}$$

$\therefore$ The required solution is

$$u(x,t)=\sum_{n=1}^{\infty}=\frac{240l}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l}e^{\frac{-a^2n^2\pi^2}{l^2}t}$$

5. ***The ends A and B of a rod l cm long have their temperatures kept at $30^\circ c$ and $80^\circ c$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ c$ and that of A is increased to $40^\circ c$. Find the temperature distribution in the rod after time t.***

Sol:

One dimensional heat equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ -----(1)

Steady state equation is $\frac{\partial^2 u}{\partial x^2} = 0$ and the steady state solution is $u(x) = ax + b$ --(2)

the boundary conditions are $(i)\ u(0) = 30 \quad (ii)\ u(l) = 80$

Apply (i) in (2), $u(0) = b = 30$

Then the equation (2) becomes $u(x) = ax + 30$ (3)

Apply (ii) in (3)

$$u(l) = al + 30 = 80$$

$$\Rightarrow a = \frac{50}{l}$$

Then the equation (3) becomes $u(x) = \frac{50x}{l} + 30$

Now consider the unsteady state condition.

In unsteady state the suitable solution which satisfying the given boundary conditions is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-a^2 p^2 t} \tag{4}$$

The boundary conditions are

$(iii)\ u(0,t) = 40 \qquad (iv)\ u(l,t) = 60 \qquad (v)\ u(x,0) = u(x) = \frac{50x}{l} + 30$

Since we have all non zero boundary conditions, we write the temperature distribution function as $u(x,t) = u_s(x) + u_t(x,t)$ (5)

$$\Rightarrow \quad u_t(x,t) = u(x,t) - u_s(x)$$

<u>To find</u> $u_s(x)$

The solution is $u_s(x) = ax + b$ (6)

The boundary conditions are $u_s(0) = 60,\ u_s(l) = 40$

$$\therefore \quad u_s(0) = b = 40,$$

$$u_s(l) = al + b = 60$$

$$\Rightarrow al + 40 = 60$$

$$\Rightarrow a = \frac{20}{l}$$

$\therefore$ the equation (6) becomes $u_s(x) = \dfrac{20x}{l} + 40$ (7)

To find $u_t(x,t)$

Given the boundary conditions are

$$(vi)\ u_t(x,t) = u(0,t) - u_s(0) = 40 - 40 = 0$$
$$(vii)\ u_t(l,t) = u(l,t) - u_s(l) = 60 - 60 = 0$$
$$(viii)\ u_t(x,0) = u(x,0) - u_s(x) = \frac{50x}{l} + 30 - \left(\frac{20x}{l} + 40\right)$$
$$= \frac{30x}{l} - 10$$

In unsteady state, the suitable solution which satisfying the given boundary conditions is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-a^2 p^2 t} \quad (8)$$

Apply the boundary condition (vi) in (8)

$$u_t(0,t) = c_1 c_3 e^{-a^2 p^2 t} = 0$$

here $c_1 = 0$

$\therefore$ the equation (8) becomes $u_t(x,t) = c_2 \sin px\, c_3 e^{-a^2 p^2 t}$ (9)

Apply the boundary condition (vii) in (9)

$$u_t(l,t) = c_2 c_3 \sin pl e^{-a^2 p^2 t} = 0$$

here

$$\sin pl = 0$$
$$\Rightarrow p = \frac{n\pi}{l}$$

$\therefore$ the equation (9) becomes $u_t(x,t) = c_2 \sin \dfrac{n\pi x}{l} x c_3 e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$

$$u_t(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$$

where $c_2 c_3 = c_n$

$$= c_n \sin \frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$$

By the super position principle, the most general solution is

$$u_t(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t} \qquad (10)$$

Apply the boundary condition (viii) in (10)

$$u_t(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}.1 = \frac{30x}{l} - 10 = f(x)$$

To find c_n, expand $f(x)$ in half range sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (11)$$

From the equations (10) & (11) we get $b_n = c_n$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \left(\frac{30x}{l} - 10\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{30x}{l} - 10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{30}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{10l}{n\pi}.1\right]$$

$$= \frac{-20}{n\pi}\left[1 + 2(-1)^n\right]$$

$$= c_n$$

$\therefore$ the equation (10) becomes

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{-20}{n\pi}\left[1 + 2(-1)^n\right]\sin\frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$$

Then the required temperature distribution function is

$$u(x,t) = \frac{20x}{l} + 40 - \sum_{n=1}^{\infty} \frac{20}{n\pi}\left[1 + 2(-1)^n\right]\sin\frac{n\pi x}{l} e^{\frac{-a^2 n^2 \pi^2}{l^2} t}$$

6. *A rectangular plate is bounded by the lines* $x=0, y=0, x=a, y=b,\ b<a$. *Its surfaces are insulated. The temperature along* $x=0$ *and* $y=0$ *are kept at* $0^\circ C$ *and the others at* $100^\circ C$. *Find the steady state temperature at any point of the plate.*

Sol:

Two dimensional heat equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

Let l be the length of the square plate

Given the boundary conditions are

$(i)\ u(0,y)=0 \qquad (ii)\ u(x,0)=0 \qquad (iii)\quad u(a,y)=100$

$(iv)\quad u(x,b)=100$

Assume the temperature distribution as

$$u(x,y)=u_1(x,y)+u_2(x,y) \qquad \text{(A)}$$

To find $u_1(x,y)$

Consider the boundary conditions

$(v)\ u_1(0,y)=0 \qquad (vi)\ u_1(x,0)=0 \qquad (vii)\quad u_1(x,b)=0$

$(viii)\quad u_1(a,y)=100$

The suitable solution which satisfying the given boundary conditions is

$$u_1(x,y)=(c_1e^{px}+c_2e^{-px})(c_3\cos py+c_4\sin py) \qquad (1)$$

Apply the boundary condition (v) in (1)

$$u_1(0,y)=(c_1+c_2)(c_3\cos py+c_4\sin py)=0$$

weget $\quad c_1+c_2=0$

$$\Rightarrow c_2=-c_1$$

then the equation (2) becomes

$$\begin{aligned}u_1(x,y)&=(c_1e^{px}-c_1e^{-px})(c_3\cos py+c_4\sin py)\\&=c_1(e^{px}-e^{-px})(c_3\cos py+c_4\sin py)\end{aligned} \qquad (2)$$

Apply the boundary condition (vi) in (2)

$$u_1(x,0)=c_1(e^{px}-e^{-px})c_3=0$$

weget $\ c_3=0$

then the equation (3) becomes

$$u_1(x,y) = c_1(e^{px} - e^{-px})\, c_4 \sin py \qquad (3)$$

Apply the boundary condition (vii) in (3)

$$u_1(x,b) = c_1(e^{px} - e^{-px})\, c_4 \sin pb = 0$$

weget $\sin pb = 0 \;\Rightarrow\; pb = n\pi \Rightarrow\; p = \frac{n\pi}{b}$

then the equation (4) becomes

$$u_1(x,y) = c_1(e^{\frac{n\pi x}{b}} + e^{-\frac{n\pi x}{b}})\, c_4 \sin \frac{n\pi y}{b}$$

$$= c_1 c_4 2\sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \qquad \text{where } c_n = 2c_1c_4$$

$$= c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

The most general solution is

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \qquad (4)$$

Apply the boundary condition (viii) in (4)

$$u_1(a,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = 100 = f(y) \qquad (5)$$

To find c_n, expand $f(x)$ in half range sine series

We know that half range Fourier sine series of $f(x)$ is

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \qquad (6)$$

From the equations (5) & (6) we get

$$b_n = c_n \sinh \frac{n\pi a}{b} \Rightarrow c_n = \frac{b_n}{\sinh \frac{n\pi a}{b}}$$

$$b_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

$$= \frac{2}{b}\int_0^b 100\sin\frac{n\pi y}{b}\,dy$$

$$= \frac{200}{b}\left[\left(\frac{-\cos\frac{n\pi y}{b}}{\frac{n\pi}{b}}\right)\right]_0^b$$

$$= \frac{200}{b}\left[\frac{b}{n\pi}\left(-\cos n\pi + 1\right)\right]$$

$$= \frac{200}{n\pi}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore c_n = \frac{400}{n\pi\sinh n\pi} \qquad \text{if n is odd}$$

then the equation (6) becomes

$$u_1(x,y) = \sum_{n=1,3,5}^{\infty} \frac{400}{n\pi\sinh\frac{n\pi a}{b}}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

To find $u_2(x,y)$

Consider the boundary conditions

(ix) $u_2(0,y)=0$ (x) $u_2(x,0)=0$ (xi) $u_2(a,y)=0$

(xii) $u_2(x,b)=100$

and the suitable solution in this case is

$$u_2(x,y) = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \qquad (7)$$

Apply the boundary condition (ix) in (7)

$$u_2(0,y) = c_5(c_6e^{py} + c_8e^{-py}) = 0$$

weget $c_5 = 0$

then the equation (8) becomes

$$u_2(x,y) = c_6\sin px(c_7e^{py} + c_8e^{-py}) \qquad (8)$$

Apply the boundary condition (x) in (8)

$$u_2(x,0) = c_6 \sin px(c_7 + c_8) = 0$$

weget $c_7 + c_8 = 0 \qquad \Rightarrow c_8 = -c_7$

then the equation (9) becomes

$$\begin{aligned} u_2(x,y) &= c_6 \sin px(c_7 e^{py} - c_7 e^{-py}) \\ &= c_6 c_7 \sin px(e^{py} - e^{-py}) \end{aligned} \qquad (9)$$

Apply the boundary condition (xi) in (9)

$$u_2(a,y) = c_6 c_7 \sin pa(e^{py} - e^{-py}) = 0$$

weget $\sin pa = 0 \quad \Rightarrow \quad pa = n\pi \Rightarrow \quad p = \frac{n\pi}{a}$

then the equation (10) becomes

$$\begin{aligned} u_2(x,y) &= c_6 c_7 \sin\frac{n\pi x}{a}(e^{\frac{n\pi y}{a}} - e^{\frac{n\pi y}{a}}) \\ &= c_6 c_7 2\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a} \qquad \text{where } c_n = 2c_6 c_7 \\ &= c_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a} \end{aligned}$$

The most general solution is

$$u_2(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a} \qquad (11)$$

Apply the boundary condition (xii) in (11)

$$u_2(x,b) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} = 100 = f(x)$$

To find c_n, expand $f(x)$ in half range sine series

Half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \qquad (12)$$

From the equations (11) & (12) we get

$$b_n = c_n \sinh\frac{n\pi b}{a} \Rightarrow c_n = \frac{b_n}{\sinh\frac{n\pi b}{a}}$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$= \frac{2}{a}\int_0^a 100\sin\frac{n\pi x}{a}dx$$

$$= \frac{200}{a}\left[\left(\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)\right]_0^a$$

$$= \frac{200}{a}\left[\frac{a}{n\pi}\left(-\cos n\pi + 1\right)\right]$$

$$= \frac{200}{n\pi}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0 & if\ \ n\ is\ \ even \\ \frac{400}{n\pi} & if\ \ n\ is\ \ odd \end{cases}$$

$$\therefore c_n = \frac{400}{n\pi\sinh\frac{n\pi b}{a}} \qquad \text{if n is odd}$$

then the equation (10) becomes

$$u_2(x,y) = \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$

Then the equation (A) becomes

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

$$= \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} + \sum_{n=1,3,5}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$